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Research Article

Quadruple Best Proximity Points with Applications to Functional and Integral Equations

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This manuscript is devoted to obtaining a quadruple best proximity point for a cyclic contraction mapping in the setting of ordinary metric spaces. The validity of the theoretical results is also discussed in uniformly convex Banach spaces. Furthermore, some examples are given to strengthen our study. Also, under suitable conditions, some quadruple fixed point results are presented. Finally, as applications, the existence and uniqueness of a solution to a system of functional and integral equations are obtained to promote our paper.

1. Introduction and Preliminaries

Fixed point (FP) theory has many applications not only in the nonlinear analysis and its trends, from solutions of differential and integral equations, functional equations arising from dynamical programming, topology, and a dynamical system, but also in economics, game theory, biological sciences, computer sciences, and chemistry, etc. [1–4].

The FP technique became more attractive and elegant when Banach [5] introduced his principle, which is stated as follows: a mapping $\Gamma : Y \longrightarrow Y$ defined on a complete metric space (MS) (Y, d) has a unique FP if Γ is a contraction, i.e., $d(\Gamma\sigma_1, \Gamma\sigma_2) \le kd(\sigma_1, \sigma_2), k \in (0, 1)$. He used this method for studying the existence of solutions for some integral equations.

The FP technique was extended to a coupled and tripled FP by Bhaskar and Lakshmikantham [6] and Berinde and Borcut [7], respectively. Many researchers have worked in these directions and obtained exciting results and life applications that serve the scientific communities, which in turn has led to the fixed points being brilliant and pioneering in the field of functional analysis until the present time. For more details, see [8–20].

Not only did the matter stop here, but Karapinar and Sadarangani [21] were able to generalize the triple point to the quadruple and generalized the previous results on this scale in terms of theories and applications. After that, this trend spreads to others; see, for instance, [22–25].

In 1978, Pathak and B. Fisher [26] were able to merge the state and decision space to clarify the importance of the FP methodology in finding the solution to the following functional system:

$$\eta(\sigma_1) = \sup_{\sigma_2 \in D} \aleph(\sigma_1, \sigma_2, \eta(\mathfrak{T}(\sigma_1, \sigma_2))), \tag{1}$$

where S and D are the state space and the decision space, respectively; $\sigma_1 \in S, \mathfrak{T}$ denotes the transformation of the process; and $\eta(\sigma_1)$ refers to the optimal return function with the initial state σ_1. The above system is called a functional equation arising from dynamical programming which is commonly used in modeling and optimization problems. To clarify the participation of fixed and coincidence points and to delve deeper into this trend, we guide the reader to read these papers [27–30].

On the other hand, the importance of the FP technique lies in the fact that it presents a unified process and an important tool in solving equations that do not have to be linear. In the case of $d(z, \Gamma z) \neq 0$, that is, a contraction mapping Γ does not possess a FP; it became necessary to search a point$z$that makes$d(z, \Gamma z)$the minimum, meaning the point$z$ is in close proximity to$\Gamma(z)$.

Assume that Z, O are nonempty closed subsets of a complete MS Y and $\Gamma : Z \longrightarrow O$ is a given mapping. A point $z \in Z$ is called the best proximity point (BPP, for short) if $d(z, \Gamma z) = d(Z, O)$, where $d(Z, O)$ is described as

$$d(Z, O) = \inf \{d(z, o) \colon z \in Z, o \in O\}. \tag{2}$$

It should be noted that if $Z \cap O \neq \emptyset$, then a BPP reduces to a usual FP.

The initial paper concerned with the BPP was presented by Fan [31]. This direction is very interesting in optimization, so many researchers have discussed and developed this direction in several applications; see, [32–36].

Now, we need the definitions below.

Definition 1 (see [37]). A Banach space Y is said to be

(i) strictly convex, if

$$\|\sigma_1\| = \|\sigma_2\| = 1 \text{ with } \sigma_1 \neq \sigma_2 \text{ implies} \left\|\frac{\sigma_1 + \sigma_2}{2}\right\| < 1, \quad \forall \sigma_1, \sigma_2 \in Y, \tag{3}$$

(ii) uniformly convex, if for any $\varepsilon \in (0, 2]$, there is $\delta > 0$ so that

$$\|\sigma_1\| \leq 1, \|\sigma_2\| \leq 1 \text{ with } \|\sigma_1 - \sigma_2\| \geq \varepsilon \text{ implies} \left\|\frac{\sigma_1 + \sigma_2}{2}\right\| < 1 - \delta, \quad \forall \sigma_1, \sigma_2 \in Y. \tag{4}$$

Clearly, a uniformly convex Banach space (UCBS) is strictly convex, but the opposite does not hold.

Definition 2 (see [21]). Assume that (Y, d) is a MS and $Z \neq \emptyset$ is a subset of Y. A point $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ is called a quadruple fixed point (QFP, for short) of the map $\Theta : Z^4 \longrightarrow Z$ if

$$\sigma_1 = \Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \sigma_2 = \Theta(\sigma_2, \sigma_3, \sigma_4, \sigma_1), \sigma_3 = \Theta(\sigma_3, \sigma_4, \sigma_1, \sigma_2), \sigma_4 = \Theta(\sigma_4, \sigma_1, \sigma_2, \sigma_3). \tag{5}$$

Our paper is arranged as follows: In Section 2, some new definitions and supporting examples are presented. Also, the convergence of quadruple best proximity (QBP, for short) points for a pair of cyclic contraction mappings without and with property UC^* are obtained in the context of metric spaces (MSs). Moreover, quadruple fixed point (QFP) results for cyclic contraction mappings are established, and an example for supporting the above results is discussed in Section 3. In Section 4, the existence of a solution for quadruple functional equations arising in dynamical programming is discussed, and an example for supporting the results is presented. Ultimately, in Section 5, the existence of solutions for a system of quadruple integral equations is given, and an example is obtained to strengthen this contribution.

2. Main Results

This part is devoted to present the convergence of QBP points for a pair of cyclic contraction mappings in the setting of ordinary MSs.

We begin this part with the definitions below.

Definition 3. Let Z and O be two nonempty closed subsets of a MS (Y, d). The pair (Z, O) is called to satisfy the property UC, if there exist $\{\sigma_1^n\}, \{\sigma_4^n\} \subset Z$ and $\{\sigma_2^n\}, \{\sigma_3^n\} \subset O$ such that

$$d(\sigma_1^n, \sigma_2^n) \longrightarrow d(Z, O), d(\sigma_1^n, \sigma_3^n) \longrightarrow d(Z, O), d(\sigma_4^n, \sigma_2^n) \longrightarrow d(Z, O), d(\sigma_4^n, \sigma_3^n) \longrightarrow d(Z, O), \tag{6}$$

as $n \longrightarrow \infty$, then

$$d(\sigma_1^n, \sigma_4^n) \longrightarrow 0, \text{as } n \longrightarrow \infty. \tag{7}$$

Example 1. Suppose that Z and O are two nonempty subsets of a MS (Y, d) with $d(Z, O) = 0$. Then, (Z, O) satisfies the property UC.

Example 2. Assume that Z, Z', O, and O' are nonempty subsets of a MS (Y, d) with $Z \subset Z'$,$O \subset O'$, and $d(Z, O) = d(Z', O')$. If the pair (Z, O) satisfies the property UC, then the pair (Z', O') satisfies also the property UC.

Example 3. Suppose that Z and O are nonempty subsets of a UCBS, then the pair (Z, O) satisfies the property UC, if one of the hypotheses below holds:

(i) Z is convex

(ii) Z is convex relatively compact

Definition 4. Assume that (Y, d) is a MS and Z, O are two nonempty subsets of Y. We say that the pair (Z, O) has the property UC* if (Z, O) has the property UC and the stipulation below is fulfilled:

If the sequences $\{\sigma_1^n\},\{\sigma_4^n\}$ in Z and $\{\sigma_2^n\},\{\sigma_3^n\}$ in O so that the following statements are fulfilled:

$(\dagger_1)$ $d(\sigma_4^n, \sigma_2^n) \longrightarrow d(Z, O)$ and $d(\sigma_4^n, \sigma_3^n) \longrightarrow d(Z, O)$

$(\dagger_2)$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ so that

$$\begin{aligned} d(\sigma_1^m, \sigma_2^n) &\leq d(Z, O) + \varepsilon, \quad \forall m > n \geq N, \\ d(\sigma_1^m, \sigma_3^n) &\leq d(Z, O) + \varepsilon, \quad \forall m > n \geq N, \end{aligned} \tag{8}$$

then there is $N_1 \in \mathbb{N}$ so that

$$d(\sigma_1^m, \sigma_4^n) \leq d(Z, O) + \varepsilon, \quad \forall m > n \geq N_1. \tag{9}$$

Example 4. For nonempty subsets Z and O of a MS (Y, d), assume that $d(Z, O) = 0$. Then, the pair (Z, O) posses the property UC*.

Example 5. Assume that all requirements of Example 2 hold. If the pair (Z, O) has the property UC*, then the pair (Z', O') has the property UC* too.

Example 6. Assume that Z,O are two nonempty subsets of a UCBS and Z is convex. Then, (Z, O) verifies the property UC*.

Definition 5. Assume that (Y, d) is a MS and Z, O are nonempty closed subsets of Y. Also assume $\Theta : Z^4 \longrightarrow O$ is a given mapping. We say that a quadruple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ is a QBP point of Θ if,

$$\begin{aligned} d(\sigma_1, \Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4)) &= d(\sigma_2, \Theta(\sigma_2, \sigma_3, \sigma_4, \sigma_1)) \\ &= d(\sigma_3, \Theta(\sigma_3, \sigma_4, \sigma_1, \sigma_2)) \\ &= d(\sigma_4, \Theta(\sigma_4, \sigma_1, \sigma_2, \sigma_3)) \\ &= d(Z, O). \end{aligned} \tag{10}$$

Clearly, when $Z = O$ in Definition 5, then a QBP point reduces to a QFP.

Definition 6. Assume that (Y, d) is a MS and Z, O are nonempty closed subsets of Y. We say that the mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ are cyclic contractions if there is $\ell \in 0, 1)$ so that the inequality below holds:

$$\begin{aligned} &d(\Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)) \\ &\quad \leq \frac{\ell}{4}(d(\sigma_1, \vartheta_1) + d(\sigma_2, \vartheta_2) + d(\sigma_3, \vartheta_3) + d(\sigma_4, \vartheta_4)) \\ &\quad + (1 - \ell) d(Z, O), \end{aligned} \tag{11}$$

for all $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$.

Notice that, if the pair (Θ, Ω) is a cyclic contraction, then the pair (Ω, Θ) is a cyclic contraction too.

Example 7. Consider $Y = \mathbb{R}$ equipped with the distance $d(\sigma_1, \sigma_2) = |\sigma_1 - \sigma_2|$. Let $Z = [3, 9]$ and $O = [-9, -3]$. Obviously, $d(Z, O) = 6$. Describe two mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ as

$$\begin{aligned} \Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= \frac{-\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - 12}{8}, \\ \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) &= \frac{-\vartheta_1 - \vartheta_2 - \vartheta_3 - \vartheta_4 + 12}{8}, \end{aligned} \tag{12}$$

for all $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$, respectively. For each $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$ and let $\ell = 1/2$, we have

$$\begin{aligned} &d(\Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)) \\ &\quad = \left| \frac{-\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - 12}{8} - \frac{-\vartheta_1 - \vartheta_2 - \vartheta_3 - \vartheta_4 + 12}{8} \right| \\ &\quad \leq \frac{|\sigma_1 - \vartheta_1| + |\sigma_2 - \vartheta_2| + |\sigma_3 - \vartheta_3| + |\sigma_4 - \vartheta_4|}{8} + 3 \\ &\quad = \frac{\ell}{4}(d(\sigma_1, \vartheta_1) + d(\sigma_2, \vartheta_2) + d(\sigma_3, \vartheta_3) + d(\sigma_4, \vartheta_4)) \\ &\quad + (1 - \ell) d(Z, O). \end{aligned} \tag{13}$$

This leads to the pair (Θ, Ω) as a cyclic contraction with $\ell = 1/2$.

Example 8. Consider $Y = \mathbb{R}^4$ endowed with

$$\begin{aligned} &d((\sigma_1, \sigma_2, \sigma_3, \sigma_4), (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)) \\ &\quad = \max \{|\sigma_1 - \vartheta_1|, |\sigma_2 - \vartheta_2|, |\sigma_3 - \vartheta_3|, |\sigma_4 - \vartheta_4|\}, \end{aligned} \tag{14}$$

for all $(\sigma_1, \sigma_2, \sigma_3, \sigma_4),(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in Y$ and suppose that

$$\begin{aligned} Z &= \left\{(\sigma_1, 0, 0, 0) \in \mathbb{R}^4 : 0 \leq \sigma_1 \leq 1\right\}, O \\ &= \left\{(\vartheta_1, 1, 1, 1) \in \mathbb{R}^4 : 0 \leq \vartheta_1 \leq 1\right\}. \end{aligned} \tag{15}$$

Clearly, $d(Z, O) = 1$. Define $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ by

$$\begin{aligned} &\Theta((\sigma_1, 0, 0, 0), (\sigma_2, 0, 0, 0), (\sigma_3, 0, 0, 0), (\sigma_4, 0, 0, 0)) \\ &\quad = \left(\frac{\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4}{4}, 1, 1, 1\right), \\ &\Omega((\vartheta_1, 1, 1, 1), (\vartheta_2, 1, 1, 1), (\vartheta_3, 1, 1, 1), (\vartheta_4, 1, 1, 1)) \\ &\quad = \left(\frac{\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4}{4}, 0, 0, 0\right), \end{aligned} \tag{16}$$

respectively. Then, we obtain

$$
\begin{aligned}
& d\begin{pmatrix} \Theta((\sigma_1,0,0,0),(\sigma_2,0,0,0),(\sigma_3,0,0,0),(\sigma_4,0,0,0)), \\ \Omega((\vartheta_1,1,1,1),(\vartheta_2,1,1,1),(\vartheta_3,1,1,1),(\vartheta_4,1,1,1)) \end{pmatrix} \\
& = d\left(\left(\frac{\sigma_1+\sigma_2+\sigma_3+\sigma_4}{4},1,1,1\right),\left(\frac{\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4}{4},0,0,0\right)\right) \\
& = 1.
\end{aligned} \tag{17}
$$

Also, if

$$
\begin{aligned}
& (\sigma_1,0,0,0),(\sigma_2,0,0,0),(\sigma_3,0,0,0),(\sigma_4,0,0,0) \in Z, \\
& (\vartheta_1,1,1,1),(\vartheta_2,1,1,1),(\vartheta_3,1,1,1),(\vartheta_4,1,1,1) \in O,
\end{aligned} \tag{18}
$$

then one can write

$$
\begin{aligned}
& \frac{\ell}{4}(d((\sigma_1,0,0,0),(\vartheta_1,1,1,1)) + d((\sigma_2,0,0,0),(\vartheta_2,1,1,1)) \\
& \quad + d((\sigma_3,0,0,0),(\vartheta_3,1,1,1)) + d((\sigma_4,0,0,0),(\vartheta_4,1,1,1))) \\
& \quad + (1-\ell)d(Z,O) = \frac{\ell}{4}(\max\{|\sigma_1-\vartheta_1|,1,1,1\} \\
& \quad + \max\{|\sigma_2-\vartheta_2|,1,1,1\} + \max\{|\sigma_3-\vartheta_3|,1,1,1\} \\
& \quad + \max\{|\sigma_4-\vartheta_4|,1,1,1\}) + (1-\ell)d(Z,O) = \frac{\ell}{4}\times 4 \\
& \quad + (1-\ell) = 1,
\end{aligned} \tag{19}
$$

for any $\ell < 1$. In addition, let

$$
\begin{aligned}
& (\sigma_1,\sigma_2,\sigma_3,\sigma_4) = ((\sigma_1,0,0,0),(\sigma_2,0,0,0),(\sigma_3,0,0,0),(\sigma_4,0,0,0)) \in Z^4, \\
& (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) = ((\vartheta_1,1,1,1),(\vartheta_2,1,1,1),(\vartheta_3,1,1,1),(\vartheta_4,1,1,1)) \in O^4,
\end{aligned} \tag{20}
$$

then it follows from (17) and (19) that

$$
\begin{aligned}
& d(\Theta(\sigma_1,\sigma_2,\sigma_3,\sigma_4),\Omega(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4)) \\
& \quad \le \frac{\ell}{4}(d(\sigma_1,\vartheta_1)+d(\sigma_2,\vartheta_2)+d(\sigma_3,\vartheta_3)+d(\sigma_4,\vartheta_4)) \\
& \quad + (1-\ell)d(Z,O).
\end{aligned} \tag{21}
$$

Thus, the pair (Θ,Ω) is a cyclic contraction.

The lemma below is very important in the sequel.

Lemma 7. *Assume that (Y,d) is a MS and Z,O are nonempty closed subsets of Y. Let $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ be two cyclic contraction mappings. If $(\sigma_1^0,\sigma_2^0,\sigma_3^0,\sigma_4^0) \in Z^4$ and the sequences $\{\sigma_1^n\},\{\sigma_2^n\},\{\sigma_3^n\},\{\sigma_4^n\}$ in Y are defined as follows:*

$$
\begin{aligned}
& \sigma_1^{2n+1} = \Theta(\sigma_1^{2n},\sigma_2^{2n},\sigma_3^{2n},\sigma_4^{2n}), \sigma_1^{2n+2} = \Omega(\sigma_1^{2n+1},\sigma_2^{2n+1},\sigma_3^{2n+1},\sigma_4^{2n+1}), \\
& \sigma_2^{2n+1} = \Theta(\sigma_2^{2n},\sigma_3^{2n},\sigma_4^{2n},\sigma_1^{2n}), \sigma_2^{2n+2} = \Omega(\sigma_2^{2n+1},\sigma_3^{2n+1},\sigma_4^{2n+1},\sigma_1^{2n+1}), \\
& \sigma_3^{2n+1} = \Theta(\sigma_3^{2n},\sigma_4^{2n},\sigma_1^{2n},\sigma_2^{2n}), \sigma_3^{2n+2} = \Omega(\sigma_3^{2n+1},\sigma_4^{2n+1},\sigma_1^{2n+1},\sigma_2^{2n+1}), \\
& \sigma_4^{2n+1} = \Theta(\sigma_4^{2n},\sigma_1^{2n},\sigma_2^{2n},\sigma_3^{2n}), \sigma_4^{2n+2} = \Omega(\sigma_4^{2n+1},\sigma_1^{2n+1},\sigma_2^{2n+1},\sigma_3^{2n+1}),
\end{aligned} \tag{22}
$$

for all $n \ge 0$, then we get

$$
\begin{aligned}
& d(\sigma_1^{2n},\sigma_1^{2n+1}) \longrightarrow d(Z,O), d(\sigma_1^{2n+1},\sigma_1^{2n+2}) \longrightarrow d(Z,O), \\
& d(\sigma_2^{2n},\sigma_2^{2n+1}) \longrightarrow d(Z,O), d(\sigma_2^{2n+1},\sigma_2^{2n+2}) \longrightarrow d(Z,O), \\
& d(\sigma_3^{2n},\sigma_3^{2n+1}) \longrightarrow d(Z,O), d(\sigma_3^{2n+1},\sigma_3^{2n+2}) \longrightarrow d(Z,O), \\
& d(\sigma_4^{2n},\sigma_4^{2n+1}) \longrightarrow d(Z,O), d(\sigma_1^{2n+1},\sigma_1^{2n+2}) \longrightarrow d(Z,O).
\end{aligned} \tag{23}
$$

Proof. Consider, for each $n \in \mathbb{N}$,

$$
\begin{aligned}
d(\sigma_1^{2n},\sigma_1^{2n+1}) &= d(\sigma_1^{2n},\Theta(\sigma_1^{2n},\sigma_2^{2n},\sigma_3^{2n},\sigma_4^{2n})) = d\left(\begin{array}{c} \Omega(\sigma_1^{2n-1},\sigma_2^{2n-1},\sigma_3^{2n-1},\sigma_4^{2n-1}), \\ \Theta\begin{pmatrix} \Omega(\sigma_1^{2n-1},\sigma_2^{2n-1},\sigma_3^{2n-1},\sigma_4^{2n-1}),\Omega(\sigma_2^{2n-1},\sigma_3^{2n-1},\sigma_4^{2n-1},\sigma_1^{2n-1}), \\ \Omega(\sigma_3^{2n-1},\sigma_4^{2n-1},\sigma_1^{2n-1},\sigma_2^{2n-1}),\Omega(\sigma_4^{2n-1},\sigma_1^{2n-1},\sigma_2^{2n-1},\sigma_3^{2n-1}) \end{pmatrix} \end{array}\right) \\
&\le \frac{\ell}{4}\left[d(\sigma_1^{2n-1},\Omega(\sigma_1^{2n-1},\sigma_2^{2n-1},\sigma_3^{2n-1},\sigma_4^{2n-1})) + d(\sigma_2^{2n-1},\Omega(\sigma_2^{2n-1},\sigma_3^{2n-1},\sigma_4^{2n-1},\sigma_1^{2n-1}))\right. \\
&\quad \left. + d(\sigma_3^{2n-1},\Omega(\sigma_3^{2n-1},\sigma_4^{2n-1},\sigma_1^{2n-1},\sigma_2^{2n-1})) + d(\sigma_4^{2n-1},\Omega(\sigma_4^{2n-1},\sigma_1^{2n-1},\sigma_2^{2n-1},\sigma_3^{2n-1}))\right] + (1-\ell)d(Z,O) \\
&= \frac{\ell}{4}\left[d\left(\begin{array}{c} \Theta(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}), \\ \Omega\begin{pmatrix} \Theta(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}),\Theta(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}), \\ \Theta(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}),\Theta(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}) \end{pmatrix} \end{array}\right) + d\left(\begin{array}{c} \Theta(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}), \\ \Omega\begin{pmatrix} \Theta(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}),\Theta(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}) \\ \Theta(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}),\Theta(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}) \end{pmatrix} \end{array}\right)\right. \\
&\quad \left. + d\left(\begin{array}{c} \Theta(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}), \\ \Omega\begin{pmatrix} \Theta(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}),\Theta(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}), \\ \Theta(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}),\Theta(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}) \end{pmatrix} \end{array}\right) + d\left(\begin{array}{c} \Theta(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}), \\ \Omega\begin{pmatrix} \Theta(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}),\Theta(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}) \\ \Theta(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}),\Theta(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}) \end{pmatrix} \end{array}\right)\right] \\
&\quad + (1-\ell)d(Z,O).
\end{aligned} \tag{24}
$$

Using (11), we have

$$
\begin{aligned}
d\left(\sigma_1^{2n},\sigma_1^{2n+1}\right) \le \frac{\ell}{4}\Bigg[&\frac{\ell}{4}\left(d\left(\sigma_1^{2n-2},\Theta\left(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}\right)\right)\right.\\
&+ d\left(\sigma_2^{2n-2},\Theta\left(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}\right)\right)\\
&+ d\left(\sigma_3^{2n-2},\Theta\left(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}\right)\right)\\
&\left.+ d\left(\sigma_4^{2n-2},\Theta\left(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}\right)\right)\right)\\
&+ (1-\ell)d(Z,O) + \frac{\ell}{4}\left(d\left(\sigma_2^{2n-2},\Theta\left(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}\right)\right)\right.\\
&+ d\left(\sigma_3^{2n-2},\Theta\left(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}\right)\right)\\
&+ d\left(\sigma_4^{2n-2},\Theta\left(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}\right)\right)\\
&\left.+ d\left(\sigma_1^{2n-2},\Theta\left(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}\right)\right)\right) + (1-\ell)d(Z,O)\\
&+ \frac{\ell}{4}\left(d\left(\sigma_3^{2n-2},\Theta\left(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}\right)\right), d\right.\\
&\cdot\left(\sigma_4^{2n-2},\Theta\left(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}\right)\right)\\
&+ d\left(\sigma_1^{2n-2},\Theta\left(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}\right)\right)\\
&\left.+ d\left(\sigma_2^{2n-2},\Theta\left(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}\right)\right)\right)\\
&+ (1-\ell)d(Z,O) + \frac{\ell}{4}\left(d\left(\sigma_4^{2n-2},\Theta\left(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}\right)\right)\right.\\
&+ d\left(\sigma_1^{2n-2},\Theta\left(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}\right)\right)\\
&+ d\left(\sigma_2^{2n-2},\Theta\left(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}\right)\right)\\
&\left.+ d\left(\sigma_3^{2n-2},\Theta\left(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}\right)\right)\right)\\
&+ (1-\ell)d(Z,O)\Bigg] + (1-\ell)d(Z,O)\\
= \frac{\ell^2}{4}&\left(d\left(\sigma_1^{2n-2},\Theta\left(\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2}\right)\right)\right.\\
&+ d\left(\sigma_2^{2n-2},\Theta\left(\sigma_2^{2n-2},\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2}\right)\right)\\
&+ d\left(\sigma_3^{2n-2},\Theta\left(\sigma_3^{2n-2},\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2}\right)\right)\\
&\left.+ d\left(\sigma_4^{2n-2},\Theta\left(\sigma_4^{2n-2},\sigma_1^{2n-2},\sigma_2^{2n-2},\sigma_3^{2n-2}\right)\right)\right)\\
&+ \left(1-\ell^2\right)d(Z,O).
\end{aligned}
\tag{25}
$$

By mathematical induction, we obtain for each $n \in \mathbb{N}$ that

$$
\begin{aligned}
d\left(\sigma_1^{2n},\sigma_1^{2n+1}\right) \le \frac{\ell^{2n}}{4}&\left(d\left(\sigma_1^0,\Theta\left(\sigma_1^0,\sigma_2^0,\sigma_3^0,\sigma_4^0\right)\right) + d\left(\sigma_2^0,\Theta\left(\sigma_2^0,\sigma_3^0,\sigma_4^0,\sigma_1^0\right)\right)\right.\\
&\left.+ d\left(\sigma_3^0,\Theta\left(\sigma_3^0,\sigma_4^0,\sigma_1^0,\sigma_2^0\right)\right) + d\left(\sigma_4^0,\Theta\left(\sigma_4^0,\sigma_1^0,\sigma_2^0,\sigma_3^0\right)\right)\right)\\
&+ \left(1-\ell^2\right)d(Z,O).
\end{aligned}
\tag{26}
$$

Passing $n \longrightarrow \infty$, we find that

$$
d\left(\sigma_1^{2n},\sigma_1^{2n+1}\right) \longrightarrow d(Z,O). \tag{27}
$$

Again, for each $n \in \mathbb{N}$, by induction, one can write

$$
\begin{aligned}
d\left(\sigma_1^{2n+1},\sigma_1^{2n+2}\right) \le \frac{\ell^{2n}}{4}&\left(d\left(\sigma_1^1,\Theta\left(\sigma_1^1,\sigma_2^1,\sigma_3^1,\sigma_4^1\right)\right) + d\left(\sigma_2^1,\Theta\left(\sigma_2^1,\sigma_3^1,\sigma_4^1,\sigma_1^1\right)\right)\right.\\
&\left.+ d\left(\sigma_3^1,\Theta\left(\sigma_3^1,\sigma_4^1,\sigma_1^1,\sigma_2^1\right)\right) + d\left(\sigma_4^1,\Theta\left(\sigma_4^1,\sigma_1^1,\sigma_2^1,\sigma_3^1\right)\right)\right)\\
&+ \left(1-\ell^2\right)d(Z,O),
\end{aligned}
\tag{28}
$$

this yields after passing $n \longrightarrow \infty$,

$$
d\left(\sigma_1^{2n+1},\sigma_1^{2n+2}\right) \longrightarrow d(Z,O). \tag{29}
$$

Analogously, we have

$$
\begin{aligned}
&d\left(\sigma_2^{2n},\sigma_2^{2n+1}\right) \longrightarrow d(Z,O), d\left(\sigma_2^{2n+1},\sigma_2^{2n+2}\right) \longrightarrow d(Z,O), d\left(\sigma_3^{2n},\sigma_3^{2n+1}\right) \longrightarrow d(Z,O),\\
&d\left(\sigma_3^{2n+1},\sigma_3^{2n+2}\right) \longrightarrow d(Z,O), d\left(\sigma_4^{2n},\sigma_4^{2n+1}\right) \longrightarrow d(Z,O), d\left(\sigma_1^{2n+1},\sigma_1^{2n+2}\right) \longrightarrow d(Z,O).
\end{aligned}
\tag{30}
$$

This finishes the required proof. □

Lemma 8. *Assume that (Y,d) is a MS and Z, O are nonempty closed subsets of Y so that (Z,O) and (O,Z) satisfy the property UC. Let the mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ be cyclic contractions. If $(\sigma_1^0,\sigma_2^0,\sigma_3^0,\sigma_4^0) \in Z^4$ and the sequences $\{\sigma_1^n\},\{\sigma_2^n\},\{\sigma_3^n\},\{\sigma_4^n\}$ in Y are defined as (22), $\forall n \in \mathbb{N} \cup \{0\}$, then for each $\varepsilon > 0$, there is $N_0 > 0$ so that*

$$
\begin{aligned}
&\frac{1}{4}\left(d\left(\sigma_1^{2m},\sigma_1^{2n+1}\right) + d\left(\sigma_2^{2m},\sigma_2^{2n+1}\right) + d\left(\sigma_3^{2m},\sigma_3^{2n+1}\right) + d\left(\sigma_4^{2m},\sigma_4^{2n+1}\right)\right)\\
&\quad < d(Z,O) + \varepsilon, \quad \forall m > n \ge N_0.
\end{aligned}
\tag{31}
$$

Proof. According to Lemma 7, we get

$$
\begin{aligned}
&d\left(\sigma_1^{2n},\sigma_1^{2n+1}\right) \longrightarrow d(Z,O), d\left(\sigma_1^{2n+1},\sigma_1^{2n+2}\right) \longrightarrow d(Z,O),\\
&d\left(\sigma_2^{2n},\sigma_2^{2n+1}\right) \longrightarrow d(Z,O), d\left(\sigma_2^{2n+1},\sigma_2^{2n+2}\right) \longrightarrow d(Z,O),\\
&d\left(\sigma_3^{2n},\sigma_3^{2n+1}\right) \longrightarrow d(Z,O), d\left(\sigma_3^{2n+1},\sigma_3^{2n+2}\right) \longrightarrow d(Z,O),\\
&d\left(\sigma_4^{2n},\sigma_4^{2n+1}\right) \longrightarrow d(Z,O), d\left(\sigma_1^{2n+1},\sigma_1^{2n+2}\right) \longrightarrow d(Z,O).
\end{aligned}
\tag{32}
$$

Because (Z,O) fulfills the property UC, then we have

$$
\begin{aligned}
d\left(\sigma_1^{2n},\sigma_1^{2n+2}\right) \longrightarrow 0, d\left(\sigma_2^{2n},\sigma_2^{2n+2}\right) \longrightarrow 0, d\left(\sigma_3^{2n},\sigma_3^{2n+2}\right)&\\
\longrightarrow 0, d\left(\sigma_4^{2n},\sigma_4^{2n+2}\right) \longrightarrow 0.&
\end{aligned}
\tag{33}
$$

Also, (O,Z) verifies the property UC, we have

$$
\begin{aligned}
d\left(\sigma_1^{2n+1},\sigma_1^{2n+3}\right) \longrightarrow 0, d\left(\sigma_2^{2n+1},\sigma_2^{2n+3}\right) \longrightarrow 0, d\left(\sigma_3^{2n+1},\sigma_3^{2n+3}\right)&\\
\longrightarrow 0, d\left(\sigma_4^{2n+1},\sigma_4^{2n+3}\right) \longrightarrow 0.&
\end{aligned}
\tag{34}
$$

Assuming (31) is not true. Then, for each $k \in \mathbb{N}$ with $m_k > n_k \ge k$, there is $\varepsilon' > 0$ so that

$$
\begin{aligned}
&\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right) + d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right) + d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right.\\
&\quad\left.+ d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right) \ge d(Z,O) + \varepsilon'.
\end{aligned}
\tag{35}
$$

Hence, we can select the smallest integer m_k with $m_k > n_k$ fulfilling (35). Therefore,

$$\begin{aligned}&\frac{1}{4}\left(d\left(\sigma_1^{2m_k-2},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k-2},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k-2},\sigma_3^{2n_k+1}\right)\right.\\&\quad\left.+d\left(\sigma_4^{2m_k-2},\sigma_4^{2n_k+1}\right)\right)<d(Z,O)+\varepsilon'.\end{aligned}\tag{36}$$

Thus, we obtain

$$\begin{aligned}d(Z,O)+\varepsilon'&\le\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)\right.\\&\quad\left.+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)\le\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2m_k-2}\right)\right.\\&\quad+d\left(\sigma_1^{2m_k-2},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k-2}\right)+d\left(\sigma_2^{2m_k-2},\sigma_2^{2n_k+1}\right)\\&\quad+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k-2}\right)+d\left(\sigma_3^{2m_k-2},\sigma_3^{2n_k+1}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k-2}\right)\\&\quad\left.+d\left(\sigma_4^{2m_k-2},\sigma_4^{2n_k+1}\right)\right)<\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2m_k-2}\right)\right.\\&\quad\left.+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k-2}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k-2}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k-2}\right)\right)\\&\quad+d(Z,O)+\varepsilon'.\end{aligned}\tag{37}$$

As $k\longrightarrow\infty$, one can write

$$\begin{aligned}&\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2m_k-2}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k-2}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k-2}\right)\right.\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k-2}\right)\right)\longrightarrow d(Z,O)+\varepsilon'.\end{aligned}\tag{38}$$

Applying the triangle inequality, we get

$$\begin{aligned}&\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right.\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)\le\frac{1}{4}\left[d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)\right.\\&\quad+d\left(\sigma_1^{2m_k+2},\sigma_1^{2n_k+3}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\&\quad+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k+2}\right)+d\left(\sigma_2^{2m_k+2},\sigma_2^{2n_k+3}\right)\\&\quad+d\left(\sigma_2^{2n_k+3},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k+2}\right)\\&\quad+d\left(\sigma_3^{2m_k+2},\sigma_3^{2n_k+3}\right)+d\left(\sigma_3^{2n_k+3},\sigma_3^{2n_k+1}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k+2}\right)\\&\quad\left.+d\left(\sigma_4^{2m_k+2},\sigma_4^{2n_k+3}\right)+d\left(\sigma_4^{2n_k+3},\sigma_4^{2n_k+1}\right)\right]=\frac{1}{4}\left[d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)\right.\\&\quad+d\left(\Omega\left(\sigma_1^{2m_k+1},\sigma_2^{2m_k+1},\sigma_3^{2m_k+1},\sigma_4^{2m_k+1}\right),\right.\\&\left.\left.\Theta\left(\sigma_1^{2n_k+2},\sigma_2^{2n_k+2},\sigma_3^{2n_k+2},\sigma_4^{2n_k+2}\right)\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\right]\\&\quad+\frac{1}{4}\left[d\left(\sigma_2^{2m_k},\sigma_2^{2m_k+2}\right)+d\left(\Omega\left(\sigma_2^{2m_k+1},\sigma_3^{2m_k+1},\sigma_4^{2m_k+1},\sigma_1^{2m_k+1}\right),\right.\right.\\&\left.\left.\Theta\left(\sigma_2^{2n_k+2},\sigma_3^{2n_k+2},\sigma_4^{2n_k+2},\sigma_1^{2n_k+2}\right)\right)++d\left(\sigma_2^{2n_k+3},\sigma_2^{2n_k+1}\right)\right]+\frac{1}{4}\\&\quad\cdot\left[d\left(\sigma_3^{2m_k},\sigma_3^{2m_k+2}\right)+d\left(\Omega\left(\sigma_3^{2m_k+1},\sigma_4^{2m_k+1},\sigma_1^{2m_k+1},\sigma_2^{2m_k+1}\right),\right.\right.\\&\quad\left.\left.\Theta\left(\sigma_3^{2n_k+2},\sigma_4^{2n_k+2},\sigma_1^{2n_k+2},\sigma_2^{2n_k+2}\right)\right)+d\left(\sigma_3^{2n_k+3},\sigma_3^{2n_k+1}\right)\right]+\frac{1}{4}\\&\qquad\cdot\left[d\left(\sigma_4^{2m_k},\sigma_4^{2m_k+2}\right)+d\left(\Omega\left(\sigma_4^{2m_k+1},\sigma_1^{2m_k+1},\sigma_2^{2m_k+1},\sigma_3^{2m_k+1}\right),\right.\right.\\&\left.\left.\Theta\left(\sigma_4^{2n_k+2},\sigma_1^{2n_k+2},\sigma_2^{2n_k+2},\sigma_3^{2n_k+2}\right)\right)+d\left(\sigma_4^{2n_k+3},\sigma_4^{2n_k+1}\right)\right].\end{aligned}\tag{39}$$

Applying (11), we obtain that

$$\begin{aligned}&\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right.\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)\le\frac{1}{4}\left[d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)\right.\\&\quad+\frac{\ell}{4}\left(d\left(\sigma_1^{2m_k+1},\sigma_1^{2n_k+2}\right)+d\left(\sigma_2^{2m_k+1},\sigma_2^{2n_k+2}\right)+d\left(\sigma_3^{2m_k+1},\sigma_3^{2n_k+2}\right)\right.\\&\quad\left.+d\left(\sigma_4^{2m_k+1},\sigma_4^{2n_k+2}\right)\right)+(1-\ell)d(Z,O)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\&\quad+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k+2}\right)+\frac{\ell}{4}\left(d\left(\sigma_2^{2m_k+1},\sigma_2^{2n_k+2}\right)\right.\\&\quad\left.+d\left(\sigma_3^{2m_k+1},\sigma_3^{2n_k+2}\right)+d\left(\sigma_4^{2m_k+1},\sigma_4^{2n_k+2}\right)+d\left(\sigma_1^{2m_k+1},\sigma_1^{2n_k+2}\right)\right)\\&\quad+(1-\ell)d(Z,O)+d\left(\sigma_2^{2n_k+3},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k+2}\right)\\&\quad+\frac{\ell}{4}\left(d\left(\sigma_3^{2m_k+1},\sigma_3^{2n_k+2}\right)+d\left(\sigma_4^{2m_k+1},\sigma_4^{2n_k+2}\right)+d\left(\sigma_1^{2m_k+1},\sigma_1^{2n_k+2}\right)\right.\\&\quad\left.+d\left(\sigma_2^{2m_k+1},\sigma_2^{2n_k+2}\right)\right)+(1-\ell)d(Z,O)+d\left(\sigma_3^{2n_k+3},\sigma_3^{2n_k+1}\right)\\&\quad+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k+2}\right)+\frac{\ell}{4}\left(d\left(\sigma_4^{2m_k+1},\sigma_4^{2n_k+2}\right)\right.\\&\quad\left.+d\left(\sigma_1^{2m_k+1},\sigma_1^{2n_k+2}\right)+d\left(\sigma_2^{2m_k+1},\sigma_2^{2n_k+2}\right)+d\left(\sigma_3^{2m_k+1},\sigma_3^{2n_k+2}\right)\right)\\&\quad\left.+(1-\ell)d(Z,O)+d\left(\sigma_4^{2n_k+3},\sigma_4^{2n_k+1}\right)\right]=\frac{1}{4}\left[d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)\right.\\&\quad+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k+2}\right)\\&\quad+d\left(\sigma_2^{2n_k+3},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k+2}\right)+d\left(\sigma_3^{2n_k+3},\sigma_3^{2n_k+1}\right)\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k+2}\right)+d\left(\sigma_4^{2n_k+3},\sigma_4^{2n_k+1}\right)\right]+\frac{\ell}{4}\left(d\left(\sigma_1^{2m_k+1},\sigma_1^{2n_k+2}\right)\right.\\&\quad\left.+d\left(\sigma_2^{2m_k+1},\sigma_2^{2n_k+2}\right)+d\left(\sigma_3^{2m_k+1},\sigma_3^{2n_k+2}\right)+d\left(\sigma_4^{2m_k+1},\sigma_4^{2n_k+2}\right)\right)\\&\quad+(1-\ell)d(Z,O).\end{aligned}\tag{40}$$

It follows that

$$\begin{aligned}&\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right.\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)\le\frac{1}{4}\left[d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)\right.\\&\quad+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k+2}\right)\\&\quad+d\left(\sigma_2^{2n_k+3},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k+2}\right)+d\left(\sigma_3^{2n_k+3},\sigma_3^{2n_k+1}\right)\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k+2}\right)+d\left(\sigma_4^{2n_k+3},\sigma_4^{2n_k+1}\right)\right]+\frac{\ell}{4}\\&\quad\cdot\left[d\left(\Theta\left(\sigma_1^{2m_k},\sigma_2^{2m_k},\sigma_3^{2m_k},\sigma_4^{2m_k}\right),\Omega\left(\sigma_1^{2n_k+1},\sigma_2^{2n_k+1},\sigma_3^{2n_k+1},\sigma_4^{2n_k+1}\right)\right)\right.\\&\quad+d\left(\Theta\left(\sigma_2^{2m_k},\sigma_3^{2m_k},\sigma_4^{2m_k},\sigma_1^{2m_k}\right),\Omega\left(\sigma_2^{2n_k+1},\sigma_3^{2n_k+1},\sigma_4^{2n_k+1},\sigma_1^{2n_k+1}\right)\right)\\&\quad+d\left(\Theta\left(\sigma_3^{2m_k},\sigma_4^{2m_k},\sigma_1^{2m_k},\sigma_2^{2m_k}\right),\Omega\left(\sigma_3^{2n_k+1},\sigma_4^{2n_k+1},\sigma_1^{2n_k+1},\sigma_2^{2n_k+1}\right)\right)\\&\quad\left.+d\left(\Theta\left(\sigma_4^{2m_k},\sigma_1^{2m_k},\sigma_2^{2m_k},\sigma_3^{2m_k}\right),\Omega\left(\sigma_4^{2n_k+1},\sigma_1^{2n_k+1},\sigma_2^{2n_k+1},\sigma_3^{2n_k+1}\right)\right)\right]\\&\quad+(1-\ell)d(Z,O)\le\frac{1}{4}\left[d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\right.\\&\quad+d\left(\sigma_2^{2m_k},\sigma_2^{2m_k+2}\right)+d\left(\sigma_2^{2n_k+3},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2m_k+2}\right)+d\left(\sigma_3^{2n_k+3},\sigma_3^{2n_k+1}\right)\\&\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2m_k+2}\right)+d\left(\sigma_4^{2n_k+3},\sigma_4^{2n_k+1}\right)\right]+\frac{\ell^2}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)\right.\\&\quad\left.+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\quad+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)+(1-\ell)d(Z,O).\end{aligned}\tag{41}$$

Letting $n \longrightarrow \infty$, we have

$$d(Z,O)+\varepsilon' \le \ell^2\left(d(Z,O)+\varepsilon'\right)+\left(1-\ell^2\right)d(Z,O)=d(Z,O)+\ell^2\varepsilon', \tag{42}$$

which is a contradiction since $\ell < 1$. This implies that (31) is fulfilled, and this finishes the proof. □

Lemma 9. *Assume that (Y,d) is a MS and Z, O are nonempty closed subsets of Y so that (Z,O) and (O,Z) satisfy the property UC^*. Let the mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : Z^4 \longrightarrow O$ be cyclic contractions. If $(\sigma_1^0,\sigma_2^0,\sigma_3^0,\sigma_4^0)\in Z^4$ and for all $n\in\mathbb{N}\cup\{0\}$, the sequences $\{\sigma_1^n\}$, $\{\sigma_2^n\}$, $\{\sigma_3^n\}$, $\{\sigma_4^n\}$ in Y are defined by (22). Then, $\{\sigma_1^{2n}\}$,$\{\sigma_2^{2n}\}$,$\{\sigma_3^{2n}\}$, and $\{\sigma_4^{2n}\}$ are Cauchy sequences.*

Proof. Based on Lemma 7, one can get

$$d\left(\sigma_1^{2n},\sigma_1^{2n+1}\right)\longrightarrow d(Z,O),\ d\left(\sigma_1^{2n+1},\sigma_1^{2n+2}\right)\longrightarrow d(Z,O). \tag{43}$$

As (Z,O) satisfies the property UC, then $d(\sigma_1^{2n},\sigma_1^{2n+2})\longrightarrow 0$. Similarly, since (O,Z) verifies the property UC, then $d(\sigma_1^{2n+1},\sigma_1^{2n+3})\longrightarrow 0$.

Now, we claim that, $\forall\varepsilon>0$, $\exists N\in\mathbb{N}$ so that

$$d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)\le d(Z,O)+\varepsilon,\quad \forall m>n\ge N. \tag{44}$$

Assume that (44) is not true. Then, $\forall k\in\mathbb{N}$, $\exists\varepsilon>0$, and $m_k>n_k\ge k$ such that

$$d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)> d(Z,O)+\varepsilon. \tag{45}$$

Therefore, we can select a smallest integer m_k with $m_k>n_k$ fulfilling (45). Hence, one can get

$$\begin{aligned} d(Z,O)+\varepsilon &< d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)\le d\left(\sigma_1^{2m_k},\sigma_1^{2m_k-2}\right)\\ &\quad + d\left(\sigma_1^{2m_k-2},\sigma_1^{2n_k+1}\right)\le d\left(\sigma_1^{2m_k},\sigma_1^{2m_k-2}\right)\\ &\quad + d(Z,O)+\varepsilon. \end{aligned} \tag{46}$$

Setting $k\longrightarrow\infty$, we have

$$d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)\longrightarrow d(Z,O)+\varepsilon. \tag{47}$$

Using Lemma 8, we can write

$$\begin{aligned} &\frac{1}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right.\\ &\quad\left.+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)<d(Z,O)+\varepsilon, \end{aligned} \tag{48}$$

for all $m_k>n_k\ge k$. Applying the triangle inequality, we get

$$\begin{aligned} d(Z,O)+\varepsilon &< d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)\le d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)\\ &\quad + d\left(\sigma_1^{2m_k+2},\sigma_1^{2n_k+3}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\ &= d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\Omega\left(\sigma_1^{2m_k+1},\sigma_2^{2m_k+1},\sigma_3^{2m_k+1},\sigma_4^{2m_k+1}\right),\right.\\ &\quad\left.\Theta\left(\sigma_1^{2n_k+2},\sigma_2^{2n_k+2},\sigma_3^{2n_k+2},\sigma_4^{2n_k+2}\right)\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\ &\le d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+\frac{\ell}{4}\left(d\left(\sigma_1^{2m_k+1},\sigma_1^{2m_k+2}\right)\right.\\ &\quad + d\left(\sigma_2^{2m_k+1},\sigma_2^{2m_k+2}\right)+d\left(\sigma_3^{2m_k+1},\sigma_3^{2m_k+2}\right)\\ &\quad \left.+ d\left(\sigma_4^{2m_k+1},\sigma_4^{2m_k+2}\right)\right)+(1-\ell)d(Z,O)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\ &= \frac{\ell}{4}\left[d\left(\Theta\left(\sigma_1^{2m_k},\sigma_2^{2m_k},\sigma_3^{2m_k},\sigma_4^{2m_k}\right),\Omega\left(\sigma_1^{2n_k+1},\sigma_2^{2n_k+1},\sigma_3^{2n_k+1},\sigma_4^{2n_k+1}\right)\right)\right.\\ &\quad + d\left(\Theta\left(\sigma_2^{2m_k},\sigma_3^{2m_k},\sigma_4^{2m_k},\sigma_1^{2m_k}\right),\Omega\left(\sigma_2^{2n_k+1},\sigma_3^{2n_k+1},\sigma_4^{2n_k+1},\sigma_1^{2n_k+1}\right)\right)\\ &\quad + d\left(\Theta\left(\sigma_3^{2m_k},\sigma_4^{2m_k},\sigma_1^{2m_k},\sigma_2^{2m_k}\right),\Omega\left(\sigma_3^{2n_k+1},\sigma_4^{2n_k+1},\sigma_1^{2n_k+1},\sigma_2^{2n_k+1}\right)\right)\\ &\quad \left.+ d\left(\Theta\left(\sigma_4^{2m_k},\sigma_1^{2m_k},\sigma_2^{2m_k},\sigma_3^{2m_k}\right),\Omega\left(\sigma_4^{2n_k+1},\sigma_1^{2n_k+1},\sigma_2^{2n_k+1},\sigma_3^{2n_k+1}\right)\right)\right]\\ &\quad + d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)+(1-\ell)d(Z,O)\\ &\le \frac{\ell}{4}\left[\frac{\ell}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right.\right.\\ &\quad \left.+ d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)+(1-\ell)d(Z,O)+\frac{\ell}{4}\left(d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)\right.\\ &\quad \left.+ d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)+d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)\right)\\ &\quad + (1-\ell)d(Z,O)\frac{\ell}{4}\left(d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right.\\ &\quad \left.+ d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)\right)\\ &\quad + (1-\ell)d(Z,O)+\frac{\ell}{4}\left(d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right.\\ &\quad \left.+ d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)+d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)\right)\\ &\quad \left.+ (1-\ell)d(Z,O)\right]+d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\ &\quad + (1-\ell)d(Z,O)=\frac{\ell^2}{4}\left(d\left(\sigma_1^{2m_k},\sigma_1^{2n_k+1}\right)+d\left(\sigma_2^{2m_k},\sigma_2^{2n_k+1}\right)\right.\\ &\quad \left.+ d\left(\sigma_3^{2m_k},\sigma_3^{2n_k+1}\right)+d\left(\sigma_4^{2m_k},\sigma_4^{2n_k+1}\right)\right)+\left(1-\ell^2\right)d(Z,O)\\ &\quad + d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)<\ell^2(d(Z,O)+\varepsilon)\\ &\quad + \left(1-\ell^2\right)d(Z,O)+d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right)\\ &= \ell^2\varepsilon+d(Z,O)+d\left(\sigma_1^{2m_k},\sigma_1^{2m_k+2}\right)+d\left(\sigma_1^{2n_k+3},\sigma_1^{2n_k+1}\right). \end{aligned} \tag{49}$$

Letting $n\longrightarrow\infty$, we obtain that

$$d(Z,O)+\varepsilon\le d(Z,O)+\ell^2\varepsilon. \tag{50}$$

This is a contradiction. This achieves the inequality (44). It follows from (44), $d(\sigma_1^{2n},\sigma_1^{2n+1})\longrightarrow d(Z,O)$ and the property UC^* of $d(Z,O)$ that $\{\sigma_1^{2n}\}$ is a Cauchy sequence. By the same manner, we can show that $\{\sigma_2^{2n}\}$, $\{\sigma_3^{2n}\}$, and $\{\sigma_4^{2n}\}$ are Cauchy sequences. This finishes the proof. □

Now, via the property UC^*, we shall discuss the existence and convergence of QBP points.

Theorem 10. *Assume that Z and O are nonempty closed subsets of Y so that the property UC^* are satisfied on (Z, O) and (O, Z). Let the mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ be cyclic contractions. If $(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0) \in Z^4$ and for all $n \in \mathbb{N} \cup \{0\}$, the sequences $\{\sigma_1^n\}, \{\sigma_2^n\}, \{\sigma_3^n\}, \{\sigma_4^n\}$ in Y are described as (22). Then, Θ has a QBP point $(\daleth_1, \daleth_2, \daleth_3, \daleth_4) \in Z^4$ and Ω has a QBP point $(\daleth_1', \daleth_2', \daleth_3', \daleth_4') \in O^4$. Moreover, we have*

$$\begin{aligned}\sigma_1^{2n} &\longrightarrow \daleth_1, \sigma_2^{2n} \longrightarrow \daleth_2, \sigma_3^{2n} \longrightarrow \daleth_3, \sigma_4^{2n} \longrightarrow \daleth_4, \sigma_1^{2n+1} \\ &\longrightarrow \daleth_1', \sigma_2^{2n+1} \longrightarrow \daleth_2', \sigma_3^{2n+1} \longrightarrow \daleth_3', \sigma_4^{2n+1} \longrightarrow \daleth_4'.\end{aligned} \tag{51}$$

In addition, if $\daleth_2 = \daleth_3 = \daleth_4$ and $\daleth_2' = \daleth_3' = \daleth_4'$, then

$$d\left(\daleth_1, \daleth_1'\right) + d\left(\daleth_2, \daleth_2'\right) + d\left(\daleth_3, \daleth_3'\right) + d\left(\daleth_4, \daleth_4'\right) = 4d(Z, O). \tag{52}$$

Proof. Based on Lemma 7, we conclude that $d(\sigma_1^{2n}, \sigma_1^{2n+1}) \longrightarrow d(Z, O)$. From Lemma 9, we find that $\{\sigma_1^{2n}\}, \{\sigma_2^{2n}\}, \{\sigma_3^{2n}\}$, and $\{\sigma_4^{2n}\}$ are Cauchy sequences. Thus, there are $(\daleth_1, \daleth_2, \daleth_3, \daleth_4) \in Z$ so that $\sigma_1^{2n} \longrightarrow \daleth_1$, $\sigma_2^{2n} \longrightarrow \daleth_2$, $\sigma_3^{2n} \longrightarrow \daleth_3$, and $\sigma_4^{2n} \longrightarrow \daleth_4$. Hence, we have

$$d(Z, O) \leq d\left(\daleth_1, \sigma_1^{2n-1}\right) \leq d\left(\daleth_1, \sigma_1^{2n}\right) + d\left(\sigma_1^{2n}, \sigma_1^{2n-1}\right). \tag{53}$$

Passing $n \longrightarrow \infty$ in (53), we find that

$$d\left(\daleth_1, \sigma_1^{2n-1}\right) \longrightarrow d(Z, O). \tag{54}$$

By the same method, we have

$$\begin{aligned}d\left(\daleth_2, \sigma_1^{2n-1}\right) &\longrightarrow d(Z, O), d\left(\daleth_3, \sigma_3^{2n-1}\right) \longrightarrow d(Z, O), d\left(\daleth_4, \sigma_4^{2n-1}\right) \\ &\longrightarrow d(Z, O).\end{aligned} \tag{55}$$

Now, consider

$$\begin{aligned}d(\sigma_1^{2n}, \Theta(\daleth_1, \daleth_2, \daleth_3, \daleth_4)) &= d\left(\Omega\left(\sigma_1^{2n-1}, \sigma_2^{2n-1}, \sigma_3^{2n-1}, \sigma_4^{2n-1}\right), \Theta(\daleth_1, \daleth_2, \daleth_3, \daleth_4)\right) \\ &\leq \frac{\ell}{4}\left(d\left(\sigma_1^{2n-1}, \daleth_1\right), d\left(\sigma_2^{2n-1}, \daleth_2\right), d\left(\sigma_3^{2n-1}, \daleth_3\right), d\right. \\ &\quad \cdot \left.\left(\sigma_4^{2n-1}, \daleth_4\right)\right) + (1 - \ell dZ, O).\end{aligned} \tag{56}$$

Passing $n \longrightarrow \infty$, we obtain

$$d(\daleth_1, \Theta(\daleth_1, \daleth_2, \daleth_3, \daleth_4)) = d(Z, O). \tag{57}$$

Analogously, we can obtain

$$\begin{aligned}d(\daleth_2, \Theta(\daleth_2, \daleth_3, \daleth_4, \daleth_1)) &= d(Z, O), d(\daleth_3, \Theta(\daleth_3, \daleth_4, \daleth_1, \daleth_2)) \\ &= d(Z, O), d(\daleth_4, \Theta(\daleth_4, \daleth_1, \daleth_2, \daleth_3)) \\ &= d(Z, O).\end{aligned} \tag{58}$$

Therefore, $(\daleth_1, \daleth_2, \daleth_3, \daleth_4)$ is a QBP point of Θ.

Analogously, we can prove that there are $\daleth_1', \daleth_2', \daleth_3', \daleth_4' \in O$ so that $\sigma_1^{2n+1} \longrightarrow \daleth_1'$, $\sigma_2^{2n+1} \longrightarrow \daleth_2'$, $\sigma_3^{2n+1} \longrightarrow \daleth_3'$, and $\sigma_4^{2n+1} \longrightarrow \daleth_4'$. Moreover, we get

$$\begin{aligned}d\left(\daleth_1', \Theta\left(\daleth_1', \daleth_2', \daleth_3', \daleth_4'\right)\right) &= d(Z, O), d\left(\daleth_2', \Theta\left(\daleth_2', \daleth_3', \daleth_4', \daleth_1'\right)\right) \\ &= d(Z, O), \\ d\left(\daleth_3', \Theta\left(\daleth_3', \daleth_4', \daleth_1', \daleth_2'\right)\right) &= d(Z, O), d\left(\daleth_4', \Theta\left(\daleth_4', \daleth_1', \daleth_2', \daleth_3'\right)\right) \\ &= d(Z, O).\end{aligned} \tag{59}$$

Hence, $(\daleth_1', \daleth_2', \daleth_3', \daleth_4')$ is a QBP point of Ω.

Ultimately, let $\daleth_2 = \daleth_3 = \daleth_4$ and $\daleth_2' = \daleth_3' = \daleth_4'$, then we claim that (52) holds. For each $n \in \mathbb{N}$, one can write

$$\begin{aligned}d\left(\sigma_1^{2n}, \sigma_1^{2n+1}\right) &= d\left(\Omega\left(\sigma_1^{2n-1}, \sigma_2^{2n-1}, \sigma_3^{2n-1}, \sigma_4^{2n-1}\right), \Theta\left(\sigma_1^{2n}, \sigma_2^{2n}, \sigma_3^{2n}, \sigma_4^{2n}\right)\right) \\ &\leq \frac{\ell}{4}\left(d\left(\sigma_1^{2n-1}, \sigma_1^{2n}\right) + d\left(\sigma_2^{2n-1}, \sigma_2^{2n}\right) + d\left(\sigma_3^{2n-1}, \sigma_3^{2n}\right)\right. \\ &\quad \left.+ d\left(\sigma_4^{2n-1}, \sigma_4^{2n}\right)\right) + (1 - \ell)d(Z, O).\end{aligned} \tag{60}$$

Letting $n \longrightarrow \infty$, we get

$$\begin{aligned}d\left(\daleth_1, \daleth_1'\right) &\leq \frac{\ell}{4}\left(d\left(\daleth_1, \daleth_1'\right) + d\left(\daleth_2, \daleth_2'\right) + d\left(\daleth_3, \daleth_3'\right)\right. \\ &\quad \left.+ d\left(\daleth_4, \daleth_4'\right)\right) + (1 - \ell)d(Z, O).\end{aligned} \tag{61}$$

Also, $\forall n \in \mathbb{N}$, we get

$$\begin{aligned}d\left(\sigma_2^{2n}, \sigma_2^{2n+1}\right) &= d\left(\Omega\left(\sigma_2^{2n-1}, \sigma_3^{2n-1}, \sigma_4^{2n-1}, \sigma_1^{2n-1}\right), \Theta\left(\sigma_2^{2n}, \sigma_3^{2n}, \sigma_4^{2n}, \sigma_1^{2n}\right)\right) \\ &\leq \frac{\ell}{4}\left(d\left(\sigma_2^{2n-1}, \sigma_2^{2n}\right) + d\left(\sigma_3^{2n-1}, \sigma_3^{2n}\right) + d\left(\sigma_4^{2n-1}, \sigma_4^{2n}\right)\right. \\ &\quad \left.+ d\left(\sigma_1^{2n-1}, \sigma_1^{2n}\right)\right) + (1 - \ell)d(Z, O).\end{aligned} \tag{62}$$

Passing $n \longrightarrow \infty$, one can obtain

$$\begin{aligned}d\left(\daleth_2, \daleth_2'\right) &\leq \frac{\ell}{4}\left(d\left(\daleth_2, \daleth_2'\right) + d\left(\daleth_3, \daleth_3'\right) + d\left(\daleth_4, \daleth_4'\right) + d\left(\daleth_1, \daleth_1'\right)\right) \\ &\quad + (1 - \ell)d(Z, O).\end{aligned} \tag{63}$$

Similarly, we obtain

$$\begin{aligned} d\left(\urcorner_3, \urcorner_3'\right) \le \frac{\ell}{4}\left(d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) + d\left(\urcorner_1, \urcorner_1'\right)\right. \\ \left. + d\left(\urcorner_2, \urcorner_2'\right)\right) + (1-\ell)d(Z, O), \\ d\left(\urcorner_4, \urcorner_4'\right) \le \frac{\ell}{4}\left(d\left(\urcorner_4, \urcorner_4'\right) + d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right)\right. \\ \left. + d\left(\urcorner_3, \urcorner_3'\right)\right) + (1-\ell)d(Z, O). \end{aligned} \tag{64}$$

It follows from (61), (63), (64), and (66) that

$$\begin{aligned} & d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) \\ & \le \ell\left(d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right)\right) \\ & \quad + 4(1-\ell)d(Z, O). \end{aligned} \tag{65}$$

This leads to

$$d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) \le 4d(Z, O). \tag{66}$$

Since

$$\begin{aligned} d(Z, O) \le d\left(\urcorner_1, \urcorner_1'\right), d(Z, O) \le d\left(\urcorner_2, \urcorner_2'\right), d(Z, O) \\ \le d\left(\urcorner_3, \urcorner_3'\right), d(Z, O) \le d\left(\urcorner_4, \urcorner_4'\right), \end{aligned} \tag{67}$$

then we have

$$d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) \ge 4d(Z, O). \tag{68}$$

According to (66) and (68), we have

$$d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) = 4d(Z, O). \tag{69}$$

This finishes the proof. □

It should be noted that every pair of nonempty closed subsets Z and O of a UCBS so that Z is convex fulfills the property UC^*, then we can state the result below.

Corollary 11. *Assume that Y is a UCBS and Z, O are nonempty closed subsets of Y. Assume also $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ are cyclic contraction mappings. If $(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0) \in Z^4$ and the sequences $\{\sigma_1^n\}, \{\sigma_2^n\}, \{\sigma_3^n\}, \{\sigma_4^n\}$ in Y are defined as (22), for each $n \in \mathbb{N} \cup \{0\}$, then Θ has a QBP point $(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4) \in Z^4$ and Ω has a QBP point $(\urcorner_1', \urcorner_2', \urcorner_3', \urcorner_4') \in Z^4$. Moreover, we get*

$$\begin{aligned} \sigma_1^{2n} \longrightarrow \urcorner_1, \sigma_2^{2n} \longrightarrow \urcorner_2, \sigma_3^{2n} \longrightarrow \urcorner_3, \sigma_4^{2n} \longrightarrow \urcorner_4, \sigma_1^{2n+1} \\ \longrightarrow \urcorner_1', \sigma_2^{2n+1} \longrightarrow \urcorner_2', \sigma_3^{2n+1} \longrightarrow \urcorner_3', \sigma_4^{2n+1} \longrightarrow \urcorner_4'. \end{aligned} \tag{70}$$

In addition, if $\urcorner_2 = \urcorner_3 = \urcorner_4$ and $\urcorner_2' = \urcorner_3' = \urcorner_4'$, then

$$d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) = 4d(Z, O). \tag{71}$$

The following example supports Corollary 11.

Example 9. Let $Y = \mathbb{R}$ be a UCBS equipped with the usual norm. Take $Z = [3, 5]$ and $O = [-5, -3]$. Obviously, $d(Z, O) = 8$. Describe two mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ as

$$\begin{aligned} \Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \frac{-\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - 16}{8}, \\ \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \frac{-\vartheta_1 - \vartheta_2 - \vartheta_3 - \vartheta_4 + 16}{8}, \end{aligned} \tag{72}$$

for each $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$, respectively. For all $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$, $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$, and fixed $\ell = 1/2$, we have

$$\begin{aligned} & d(\Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)) \\ & = \left| \frac{-\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - 16}{8} - \frac{-\vartheta_1 - \vartheta_2 - \vartheta_3 - \vartheta_4 + 16}{8} \right| \\ & \le \frac{|\sigma_1 - \vartheta_1| + |\sigma_2 - \vartheta_2| + |\sigma_3 - \vartheta_3| + |\sigma_4 - \vartheta_4|}{8} + 4 \\ & = \frac{\ell}{4}(d(\sigma_1, \vartheta_1) + d(\sigma_2, \vartheta_2) + d(\sigma_3, \vartheta_3) + d(\sigma_4, \vartheta_4)) \\ & \quad + (1-\ell)d(Z, O). \end{aligned} \tag{73}$$

Thus, the mappings Θ and Ω are cyclic contractions with $\ell = 1/2$. Because Z and O are closed convex, the pairs (Z, O) and (O, Z) justify the property UC^*. Therefore, all requirements of Corollary 11 are fulfilled. Thus, Θ has a QBP point and Ω has a QBP point. We note that a point $(4, 4, 4, 4) \in Z^4$ is a unique QBP point of Θ and a point $(-4, -4, -4, -4) \in V^4$ is a unique QBP point of Ω. Therefore, we obtain

$$d(4, -4) + d(4, -4) + d(4, -4) + d(4, -4) = 32 = 4d(Z, O). \tag{74}$$

In a compact subset of a MS, we can obtain the QBP point result as follows.

Theorem 12. *Assume that (Y, d) is a MS and Z, O are nonempty compact subsets of Y. Assume also $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ are cyclic mappings. If $(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0) \in Z^4$ and the sequences $\{\sigma_1^n\}, \{\sigma_2^n\}, \{\sigma_3^n\}, \{\sigma_4^n\}$ in Y are defined* *in (22), for each $n \geq 0$. Then Θ has a QBP point $(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4) \in Z^4$ and Ω has a QBP point $(\urcorner_1', \urcorner_2', \urcorner_3', \urcorner_4') \in Z^4$. Moreover, we get*

$$\begin{aligned}\sigma_1^{2n} &\longrightarrow \urcorner_1, \sigma_2^{2n} \longrightarrow \urcorner_2, \sigma_3^{2n} \longrightarrow \urcorner_3, \sigma_4^{2n} \longrightarrow \urcorner_4, \sigma_1^{2n+1} \\ &\longrightarrow \urcorner_1', \sigma_2^{2n+1} \longrightarrow \urcorner_2', \sigma_3^{2n+1} \longrightarrow \urcorner_3', \sigma_4^{2n+1} \longrightarrow \urcorner_4'.\end{aligned} \tag{75}$$

In addition, if $\urcorner_2 = \urcorner_3 = \urcorner_4$ and $\urcorner_2' = \urcorner_3' = \urcorner_4'$, then

$$d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) = 4d(Z, O). \tag{76}$$

Proof. Since $\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0 \in Z$ and (22) holds for each $n \in \mathbb{N} \cup \{0\}$, we get

$$\sigma_1^{2n}, \sigma_2^{2n}, \sigma_3^{2n}, \sigma_4^{2n} \in Z, \sigma_1^{2n+1}, \sigma_2^{2n+1}, \sigma_3^{2n+1}, \sigma_4^{2n+1} \in O. \tag{77}$$

The compactness of Z illustrates that the sequences $\sigma_1^{2n}, \sigma_2^{2n}, \sigma_3^{2n}$, and σ_4^{2n} have the convergent subsequences $\sigma_1^{2n_k}, \sigma_2^{2n_k}, \sigma_3^{2n_k}$, and $\sigma_4^{2n_k}$, respectively, so that

$$\begin{aligned}\sigma_1^{2n_k} &\longrightarrow \urcorner_1 \in Z, \sigma_2^{2n_k} \longrightarrow \urcorner_2 \in Z, \sigma_3^{2n_k} \longrightarrow \urcorner_3 \in Z, \sigma_4^{2n_k} \\ &\longrightarrow \urcorner_4 \in Z.\end{aligned} \tag{78}$$

Now, we have

$$d(Z, O) \leq d\left(\urcorner_1, \sigma_1^{2n_k-1}\right) \leq d\left(\urcorner_1, \sigma_1^{2n_k}\right) + d\left(\sigma_1^{2n_k}, \sigma_1^{2n_k-1}\right). \tag{79}$$

Applying Lemma 7, we find that

$$d\left(\sigma_1^{2n_k}, \sigma_1^{2n_k-1}\right) \longrightarrow d(Z, O). \tag{80}$$

Taking $n \longrightarrow \infty$ in (79), we get

$$d\left(\urcorner_1, \sigma_1^{2n_k-1}\right) \longrightarrow d(Z, O). \tag{81}$$

By the same manner, one can obtain

$$\begin{aligned}d\left(\urcorner_2, \sigma_2^{2n_k-1}\right) &\longrightarrow d(Z, O), d\left(\urcorner_3, \sigma_3^{2n_k-1}\right) \\ &\longrightarrow d(Z, O), d\left(\urcorner_4, \sigma_4^{2n_k-1}\right) \\ &\longrightarrow d(Z, O).\end{aligned} \tag{82}$$

Notice that

$$\begin{aligned}d(Z, O) &= d\left(\sigma_1^{2n_k}, \Theta(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4)\right) \\ &= d\left(\Omega\left(\sigma_1^{2n_k-1}, \sigma_2^{2n_k-1}, \sigma_3^{2n_k-1}, \sigma_4^{2n_k-1}\right), \Theta(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4)\right) \\ &\leq \frac{\ell}{4}\left(d\left(\sigma_1^{2n_k-1}, \urcorner_1\right), d\left(\sigma_2^{2n-1}, \urcorner_2\right), d\left(\sigma_3^{2n-1}, \urcorner_3\right), d\left(\sigma_4^{2n-1}, \urcorner_4\right)\right) \\ &\quad + (1 - \ell)d(Z, O).\end{aligned} \tag{83}$$

As $n \longrightarrow \infty$, we have

$$d(\urcorner_1, (\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4)) = d(Z, O). \tag{84}$$

Analogously, we have

$$\begin{aligned}d(\urcorner_2, (\urcorner_2, \urcorner_3, \urcorner_4, \urcorner_1)) &= d(Z, O), d(\urcorner_3, (\urcorner_3, \urcorner_4, \urcorner_1, \urcorner_2)) \\ &= d(Z, O), d(\urcorner_4, (\urcorner_4, \urcorner_1, \urcorner_2, \urcorner_3)) \\ &= d(Z, O).\end{aligned} \tag{85}$$

Thus, Θ has a QBP point $(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4) \in Z^4$. By the same argument, since O is compact, we can also claim that Ω has a QBP point $(\urcorner_1', \urcorner_2', \urcorner_3', \urcorner_4') \in O^4$. To prove

$$d\left(\urcorner_1, \urcorner_1'\right) + d\left(\urcorner_2, \urcorner_2'\right) + d\left(\urcorner_3, \urcorner_3'\right) + d\left(\urcorner_4, \urcorner_4'\right) = 4d(Z, O). \tag{86}$$

we can follow the same approach used in the proof of Theorem 10. □

3. Quadruple Fixed Point Technique

This part is devoted to present new QFP consequences in the sense of cyclic contraction mappings.

Theorem 13. *Assume that (Y, d) is a MS and Z, O are nonempty closed subsets of Y. Let the mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ be cyclic contractions. If $(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0) \in U^4$ and the sequences $\{\sigma_1^n\}, \{\sigma_2^n\}, \{\sigma_3^n\}, \{\sigma_4^n\}$ in Y are described as (22), for each $n \geq 0$. If $d(Z, O) = 0$, then Θ has a QFP point $(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4) \in Z^4$ and Ω has a QFP point $(\urcorner_1', \urcorner_2', \urcorner_3', \urcorner_4') \in O^4$. Moreover, we get*

$$\begin{aligned}\sigma_1^{2n} &\longrightarrow \urcorner_1, \sigma_2^{2n} \longrightarrow \urcorner_2, \sigma_3^{2n} \longrightarrow \urcorner_3, \sigma_4^{2n} \longrightarrow \urcorner_4, \sigma_1^{2n+1} \\ &\longrightarrow \urcorner_1', \sigma_2^{2n+1} \longrightarrow \urcorner_2', \sigma_3^{2n+1} \longrightarrow \urcorner_3', \sigma_4^{2n+1} \longrightarrow \urcorner_4'.\end{aligned} \tag{87}$$

In addition, if $\urcorner_2 = \urcorner_3 = \urcorner_4$ and $\urcorner_2' = \urcorner_3' = \urcorner_4'$, then Θ and Ω have a common QFP in $(Z \cap O)^4$.

Proof. Because $d(Z, O) = 0$, we find that the pairs (Z, O) and (O, Z) justify the property UC^*. Using Theorem 10, we see that Θ has a QBP point $(\daleth_1, \daleth_2, \daleth_3, \daleth_4) \in Z^4$, that is,

$$\begin{aligned} d(\daleth_1, \Theta(\daleth_1, \daleth_2, \daleth_3, \daleth_4)) &= d(\daleth_2, \Theta(\daleth_2, \daleth_3, \daleth_4, \daleth_1)) \\ &= d(\daleth_3, \Theta(\daleth_3, \daleth_4, \daleth_1, \daleth_2)) \\ &= d(\daleth_4, \Theta(\daleth_4, \daleth_1, \daleth_2, \daleth_3)) \\ &= d(Z, O), \end{aligned} \tag{88}$$

and Ω has a QBP point $(\daleth'_1, \daleth'_2, \daleth'_3, \daleth'_4) \in O^4$, that is,

$$\begin{aligned} d\left(\daleth'_1, \left(\daleth'_1, \daleth'_2, \daleth'_3, \daleth'_4\right)\right) &= d\left(\daleth'_2, \left(\daleth'_2, \daleth'_3, \daleth'_4, \daleth'_1\right)\right) \\ &= d\left(\daleth'_3, \Omega\left(\daleth'_3, \daleth'_4, \daleth'_1, \daleth'_2\right)\right) \\ &= d\left(\daleth'_4, \Omega\left(\daleth'_4, \daleth'_1, \daleth'_2, \daleth'_3\right)\right) \\ &= d(Z, O). \end{aligned} \tag{89}$$

From (88) and since $d(Z, O) = 0$, we obtain

$$\begin{aligned} \daleth_1 &= \Theta(\daleth_1, \daleth_2, \daleth_3, \daleth_4), \daleth_2 = \Theta(\daleth_2, \daleth_3, \daleth_4, \daleth_1), \daleth_3 \\ &= \Theta(\daleth_3, \daleth_4, \daleth_1, \daleth_2), \daleth_4 = \Theta(\daleth_4, \daleth_1, \daleth_2, \daleth_3). \end{aligned} \tag{90}$$

This means that $(\daleth_1, \daleth_2, \daleth_3, \daleth_4)$ is a QFP of Θ.

Again, from (89) and since $d(Z, O) = 0$, we get

$$\begin{aligned} \daleth'_1 &= \Omega\left(\daleth'_1, \daleth'_2, \daleth'_3, \daleth'_4\right), \daleth'_2 = \Omega\left(\daleth'_2, \daleth'_3, \daleth'_4, \daleth'_1\right), \daleth'_3 \\ &= \Omega\left(\daleth'_3, \daleth'_4, \daleth'_1, \daleth'_2\right), \daleth'_4 = \Omega\left(\daleth'_4, \daleth'_1, \daleth'_2, \daleth'_3\right). \end{aligned} \tag{91}$$

This means that $(\daleth'_1, \daleth'_2, \daleth'_3, \daleth'_4)$ is a QFP of Ω.

Now, let $\daleth_2 = \daleth_3 = \daleth_4$ and $\daleth'_2 = \daleth'_3 = \daleth'_4$. From Theorem 10, one can write

$$d\left(\daleth_1, \daleth'_1\right) + d\left(\daleth_2, \daleth'_2\right) + d\left(\daleth_3, \daleth'_3\right) + d\left(\daleth_4, \daleth'_4\right) = 4d(Z, O). \tag{92}$$

Since $d(Z, O) = 0$, we have

$$d\left(\daleth_1, \daleth'_1\right) + d\left(\daleth_2, \daleth'_2\right) + d\left(\daleth_3, \daleth'_3\right) + d\left(\daleth_4, \daleth'_4\right) = 0. \tag{93}$$

It follows that

$$\daleth_1 = \daleth'_1 = \daleth_2 = \daleth'_2, \daleth_3 = \daleth'_3 = \daleth_4 = \daleth'_4. \tag{94}$$

Therefore, the quadruple $(\daleth_1, \daleth_2, \daleth_3, \daleth_4) \in (Z \cap O)^4$ is a common QFP of Θ and Ω. This is enough to end the proof. □

Example 10. Consider $Y = \mathbb{R}$ equipped with the usual norm. Take $Z = [-4, 0]$ and $O = [0, 4]$. Describe two mappings $\Theta : Z^4 \longrightarrow O$ and $\Omega : O^4 \longrightarrow Z$ as

$$\begin{aligned} \Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= -\frac{\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4}{8}, \\ \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) &= -\frac{\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4}{8}, \end{aligned} \tag{95}$$

for each $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$, respectively. Then, $d(Z, O) = 0$ and Θ, Ω are cyclic contractions with $\ell = 1/2$. Moreover, for each $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in Z^4$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in O^4$, we get

$$\begin{aligned} & d(\Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Omega(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)) \\ &\quad = \left| -\frac{\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4}{8} + \frac{\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4}{8} \right| \\ &\quad \le \frac{1}{8}(|\sigma_1 - \vartheta_1| + |\sigma_2 - \vartheta_2| + |\sigma_3 - \vartheta_3| + |\sigma_4 - \vartheta_4|) \\ &\quad = \frac{\ell}{4}(d(\sigma_1, \vartheta_1) + d(\sigma_2, \vartheta_2) + d(\sigma_3, \vartheta_3) + d(\sigma_4, \vartheta_4)) \\ &\qquad + (1 - \ell) d(Z, O). \end{aligned} \tag{96}$$

Therefore, all postulates of Theorem 13 are justified. Then, Θ and Ω have a common QFP $(0, 0, 0, 0) \in (Z \cap O)^4$.

Putting $Z = O$ in the above theorem, we have the result below.

Corollary 14. *Assume that (Y, d) is a complete MS and $Z \neq \emptyset$ is a closed subset of Y. Assume also that $\Theta : Z^4 \longrightarrow Z$ and $\Omega : Z^4 \longrightarrow Z$ are cyclic contraction mappings so that if $(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0) \in Z^4$ and the sequences $\{\sigma_1^n\}, \{\sigma_2^n\}, \{\sigma_3^n\}, \{\sigma_4^n\}$ in Y are defined as (22), for each $n \ge 0$, then Θ has a QFP $(\daleth_1, \daleth_2, \daleth_3, \daleth_4) \in Z^4$ and Ω has a QFP $(\daleth'_1, \daleth'_2, \daleth'_3, \daleth'_4) \in Z^4$. Also, we obtain*

$$\begin{aligned} & \sigma_1^{2n} \longrightarrow \daleth_1, \sigma_2^{2n} \longrightarrow \daleth_2, \sigma_3^{2n} \longrightarrow \daleth_3, \sigma_4^{2n} \longrightarrow \daleth_4, \sigma_1^{2n+1} \\ &\quad \longrightarrow \daleth'_1, \sigma_2^{2n+1} \longrightarrow \daleth'_2, \sigma_3^{2n+1} \longrightarrow \daleth'_3, \sigma_4^{2n+1} \longrightarrow \daleth'_4. \end{aligned} \tag{97}$$

Moreover, if $\daleth_2 = \daleth_3 = \daleth_4$ and $\daleth'_2 = \daleth'_3 = \daleth'_4$, then Θ and Ω have a common QFP in Z^4.

The following corollary is very important in the application part. We get this result by placing $\Theta = \Omega$ in Corollary 14.

Corollary 15. *Let (Y, d) be a complete MS and Z be a nonempty closed subset of Y. Assume also the mapping $\Theta : Z^4 \longrightarrow Z$ verifying*

$$\begin{aligned} d(\Theta(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Theta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)) &\le \frac{\ell}{4}(d(\sigma_1, \vartheta_1) \\ &\quad + d(\sigma_2, \vartheta_2) + d(\sigma_3, \vartheta_3) + d(\sigma_4, \vartheta_4)), \end{aligned} \tag{98}$$

for all $(\sigma_1, \sigma_2, \sigma_3, \sigma_4), (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) \in Z^4$, *and* $\ell \in 0, 1)$. *Then,* Θ *has a unique QFP* $(\urcorner_1, \urcorner_2, \urcorner_3, \urcorner_4) \in Z^4$.

where S and D are a state and decision space, respectively; $\sigma_1 \in S, g : S \times D \longrightarrow \mathbb{R}$; $\aleph : S \times D \times \mathbb{R}^4 \longrightarrow \mathbb{R}$; and $\mathfrak{T} : S \times D \longrightarrow S$(where $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$).

4. Solving a System of Functional Equations

Here, we apply Corollary 15 to discuss the existence of the solution for the following quadruple functional equations:

$$
\begin{cases}
\eta_1(\sigma_1) = \sup_{\sigma_2 \in D} \{ g(\sigma_1, \sigma_2) + \aleph(\sigma_1, \sigma_2, \eta_1(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4(\mathfrak{T}(\sigma_1, \sigma_2))) \}, \\
\eta_2(\sigma_1) = \sup_{\sigma_2 \in D} \{ g(\sigma_1, \sigma_2) + \aleph(\sigma_1, \sigma_2, \eta_2(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_1(\mathfrak{T}(\sigma_1, \sigma_2))) \}, \\
\eta_3(\sigma_1) = \sup_{\sigma_2 \in D} \{ g(\sigma_1, \sigma_2) + \aleph(\sigma_1, \sigma_2, \eta_3(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_1(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2(\mathfrak{T}(\sigma_1, \sigma_2))) \}, \\
\eta_4(\sigma_1) = \sup_{\sigma_2 \in D} \{ g(\sigma_1, \sigma_2) + \aleph(\sigma_1, \sigma_2, \eta_4(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_1(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3(\mathfrak{T}(\sigma_1, \sigma_2))) \},
\end{cases}
\tag{99}
$$

Assume that $\mathfrak{R}_S$ is the set of all bounded real-valued functions on S. Consider

$$
\|v\| = \sup_{\sigma_1 \in S} |v(\sigma_1)|, \quad \forall v \in \mathfrak{R}_S. \tag{100}
$$

Moreover, define a distance on $\mathfrak{R}_S$ in the form of

$$
\mho(\mu, \nu) = \sup_{\sigma_1 \in S} |\mu(\sigma_1) - \nu(\sigma_1)|, \quad \text{for all } \mu, \nu \in \mathfrak{R}_S. \tag{101}
$$

Obviously, the pair $(\mathfrak{R}_S, \mho)$ is a complete MS.

In the theorem below, we will discuss the existence of the solution for the system (99).

Theorem 16. *Assume that the following postulates are fulfilled:*

(a_1) the functions $g : S \times D \longrightarrow \mathbb{R}$ *and* $\mathfrak{T} : S \times D \longrightarrow S$ *are bounded*

(a_2) for each $\sigma_1 \in S, \sigma_2 \in D$, *and* $\eta_1, \eta_2, \eta_3, \eta_4, \eta_1', \eta_2', \eta_3', \eta_4' \in \mathbb{R}$, *we have*

$$
\left| \aleph(\sigma_1, \sigma_2, \eta_1, \eta_2, \eta_3, \eta_4) - \aleph\left(\sigma_1, \sigma_2, \eta_1', \eta_2', \eta_3', \eta_4'\right) \right| \le \frac{1}{8} \left(|\eta_1 - \eta_1'| + |\eta_2 - \eta_2'| + |\eta_3 - \eta_3'| + |\eta_4 - \eta_4'| \right). \tag{102}
$$

Then, system (99) has a unique bounded solution (UBS).

Proof. Describe an operator Ψ on the space $\mathfrak{R}_S$ by

$$
\Psi(\eta_1, \eta_2, \eta_3, \eta_4)(\sigma_1) = \sup_{\sigma_2 \in D} \{ g(\sigma_1, \sigma_2) + \aleph(\sigma_1, \sigma_2, \eta_1(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2 \cdot (\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4(\mathfrak{T}(\sigma_1, \sigma_2))) \}, \tag{103}
$$

for each $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathfrak{R}_S$ and $\sigma_1 \in S, \sigma_2 \in D$. The existence solution for system (99) is equivalent to find a QFP of the operator Ψ.

Clearly, the mapping Ψ is well-defined (because the functions g and $\aleph$ are bounded).

Hence, by the postulate (a_2), we obtain

$$
\begin{aligned}
\mho\left(\Psi(\eta_1, \eta_2, \eta_3, \eta_4), \Psi\left(\eta_1', \eta_2', \eta_3', \eta_4'\right)\right) &= \sup_{\sigma_1 \in S} \left| \Psi(\eta_1, \eta_2, \eta_3, \eta_4)(\sigma_1) - \Psi\left(\eta_1', \eta_2', \eta_3', \eta_4'\right)(\sigma_1) \right| = \sup_{\sigma_1 \in S} \Bigg| \sup_{\sigma_2 \in D} \{ g(\sigma_1, \sigma_2) + \aleph(\sigma_1, \sigma_2, \eta_1(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2 \\
&\quad \times (\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4(\mathfrak{T}(\sigma_1, \sigma_2))) \} - \sup_{\sigma_2 \in D} \left\{ g(\sigma_1, \sigma_2) + \aleph\left(\sigma_1, \sigma_2, \eta_1'(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2'(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3'(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4'(\mathfrak{T}(\sigma_1, \sigma_2))\right) \right\} \Bigg| \\
&= \sup_{\sigma_1 \in S} \left\{ \sup_{\sigma_2 \in D} \left| \aleph(\sigma_1, \sigma_2, \eta_1(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4(\mathfrak{T}(\sigma_1, \sigma_2))) - \aleph\left(\sigma_1, \sigma_2, \eta_1'(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_2'(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_3'(\mathfrak{T}(\sigma_1, \sigma_2)), \eta_4'(\mathfrak{T}(\sigma_1, \sigma_2))\right) \right| \right\} \\
&\le \sup_{\sigma_1 \in S} \left\{ \frac{1}{8} \sup_{\sigma_2 \in D} \left(|\eta_1 - \eta_1'| + |\eta_2 - \eta_2'| + |\eta_3 - \eta_3'| + |\eta_4 - \eta_4'| \right) \right\} \le \frac{1}{8} \sup_{\sigma_1 \in S} \left\{ \|\eta_1 - \eta_1'\| + \|\eta_2 - \eta_2'\| + \|\eta_3 - \eta_3'\| + \|\eta_4 - \eta_4'\| \right\} \\
&= \frac{\ell}{4} \left(\mho\left(\eta_1, \eta_1'\right) + \mho\left(\eta_2, \eta_2'\right) + \mho\left(\eta_3, \eta_3'\right) + \mho\left(\eta_4, \eta_4'\right) \right).
\end{aligned}
\tag{104}
$$

This implies that the contractive stipulation of Corollary 15 holds with $\ell = 1/2$. Then, the mapping Ψ has a unique QFP, which is a UBS of problem (99). □

The example below justifies Theorem 16.

Example 11. Consider a quadruple system of functional equations below:

$$\begin{cases} \eta_1(\sigma_1) = \sup_{\sigma_2\in\mathbb{R}} \left\{ \arctan(\sigma_1 + 4\sigma_2) + \frac{1}{2+(\sigma_1)^2} + \frac{1}{3+e^{3\sigma_2}} + \frac{1}{8}\frac{|\eta_1(\mathfrak{I})|}{1+|\eta_1(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_2(\mathfrak{I})|}{1+e^{|\eta_2(\mathfrak{I})|}} + \frac{1}{8}\frac{|\eta_3(\mathfrak{I})|}{1+\ln|\eta_3(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_4(\mathfrak{I})|}{1+\sqrt{|\eta_4(\mathfrak{I})|}} \right\}, \\ \eta_2(\sigma_1) = \sup_{\sigma_2\in\mathbb{R}} \left\{ \arctan(\sigma_1 + 4\sigma_2) + \frac{1}{2+(\sigma_1)^2} + \frac{1}{3+e^{3\sigma_2}} + \frac{1}{8}\frac{|\eta_2(\mathfrak{I})|}{1+|\eta_2(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_3(\mathfrak{I})|}{1+e^{|\eta_3(\mathfrak{I})|}} + \frac{1}{8}\frac{|\eta_4(\mathfrak{I})|}{1+\ln|\eta_4(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_1(\mathfrak{I})|}{1+\sqrt{|\eta_1(\mathfrak{I})|}} \right\}, \\ \eta_3(\sigma_1) = \sup_{\sigma_2\in\mathbb{R}} \left\{ \arctan(\sigma_1 + 4\sigma_2) + \frac{1}{2+(\sigma_1)^2} + \frac{1}{3+e^{3\sigma_2}} + \frac{1}{8}\frac{|\eta_3(\mathfrak{I})|}{1+|\eta_3(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_4(\mathfrak{I})|}{1+e^{|\eta_4(\mathfrak{I})|}} + \frac{1}{8}\frac{|\eta_1(\mathfrak{I})|}{1+\ln|\eta_1(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_2(\mathfrak{I})|}{1+\sqrt{|\eta_2(\mathfrak{I})|}} \right\}, \\ \eta_4(\sigma_1) = \sup_{\sigma_2\in\mathbb{R}} \left\{ \arctan(\sigma_1 + 4\sigma_2) + \frac{1}{2+(\sigma_1)^2} + \frac{1}{3+e^{3\sigma_2}} + \frac{1}{8}\frac{|\eta_4(\mathfrak{I})|}{1+|\eta_4(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_1(\mathfrak{I})|}{1+e^{|\eta_1(\mathfrak{I})|}} + \frac{1}{8}\frac{|\eta_2(\mathfrak{I})|}{1+\ln|\eta_2(\mathfrak{I})|} + \frac{1}{8}\frac{|\eta_3(\mathfrak{I})|}{1+\sqrt{|\eta_3(\mathfrak{I})|}} \right\}, \end{cases} \tag{105}$$

for all $\sigma_1 \in [0, 1]$.

Clearly, system (105) is comparable to system (99) with $S = [0, 1]$ and $D = \mathbb{R}$. Clearly, the postulate (a_1) of Theorem 16 is fulfilled. To achieve the postulate (a_2), we have

$$\begin{aligned} &|\aleph(\sigma_1, \sigma_2, \eta_1(\mathfrak{I}(\sigma_1,\sigma_2)), \eta_2(\mathfrak{I}(\sigma_1,\sigma_2)), \eta_3(\mathfrak{I}(\sigma_1,\sigma_2)), \eta_4(\mathfrak{I}(\sigma_1,\sigma_2))) \\ &\quad -\aleph\left(\sigma_1, \sigma_2, \eta_1'(\mathfrak{I}(\sigma_1,\sigma_2)), \eta_2'(\mathfrak{I}(\sigma_1,\sigma_2)), \eta_3'(\mathfrak{I}(\sigma_1,\sigma_2)), \eta_4'(\mathfrak{I}(\sigma_1,\sigma_2))\right)\Big| \\ &\le \frac{1}{8}\left|\frac{|\eta_1(\mathfrak{I})|}{1+|\eta_1(\mathfrak{I})|} - \frac{|\eta_1'(\mathfrak{I})|}{1+|\eta_1'(\mathfrak{I})|}\right| + \frac{1}{8}\left|\frac{|\eta_2(\mathfrak{I})|}{1+e^{|\eta_2(\mathfrak{I})|}} - \frac{|\eta_2'(\mathfrak{I})|}{1+e^{|\eta_2{}'(\mathfrak{I})|}}\right| \\ &\quad + \frac{1}{8}\left|\frac{|\eta_3'(\mathfrak{I})|}{1+\ln|\eta_3'(\mathfrak{I})|} - \frac{|\eta_3'(\mathfrak{I})|}{1+\ln|\eta_3'(\mathfrak{I})|}\right| + \frac{1}{8}\left|\frac{|\eta_4(\mathfrak{I})|}{1+\sqrt{|\eta_4(\mathfrak{I})|}} - \frac{|\eta_4'(\mathfrak{I})|}{1+\sqrt{|\eta_4'(\mathfrak{I})|}}\right| \\ &\le \frac{1}{8}\left(\big||\eta_1(\mathfrak{I})| - |\eta_1'(\mathfrak{I})|\big| + \big||\eta_2(\mathfrak{I})| - |\eta_2'(\mathfrak{I})|\big| + \big||\eta_3(\mathfrak{I})| \right. \\ &\quad \left. - |\eta_3'(\mathfrak{I})|\big| + \big||\eta_4(\mathfrak{I})| - |\eta_4'(\mathfrak{I})|\big|\right) \\ &\le \frac{1}{8}\left(|\eta_1 - \eta_1'| + |\eta_2 - \eta_2'| + |\eta_3 - \eta_3'| + |\eta_4 - \eta_4'|\right). \end{aligned} \tag{106}$$

Therefore, the postulate (a_2) of Theorem 16 is fulfilled. Thus, problem (105) has a UBS in $\mathfrak{R}_S$.

5. Solving a System of Integral Equations

The existence of solutions for a system of quadruple integral equations is presented here by using the results of Corollary 15.

Consider the following problem:

$$\begin{cases} \sigma_1(\beta) = \int_0^{\eth} \hbar(\beta,\zeta)\widehat{J}(\beta,\zeta,\sigma_1(\zeta),\sigma_2(\zeta),\sigma_3(\zeta),\sigma_4(\zeta))d\zeta, \\ \sigma_2(\beta) = \int_0^{\eth} \hbar(\beta,\zeta)\widehat{J}(\beta,\zeta,\sigma_2(\zeta),\sigma_3(\zeta),\sigma_4(\zeta),\sigma_1(\zeta))d\zeta, \\ \sigma_3(\beta) = \int_0^{\eth} \hbar(\beta,\zeta)\widehat{J}(\beta,\zeta,\sigma_3(\zeta),\sigma_4(\zeta),\sigma_1(\zeta),\sigma_2(\zeta))d\zeta, \\ \sigma_4(\beta) = \int_0^{\eth} \hbar(\beta,\zeta)\widehat{J}(\beta,\zeta,\sigma_4(\zeta),\sigma_1(\zeta),\sigma_2(\zeta),\sigma_3(\zeta))d\zeta, \end{cases} \tag{107}$$

where $\beta, \zeta \in [0, \eth]$ with $\eth > 0$.

Suppose that $\beth = C([0, \eth], \mathbb{R}^n)$ is endowed with

$$\|\sigma_1\| = \sup_{\beta\in[0,\eth]} |\sigma_1(\beta)|, \quad \text{for all } \sigma_1 \in \beth. \tag{108}$$

Moreover, define a distance on $\beth$ in the form of

$$\wp(\check{r}, \tilde{u}) = \sup_{\beta\in[0,\eth]} |\check{r}(\beta) - \tilde{u}(\beta)|, \quad \text{for all } \check{r}, \tilde{u} \in \beth. \tag{109}$$

Hence, $(\beth, \eth)$ is a complete MS.

Theorem 17. *Suppose that the following hypotheses hold:*

($\blacktriangle_1$) *The functions* $\widehat{J} : [0, \eth] \times [0, \eth] \times (\mathbb{R}^n)^4 \longrightarrow \mathbb{R}^n$ *and* $\hbar : [0, \eth] \times [0, \eth] \longrightarrow \mathbb{R}^n$ *are continuous so that*

$$\int_0^{\eth} \hbar(\beta, \zeta) d\zeta \leq \frac{\eth}{16}, \quad \text{for all } \beta, \zeta \in [0, \eth]. \tag{110}$$

($\blacktriangle_2$) *For all* $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_4', \sigma_2', \sigma_3', \sigma_4' \in \mathbb{R}^n$, *we get*

$$\begin{aligned} &\left|\widehat{J}(\beta, \zeta, \sigma_1(\zeta), \sigma_2(\zeta), \sigma_3(\zeta), \sigma_4(\zeta)) - \widehat{J}\left(\beta, \zeta, \sigma_1'(\zeta), \sigma_2'(\zeta), \sigma_3'(\zeta), \sigma_4'(\zeta)\right)\right| \\ &\quad \leq \frac{1}{\eth}\left(\left|\sigma_1 - \sigma_1'\right| + \left|\sigma_2 - \sigma_2'\right| + \left|\sigma_3 - \sigma_3'\right| + \left|\sigma_4 - \sigma_4'\right|\right), \quad \text{for all } \beta, \zeta \in [0, \eth]. \end{aligned} \tag{111}$$

Then, problem (107) has a unique solution on $C([0, \eth], \mathbb{R}^n)$.

Proof. Define the mapping $\Lambda : \beth^4 \longrightarrow \beth$ by

$$\Lambda(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(\beta) = \int_0^{\eth} \hbar(\beta, \zeta) \widehat{J}(\beta, \zeta, \sigma_1(\zeta), \sigma_2(\zeta), \sigma_3(\zeta), \sigma_4(\zeta)) d\zeta, \quad \beta \in [0, \eth]. \tag{112}$$

The existence solution of (107) corresponds to finding a QFP of Λ.

Assume that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1', \sigma_2', \sigma_3', \sigma_4' \in \mathbb{R}^n$, we get

$$\begin{aligned} \wp\left(\Lambda(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Lambda\left(\sigma_1', \sigma_2', \sigma_3', \sigma_4'\right)\right) &= \sup_{\beta \in [0, \eth]} \left|\Lambda(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(\beta) - \Lambda\left(\sigma_1', \sigma_2', \sigma_3', \sigma_4'\right)(\beta)\right| \\ &= \sup_{\beta \in [0, \eth]} \left|\int_0^{\eth} \hbar(\beta, \zeta) \widehat{J}(\beta, \zeta, \sigma_1(\zeta), \sigma_2(\zeta), \sigma_3(\zeta), \sigma_4(\zeta)) d\zeta - \int_0^{\eth} \hbar(\beta, \zeta) \widehat{J}\left(\beta, \zeta, \sigma_1'(\zeta), \sigma_2'(\zeta), \sigma_3'(\zeta), \sigma_4'(\zeta)\right) d\zeta\right| \\ &\leq \sup_{\beta \in [0, \eth]} \int_0^{\eth} \hbar(\beta, \zeta) \left|\widehat{J}(\beta, \zeta, \sigma_1(\zeta), \sigma_2(\zeta), \sigma_3(\zeta), \sigma_4(\zeta)) - \widehat{J}\left(\beta, \zeta, \sigma_1'(\zeta), \sigma_2'(\zeta), \sigma_3'(\zeta), \sigma_4'(\zeta)\right)\right| d\zeta \\ &\leq \sup_{\beta \in [0, \eth]} \int_0^{\eth} \hbar(\beta, \zeta) d\zeta \left(\frac{1}{\eth}\left(\left|\sigma_1 - \sigma_1'\right| + \left|\sigma_2 - \sigma_2'\right| + \left|\sigma_3 - \sigma_3'\right| + \left|\sigma_4 - \sigma_4'\right|\right)\right) \\ &\leq \frac{1}{16} \sup_{\beta \in [0, \eth]} \left(\left|\sigma_1 - \sigma_1'\right| + \left|\sigma_2 - \sigma_2'\right| + \left|\sigma_3 - \sigma_3'\right| + \left|\sigma_4 - \sigma_4'\right|\right) \\ &= \frac{\ell}{4}\left(\wp\left(\sigma_1, \sigma_1'\right) + \wp\left(\sigma_2, \sigma_2'\right) + \wp\left(\sigma_3, \sigma_3'\right) + \wp\left(\sigma_4, \sigma_4'\right)\right). \end{aligned} \tag{113}$$

Hence, the stipulation of Corollary 15 holds with $\ell = 1/4$. Therefore, Λ has a QFP, which in turn is considered the unique solution to (107). □

Example 12. Consider a system of quadruple integral equations below:

$$\begin{cases} \sigma_1(\beta) = \displaystyle\int_0^{16} \left(\beta^3 + 2\zeta^2 + \frac{1}{16}\frac{|\sigma_1(\zeta)|}{1 + \sin(|\sigma_1(\zeta)|)} + \frac{1}{16}\frac{|\sigma_2(\zeta)|}{2 + \cos(|\sigma_2(\zeta)|)} + \frac{1}{16}\frac{|\sigma_3(\zeta)|}{1 + |\sigma_3(\zeta)|} + \frac{1}{16}\frac{|\sigma_4(\zeta)|}{3 + (|\sigma_4(\zeta)|)^2}\right) d\zeta, \\ \sigma_2(\beta) = \displaystyle\int_0^{16} \left(\beta^3 + 2\zeta^2 + \frac{1}{16}\frac{|\sigma_2(\zeta)|}{1 + \sin(|\sigma_2(\zeta)|)} + \frac{1}{16}\frac{|\sigma_3(\zeta)|}{2 + \cos(|\sigma_3(\zeta)|)} + \frac{1}{16}\frac{|\sigma_4(\zeta)|}{1 + |\sigma_4(\zeta)|} + \frac{1}{16}\frac{|\sigma_1(\zeta)|}{3 + (|\sigma_1(\zeta)|)^2}\right) d\zeta, \\ \sigma_3(\beta) = \displaystyle\int_0^{16} \left(\beta^3 + 2\zeta^2 + \frac{1}{16}\frac{|\sigma_3(\zeta)|}{1 + \sin(|\sigma_3(\zeta)|)} + \frac{1}{16}\frac{|\sigma_4(\zeta)|}{2 + \cos(|\sigma_4(\zeta)|)} + \frac{1}{16}\frac{|\sigma_1(\zeta)|}{1 + |\sigma_1(\zeta)|} + \frac{1}{16}\frac{|\sigma_2(\zeta)|}{3 + (|\sigma_2(\zeta)|)^2}\right) d\zeta, \\ \sigma_4(\beta) = \displaystyle\int_0^{16} \left(\beta^3 + 2\zeta^2 + \frac{1}{16}\frac{|\sigma_4(\zeta)|}{1 + \sin(|\sigma_4(\zeta)|)} + \frac{1}{16}\frac{|\sigma_1(\zeta)|}{2 + \cos(|\sigma_1(\zeta)|)} + \frac{1}{16}\frac{|\sigma_2(\zeta)|}{1 + |\sigma_2(\zeta)|} + \frac{1}{16}\frac{|\sigma_3(\zeta)|}{3 + (|\sigma_3(\zeta)|)^2}\right) d\zeta, \end{cases} \tag{114}$$

for all $\beta \in [0, 16]$.

Problem (114) is another shape of problem (107) with $\eth = 1, \hbar(\beta, \zeta) = \beta^3 + 2\zeta^2$ and

$$\widehat{J}(\beta, \zeta, \sigma_1(\zeta), \sigma_2(\zeta), \sigma_3(\zeta), \sigma_4(\zeta)) = \frac{1}{16}\frac{|\sigma_1(\zeta)|}{1 + \sin(|\sigma_1(\zeta)|)} + \frac{1}{16}\frac{|\sigma_2(\zeta)|}{2 + \cos(|\sigma_2(\zeta)|)} + \frac{1}{16}\frac{|\sigma_3(\zeta)|}{1 + |\sigma_3(\zeta)|} + \frac{1}{16}\frac{|\sigma_4(\zeta)|}{3 + (|\sigma_4(\zeta)|)^2}. \quad (115)$$

Obviously, the hypothesis ($\blacktriangle_1$) of Theorem 17 holds. For the hypothesis ($\blacktriangle_2$), one can write

$$\begin{aligned}
&\left|\widehat{J}(\beta, \zeta, \sigma_1(\zeta), \sigma_2(\zeta), \sigma_3(\zeta), \sigma_4(\zeta)) - \widehat{J}\left(\beta, \zeta, \sigma_1'(\zeta), \sigma_2'(\zeta), \sigma_3'(\zeta), \sigma_4'(\zeta)\right)\right| \\
&\le \frac{1}{16}\left|\frac{|\sigma_1(\zeta)|}{1 + \sin(|\sigma_1(\zeta)|)} - \frac{|\sigma_1'(\zeta)|}{1 + \sin\left(|\sigma_1'(\zeta)|\right)}\right| \\
&\quad + \frac{1}{16}\left|\frac{|\sigma_2(\zeta)|}{2 + \cos(|\sigma_2(\zeta)|)} - \frac{|\sigma_2'(\zeta)|}{2 + \cos\left(|\sigma_2'(\zeta)|\right)}\right| \\
&\quad + \frac{1}{16}\left|\frac{|\sigma_3(\zeta)|}{1 + |\sigma_3(\zeta)|} - \frac{|\sigma_3'(\zeta)|}{1 + |\sigma_3'(\zeta)|}\right| \\
&\quad + \frac{1}{16}\left|\frac{|\sigma_4(\zeta)|}{3 + (|\sigma_4(\zeta)|)^2} - \frac{|\sigma_4'(\zeta)|}{3 + \left(|\sigma_4'(\zeta)|\right)^2}\right| \\
&\le \frac{1}{16}\Big(\left||\sigma_1(\zeta)| - |\sigma_1'(\zeta)|\right| + \left||\sigma_2(\zeta)| - |\sigma_2'(\zeta)|\right| \\
&\quad + \left||\sigma_3(\zeta)| - |\sigma_3'(\zeta)|\right| + \left||\sigma_4(\zeta)| - |\sigma_4'(\zeta)|\right|\Big) \\
&\le \frac{1}{16}\left(|\sigma_1 - \sigma_1'| + |\sigma_2 - \sigma_2'| + |\sigma_3 - \sigma_3'| + |\sigma_4 - \sigma_4'|\right).
\end{aligned} \quad (116)$$

Hence, the hypothesis ($\blacktriangle_2$) of Theorem 17 is justified with $\eth = 16$. Therefore, the mapping $\widehat{J}$ has a unique QFP which in turn is considered the unique solution to (114).

6. Conclusion

One of the central problems in approximation theory is to determine points that minimize the distance to a given point or subset. The best approximation has always attracted analysts because it carries enough potential to be extended especially with the functional analytic approach in nonlinear analysis. The best proximity point has many applications such as obtaining the existence of a unique solution for a variational inequality problem, integral and differential equations, and many other directions. The fixed point method is considered one of the distinguished methods for obtaining these points under cyclic contraction mappings, due to its smoothness and clarity. So, in this paper, the existence of a quadruple best proximity point for a cyclic contraction mapping is introduced in ordinary metric space. The validity of theoretical results in a uniformly convex Banach space was also discussed. Moreover, several examples are given to strengthen the theoretical results. Finally, our paper has been provided with applications on the existence and uniqueness of the solution to a system of functional and integral equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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