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Research Article

Analysis of Two-Dimensional Heat Transfer Problem Using the Boundary Integral Equation

Nimona Ketema Kebeba and Gizaw Debito Haifo

Department of Mathematics, College of Natural and Computational Science, Mizan Tepi University, Tepi, Ethiopia

Correspondence should be addressed to Nimona Ketema Kebeba; nimonaketema2017@gmail.com

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In this paper, we examine the problem of two-dimensional heat equations with certain initial and boundary conditions being considered. In a two-dimensional heat transport problem, the boundary integral equation technique was applied. The problem is expressed by an integral equation using the fundamental solution in Green's identity. In this study, we transform the boundary value problem for the steady-state heat transfer problem into a boundary integral equation and drive the solution of the two-dimensional heat transfer problem using the boundary integral equation for the mixed boundary value problem by using Green's identity and fundamental solution.

1. Introduction

For partial differential equations, the boundary integral equation is a basic method for analyzing boundary value problems [1]. Various schemes have emerged to discretize time domain boundary integral equations associated to parabolic problem [2]. In the inception of the boundary integral equation method, the thermal engineering community has been exploiting its potential in solving transient heat conduction problems [3, 4]. Any approach for the approximate numerical solution of the boundary integral equations is referred to as a boundary element method [5]. The accurate solution of the differential equation of a two-dimensional heat transfer problem in the domain acquired by the boundary element method distinguishes the approximate solution of the boundary value problem produced by the boundary element method [6–9]. Only the domain's boundary needs to be discretized, notably in two-dimensional heat transfer problems with a simple circle boundary.

In some applications, the physical relevant data are provided by the boundary value of the solution or its derivatives rather than the solution in the domain boundary [10]. These data can be derived directly from the boundary integral equation's solution.

The advantage of using boundary integral formulation of partial differential equation problems is that we require only N^{d-1} unknown to discretize the boundary Γ, where N is the number of variables in each space dimension [5, 6]. Many different formulations have been proposed for the treatment of heat conduction (diffusion) problems by the boundary integral equation BIE method, the most efficient of which is the one which employs a time-dependent fundamental solution. The formulation adopted for this analysis employs Green's identity to derive the boundary integral equation in [4, 11]. A fundamental solution is generally not available if the coefficients of the original partial differential equation are not constant. One can use, in this case, a parametrix (Levi function), which is usually available, instead of fundamental solution Green formulae [3, 12, 13].

The solution exactly satisfies the differential equation inside the domain; nevertheless, approximate solutions exist because boundary conditions are only approximately satisfied. Because functions are defined globally, there is no need to divide the domain into elements [14–16].

The solution also meets the criterion at infinity, so dealing with infinite domains, where the finite element method must apply either truncation or approximate infinite elements, is not an issue [10, 17, 18]. As a result, the goal of this work is to use Green's identity and fundamental solution to transform the boundary value problem for steady-state heat transfer into a boundary integral equation and solve the boundary integral equation for the mixed boundary value problem [8, 19–21].

Using a boundary integral expression for a two-dimensional heat transfer problem, we obtain a unique weak solution and a variational solution in the Sobolev space of order one, $H^1(\Omega)$ [15, 22]. The remainder of the current document is as follows: some basic definitions, theorems, and properties of the Laplace equation that arise as a steady-state problem for heat equation are mentioned in Section 2. Section 3 illustrates the details of the statement of the steady-state heat transfer problem, the boundary integral equation for the classical solution, and the boundary integral expression for the weak solution. Section 4 provides the conclusions of the paper.

2. Preliminaries

2.1. Laplace Equation in Two Dimensions. Let $\Omega \subset \mathbb{R}^2$ open and $u : \Omega \longrightarrow \mathbb{R}^2$. The Laplace equation for u is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

For a heat equation that does not change with time, the Laplace equation arises as a steady-state problem [20].

Equation (1) has no dependence on time, just on the spatial variables x and y. This means that the Laplace equation described steady state situated on the temperature distribution.

The steady-state solution satisfies $\Delta u = 0$ and boundary condition, u is prescribed on $\partial\Omega$, and then, we consider the domain Ω that are circular [23].

2.2. Sobolev Space

Definition 1 (see [8]). Let $1 \leq p \leq \infty$ and $r \in \mathbb{N}_0$, and let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. The Sobolev space w_p^r order r based on $L_p(\Omega)$ is defined by

$$w_p^r(\Omega) = \left\{ u \in L_p(\Omega) \colon \partial^\alpha u \in L_p(\Omega), \text{for} |\alpha| \leq r \right\}. \tag{2}$$

Remark 2. $\partial^\alpha u$ is viewed as a distribution on Ω, so the condition $\partial^\alpha u \in L_p(\Omega)$ means that there exists a function $g^\alpha u \in L_p(\Omega)$ such that $\langle u, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle g^\alpha, \varphi \rangle, \forall \varphi \in D'(\Omega)$, such that a function g^α is defined as a weak derivatives of u.

The complement of $L_p(\Omega)$ implies that $w_p^r(\Omega)$ becomes a Banach space on putting the norm $w_p^r(\Omega)$ as

$$\begin{aligned} \|u\|_{W_p^r(\Omega)} &= \left(\sum_{|\alpha| \leq r} \int_\Omega |\partial^\alpha u|^p dx \right)^{1/p}, \\ \|u\|_{W_p^r(\Omega)} &= \sum_{|\alpha| \leq r} \|\partial^\alpha u(x)\|^p L_p(\Omega). \end{aligned} \tag{3}$$

For $p = 2$, $W_2^r(\Omega) = H^r(\Omega)$ is a Hilbert space with the inner product.

$$\begin{aligned} \langle u, v \rangle_{H^r(\Omega)} &= \sum_{|\alpha| \leq r} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L_2(\Omega)}, \\ \langle u, v \rangle_{H^r(\Omega)} &= \int_\Omega \left(\sum_{|\alpha| \leq r} \left(\partial^\alpha u \overline{\partial^\alpha} v \right) \right) dx. \end{aligned} \tag{4}$$

The norm induced by the inner product is

$$\|u\|_{H^r(\Omega)} = \left(\langle u, v \rangle_{H^r(\Omega)} \right)^{1/2}. \tag{5}$$

Definition 3 (see [24]). In a particular case, the Sobolev space $H^1(\Omega)$ is the set of all $f \in L_2(\Omega)$ such that all the first partial derivative $\partial f / \partial x_i$ belongs to $L_2(\Omega)$. The inner product in $H^1(\Omega)$ is

$$(f, g)_1 = \int (f\bar{g} + \nabla f . \nabla \bar{g}) d^n x, \tag{6}$$

where $\nabla f . \nabla \bar{g}$ denotes $\sum_{i=1}^n (\partial f / \partial x_i).(\partial \bar{g} / \partial x_i)$. This inner product clearly gives the norm

$$\|u\| = \left[\int \left(|u|^2 + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right) d^n x \right]^{1/2}. \tag{7}$$

Then, we denote the L_2 inner product by a subscript zero.

$$(f, g)_0 = \int f \bar{g} d^2 x. \tag{8}$$

Then, equation (6) reads

$$(f, g)_1 = (f, g)_0 + (\nabla f, \nabla g)_0, \tag{9}$$

where $(\nabla f, \nabla g)$ is an abbreviation for $\sum (\partial f / \partial x_i, \partial g / \partial x_i)$. In particular,

$$\|f\|_1^2 = \|f\|_0^2 + \|\nabla f\|_0^2. \tag{10}$$

Then, the Cauchy sequence in $H^1(\Omega)$ converges to the element of $H^1(\Omega)$. In other words, $H^1(\Omega)$ is a Hilbert space. It is in fact the Hilbert space obtained by completing the set of smooth function with respect to the $\|,\|_1$, in the same way that $L_2(\Omega)$ is the Hilbert space obtained by completing the set of smooth functions with respect to the L_2 norm [24].

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx \quad \forall \phi \in D(\Omega). \tag{11}$$

2.3. Weak Solution [1, 15]. Consider a partial differential operator L of order m in N variables $L = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ where $\alpha = (\alpha_1, \alpha_2 \cdots \alpha_N)$ is a multi-index $|\alpha| = \alpha_1 + \alpha_2 \cdots + \alpha_N$ and $a_\alpha = a_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \cdots x_N)$ are functions in $C^\infty(\mathbb{R}^2)$.

Considering a differential equation $Lu(x) = f$ in the sense of distribution, then the following is true.

Let $\varphi \in D(\Omega)$; then, $\langle Lu, \varphi\rangle = \langle f, \varphi\rangle$ in Ω.

This implies $\langle u, L^* \varphi\rangle = \langle f, \varphi\rangle$ where $L^*\phi = \sum (-1)^{|\alpha|} D^\alpha (a(x)\phi)$; here, the operation L^* is the adjoint operator of L.

If the original problem was to find $|\alpha|$-times differentiable function u defined on the open set Ω such that $Lu(x) = f$ for all $x \in \Omega$, called the classical solution, then an integrable function u is said to be a weak solution if

$$\int_\Omega u(x) L^* \varphi dx = \int_\Omega f \varphi dx \, \forall \varphi \in D(\Omega). \tag{12}$$

2.4. Fundamental Solutions

Definition 4 (see [25]). A distribution D' is a fundamental solution of the differential operator L, if and only if

$$\mathrm{LD} = \delta. \tag{13}$$

The fundamental solution D of the differential operator L satisfies the equation; however, D need not fulfill the provided boundary conditions. A fundamental solution that satisfies the given boundary condition is known as Green's function [20, 21, 25].

Let $Lu = f$ be Green's function $G(x, \xi)$; it satisfies the equation

$$\mathrm{LG}(x, \xi) = \delta(x - \xi). \tag{14}$$

Physically, Green's function $G(x, \xi)$ represents the effect at the point x of a Dirac delta function source at the point $x = \xi$ [20].

Multiply equation (14) by $f(\xi)$ and integrate over the area A of the ξ circle so that

$$dA = d\xi d\eta. \tag{15}$$

Then, we have

$$L\left\{\int_A G(x, \xi) f(\xi) dA\right\} = \int \delta(x - \xi) f(\xi) dA = f(x). \tag{16}$$

Since $Lu = f$, we have

$$u(x) = \int G(x, \xi) f(\xi) dA. \tag{17}$$

The fundamental solution for the Laplace operator is as follows.

Definition 5 (see [26]). Let $E \in D'(R^2)$ such that

$$\Delta E = -\delta(x - \xi) \quad x, \xi \in \mathbb{R}^2, \tag{18}$$

with δ being the Dirac delta function. In general dimension, the $D'(R^2)$ (distributional space in R^2) is a solution of equation (18) which is called a fundamental solution of Laplace's equation at $x = \xi$. In the context of the heat equation, the fundamental solution of the Laplace equation is crucial to the heat kernel. In two dimensions, the fundamental radial solution of the Laplace equation is

$$E = -c \log r, \tag{19}$$

where c is the arbitrary constant and r is the distance from x to ξ.

It is also known as a heat kernel, which is a solution to the heat equation that corresponds to the initial condition of an initial point source at a specified place. This method can be used to discover a general solution to the heat equation for a given domain [21, 25, 26].

2.5. Green's Second [16, 26]. Let $u, v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and Green's first identity for the pair u and v; then,

$$\iint_\Omega v \Delta u dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds - \iint_\Omega \nabla u . \nabla v dx, \tag{20}$$

and again for the pair v and u,

$$\iint_\Omega u \Delta v dx = \int_{\partial\Omega} u \frac{\partial v}{\partial n} ds - \iint_\Omega \nabla v . \nabla u dx. \tag{21}$$

By subtracting equation (21) from equation (20), we get Green's second identity [23].

$$\iint_\Omega (u \Delta v - v \Delta u) dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds. \tag{22}$$

It is valid for the pair of functions u and v.

The above integral is a line integral over the boundary curve of two-dimensional region Ω, and ds denotes the arc length of the boundary [16, 26].

2.6. Boundary Integral Equation. In a variety of applications, the efficient numerical solution of partial differential equations (PDE) using boundary integral formulation is critical [27, 28].

Consider as an example a Laplace problem of the form

$$-\Delta u(x) = 0. \tag{23}$$

In some domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth Lipschitz boundary Γ, Green's representation theorem allows us to write the solution u as

$$u(x) = \int_{\Gamma} G(x,y)\frac{\partial}{\partial n}u(y)d\Gamma(y) - \int_{\Gamma}\frac{\partial}{\partial n(y)}G(x,y)u(y)d\Gamma(y), \text{for } x \in \Omega, \tag{24}$$

where n is the unit outward pointing normal at Γ and $G(x, y)$ is a fundamental solution defined as

$$G(x,y) = -\frac{1}{2\pi}\log|x-y|. \tag{25}$$

Hence, in principle, if either u or $\partial u/\partial n$ is known on G, we can recover the unknown quantity by restricting equation (24) to the boundary and solving to the unknown boundary (see, e.g., [6, 16]).

2.7. Variational Formulation. The variational approach to the problem not only lays the groundwork for mathematical proofs of existence and uniqueness but also strong numerical methods like the finite element method [15, 29]. Using the boundary conditions mentioned above in an appropriate space of functions, we look for a unique weak solution u of the Laplace equation $\Delta u = 0$ in S [15, 29].

$$k = \{v : v \in H_0^1, v = 0 \text{ on } \Gamma_3\}. \tag{26}$$

The problem is written in a weak form as follows:

(1) Multiply on both sides of the Laplace equation $\Delta u = 0$ by a function v in H_0^1 and integrate over Ω

$$\int_{\Omega}\Delta uvdx = 0 \tag{27}$$

(2) Apply integration by parts to arrive at

$$\int_{\Omega}\nabla u\nabla vdx + \int_{\Gamma}\frac{\partial u}{\partial n}vds = 0 \tag{28}$$

3. Statement of the Steady-State Heat Transfer Problem

Consider a heat-conducting body that is homogeneous and isotropic; Ω is a simple connected and bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ when $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are disjoint parts of Γ. Convection in the ambient medium is thought to occur at the boundary Γ_1 and Γ_2, temperature is kept constant at the prescribed value T_3 on Γ_3, and Γ_4 is insulated. The mixed boundary value problem describes the system's state equation as

$$\Delta\theta = 0, \text{in } \Omega, \tag{29}$$

$$\frac{\partial\theta}{\partial n} = -\frac{h}{k}(\theta - \theta_\infty), \text{on } \Gamma_1 \cup \Gamma_2, \tag{30}$$

$$\theta = 0, \text{on } \Gamma_3, \tag{31}$$

$$\frac{\partial\theta}{\partial n} = 0, \text{on } \Gamma_4, \tag{32}$$

where $\theta = T - T_3$, $\theta_\infty = T_\infty - T_3$ when T is temperature in the domain, T_∞ is the ambient temperature, n is the outward unit normal vector, h is the convection coefficient, and k is the conduction coefficient.

Since the classical solution $\theta \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ to the problem does not exist if $x = \xi$ for equation (19), then ξ is a singular point [1, 25, 30], where $\bar{\Omega}$ is closure of the domain Ω; we can be concerned with the variational solution $H^1(\Omega)$.

3.1. Boundary Integral Equation for the Classical Solution. The boundary integral equation formulation for the heat transfer problem is based on Green's formula with the fundamental solution [20, 28, 31]. The simplest method for transforming variables to boundary variables is to use Green's second identity [1, 25, 32].

Let u and v be $C^1(\bar{\Omega}) \cap C^2(\Omega)$ function; then, Green's formula of equation (22) holds. If the classical solution $\theta \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ exists, we can substitute u by θ in equation (22). However, the singularity of E in equation (19) is preventing one from substituting v by E in equation (22). One way of overcoming the difficulty is to replace Ω by $\Omega - B_\rho(\xi)$ where $B_\rho(\xi)$ is a circle with the small radius ρ centered at a singular point ξ.

One can conclude from equation (22) that

$$\int_{\Omega - B_\rho}(E\Delta\theta - \theta\Delta E)dx = \int_{\Gamma}\left(E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n}\right)ds + \int_{\partial B_\rho}\left(E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n}\right)ds, \tag{33}$$

for

$$\theta \in C^1(\bar{\Omega}) \cap C^2(\Omega), \tag{34}$$

where $\Omega - B_\rho$ is the boundary of B_ρ in equation (34). Since $\Delta\theta = 0$ and $\Delta E = -\delta(x - \xi) = 0$ on $\Omega - B_\rho$, we have

$$\int_{\Omega - B_\rho}E\,\Delta\theta dx = 0, \tag{35}$$

$$\int_{\Omega - B_\rho}\theta\,\Delta E dx = 0. \tag{36}$$

The first term in the integral over ∂B_ρ in equation (34) becomes

$$\int_{\partial B_\rho}E\frac{\partial\theta}{\partial n}ds = E(\rho)\int_{\partial B_\rho}\frac{\partial\theta}{\partial n}ds \le E(\rho)\omega_2\rho\sup_{\partial B}\left|\frac{\partial\theta}{\partial n}\right| \longrightarrow 0, \text{as } \rho \longrightarrow 0. \tag{37}$$

Since

$$\left| C\rho^{-1} \int_{\partial B_\rho} (\theta(x) - \theta(\xi)) ds \right| \le C\rho^{-1} \int_{\partial B_\rho} |(\theta(x) - \theta(\xi))| ds, \\ \le C\omega_2 \max_{x \in \partial B_\rho} |\theta(x) - \theta(\xi)| \longrightarrow 0, \text{ as } \rho \longrightarrow 0. \tag{38}$$

The second term in integration over ∂B_ρ in equation (56) becomes

$$\int_{\partial B_\rho} \theta \frac{\partial E}{\partial n} ds = C\rho^{-1} \int_{\partial B_\rho} \theta(x) ds \longrightarrow C\rho^{-1} \int_{\partial B_\rho} \theta(\xi) ds \\ = C\omega_2 \theta(\xi), \text{ as } \rho \longrightarrow 0 = \theta(\xi), \tag{39}$$

where $\omega_2 = 2\pi$ is the boundary length of the unit circle in $\mathbb{R}^2$ and $\omega_2 \rho$ is the boundary of the circle with the radius ρ. If one chooses $C = 1/\omega_2$ and substitutes equations (35), (36), (37), and (39) in equation (34), then

$$\theta(\xi) = \int_\Gamma \left(E \frac{\partial \theta}{\partial n} - \theta \frac{\partial E}{\partial n} \right) ds, \text{ for all } \xi \in \Omega, \theta \in C^1(\bar{\Omega}) \cap C^2(\Omega) \tag{40}$$

holds as ρ goes to zero. If ξ is on Γ, equation (40) has a singularity. Then, we can divide the boundary Γ by Γ_ε and $\Gamma - \Gamma_\varepsilon$ where Γ_ε is half circle with small radius ε centered at a singular ξ. Then, equation (40) becomes

$$\theta(\xi) = \int_{\Gamma_\varepsilon} \left(E \frac{\partial \theta}{\partial n} - \theta \frac{\partial E}{\partial n} \right) ds - \int_{\Gamma - \Gamma_\varepsilon} \left(E \frac{\partial \theta}{\partial n} - \theta \frac{\partial E}{\partial n} \right) ds. \tag{41}$$

The first term of the boundary integration over Γ_ε in equation (41) becomes

$$\int_{\Gamma_\varepsilon} E \frac{\partial \theta}{\partial n} ds = E(\varepsilon) \int_{\Gamma_\varepsilon} \frac{\partial \theta}{\partial n} ds \le E(\varepsilon) \omega_2 \varepsilon \sup_{B_\varepsilon} \left| \frac{\partial \theta}{\partial n} \right| \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0. \tag{42}$$

The second term becomes

$$-\int_{\Gamma_\varepsilon} \theta \frac{\partial E}{\partial n} ds = C\varepsilon^{-1} \int_{\Gamma_\varepsilon} \theta(x) ds. \tag{43}$$

Since

$$\left| C\varepsilon^{-1} \int_{\Gamma_\varepsilon} (\theta(x) - \theta(\xi)) ds \right| \le C\varepsilon^{-1} \int_{\Gamma_\varepsilon} |\theta(x) - \theta(\xi)| ds, \\ \le \frac{1}{2} C\omega_2 \max_{x \in \Gamma_\varepsilon} |\theta(x) - \theta(\xi)| \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0. \tag{44}$$

Assume $C = 1/\omega_2$; then, equation (42) becomes

$$C\varepsilon^{-1} \int_{\Gamma_\varepsilon} \theta(x) ds \longrightarrow C\varepsilon^{-1} \int_{\Gamma_\varepsilon} \theta(\xi) ds = \frac{1}{2} C\omega_2 \theta(\xi) = \frac{1}{2} \theta(\xi). \tag{45}$$

By substituting equation (42) and equation (45) into equation (41) and let ε go to zero, then we obtain

$$\frac{1}{2} \theta(\xi) = \int_\Gamma \left(E \frac{\partial \theta}{\partial n} - \theta \frac{\partial E}{\partial n} \right) ds \text{ for all } \xi \in \Gamma, \theta \in C^1(\bar{\Omega}) \cap C^2(\Omega). \tag{46}$$

When we use a boundary element method for the problem with $\xi \in \Gamma$, $\theta(\xi)$is obtained numerically from equation (46), while it is obtained from equation (40) when $\xi \in \Omega$. By dividing the boundary into small segments, the classical solution, if it exists, can be approximated numerically using boundary integral equations (40) and (46) as illustrated above. However, in the mixed boundary value problem, the classical solution does not exist when x and ξ are at the same point; then, it has a singularity and ξ is singularity of the fundamental solution. Therefore, we cannot use equations (40) and (46) directly.

3.2. Boundary Integral Expression for the Weak Solution. The state equation of equations (29)–(32) is written in a variational form as

$$\int_\Omega \nabla\theta \nabla v dx + \int_{\Gamma_1 \cup \Gamma_2} \frac{h}{k} (\theta - \theta_\infty) v ds = 0, \text{ for all } v \in K, \tag{47}$$

where K is the admissible set given by $K = \{v / v \in H^1(\Omega), v = 0 \text{ on } \Gamma_3\}$. The weak solution of equation (47) is unique in $H^1(\Omega)$ by using equations (27) and (28) and applying the Lax-Milgram theorem. For every $u \in H_0^1(\Omega)$, there exists a unique solution $\theta \in H_0^1(\Omega)$.

By using the Cauchy-Schwarz inequality, let us check the continuity of $B[\theta, v]$:

$$\int_\Omega \nabla\theta \nabla v dx \le C \|\theta\| \|v\|. \tag{48}$$

On the boundary $\Gamma_1 \cup \Gamma_2$ by the Cauchy-Schwarz inequality,

$$\int_{\Gamma_1 \cup \Gamma_2} \frac{h}{k} (\theta - \theta_\infty) v ds \le C \|\theta\| \|v\|. \tag{49}$$

Then, from equations (47) and (48), we have continuity

$$B[\theta, v] \le C \|\theta\| \|v\|. \tag{50}$$

The following is the bilinear form of $B[\theta, v]$:

$$-B[\theta, \theta] = \int_\Omega \nabla\theta^2 dx + \int_{\Gamma_1 \cup \Gamma_2} \frac{h}{k} (\theta - \theta_\infty) \theta ds. \tag{51}$$

Poincare's inequality indicates that

$$\int_{\Omega} \nabla\theta^2 dx \geq C\|\theta\|^2. \tag{52}$$

Then, we have

$$B[\theta, \theta] \geq C\|\theta\|^2. \tag{53}$$

Therefore, the condition of the Lax-Milgram theorem is satisfied, and there exists a unique weak solution on $\theta \in H_0^1(\Omega)$ [2, 8, 9, 14].

To represent the boundary integral equation for the variational weak solution $\theta \in H^1(\Omega)$, then we need the following theorem [8].

Theorem 6 (see [8, 21]. *Green's formula in the Sobolev space*

$$\left(\int_{\Omega} v\Delta u - u\Delta v\right) dx = \int_{\Gamma} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds \tag{54}$$

holds for the domain Ω *with the Lipschitz boundary* Γ *if* u, $v \in H^1(\Omega, \Delta)$
where $H^1(\Omega, \Delta) = \{u/u \in H^1(\Omega)$ *such that* $\Delta \in L^2(\Omega)\}$.

The variational solution θ is in $H^1(\Omega, \Delta)$, but the fundamental solution is not. In fact, it is in $C^{\infty}(\mathbb{R}^2 - \{\xi\})$ [8]. Then, u in equation (49) can be substituted by θ but v cannot by E. This difficulty is removed by replacing Ω by $\Omega - B_\rho$, since E is the $H^1(\Omega, \Delta)$ in $\Omega - B_\rho$. Then, we can conclude from equation (54) that

$$\int_{\Omega-B_\rho} (E\Delta\theta - \theta\Delta E) dx = \int_{\Gamma} \left(E\frac{\partial \theta}{\partial n} - \theta\frac{\partial E}{\partial n}\right) ds + \int_{\partial B_\rho} \left(E\frac{\partial \theta}{\partial n} - \theta\frac{\partial E}{\partial n}\right) ds, \text{ for } \theta \in H^1(\Omega, \Delta). \tag{55}$$

The left-hand side term of integration over $\Omega - B_\rho$ is zero, and the first term in the integration over ∂B_ρ of equation (55) becomes

$$\int_{\partial B_\rho} E\frac{\partial \theta}{\partial n} ds = E(\rho)\int_{\partial B_\rho} \frac{\partial \theta}{\partial n} ds \leq E(\rho)\omega_2\rho \sup_{B_\rho}\left|\frac{\partial \theta}{\partial x_i}\right| \longrightarrow 0, \text{ as } \rho \longrightarrow 0. \tag{56}$$

The second term in the integration over ∂B_ρ of equation (55) becomes

$$\begin{aligned} \int_{\partial B_\rho} \theta\frac{\partial E}{\partial n} ds &= C\rho^{-1}\int_{\partial B_\rho} \theta(x) dx \longrightarrow C\rho^{-1}\int_{\partial B_\rho} \theta(\xi) ds \\ &= C\omega_2\theta(\xi), \text{ as } \rho \longrightarrow 0 = \theta(\xi). \end{aligned} \tag{57}$$

Then, by substituting equations (56) and (57) in equation (54), we obtain

$$\theta(\xi) = \int_{\Gamma} \left(E\frac{\partial \theta}{\partial n} - \theta\frac{\partial E}{\partial n}\right) ds, \text{ for all } \xi \in \Omega, \theta \in H^1(\Omega, \Delta), \tag{58}$$

as ρ goes to zero.

With the similar way stated in Section 3.1 in equations (41), (42), (43), (44), and (45), if ξ is on Γ, equation (55) becomes

$$\frac{1}{2}\theta(\xi) = \int_{\Gamma} \left(E\frac{\partial \theta}{\partial n} - \theta\frac{\partial E}{\partial n}\right) ds, \text{ for all } \xi \in \Gamma, \theta \in H^{1/2}(\Gamma). \tag{59}$$

If we insert the boundary condition of equations (30)–(32) into equations (58) and (59), respectively, we can get

$$\theta(\xi) = \int_{\Gamma_1\cup\Gamma_2} E\frac{h}{k}(\theta - \theta_\infty) + \int_{\Gamma_3} E\frac{\partial \theta}{\partial n} ds - \int_{\Gamma_1\cup\Gamma_2\cup\Gamma_4} \theta\frac{\partial E}{\partial n} ds, \tag{60}$$

$$\frac{1}{2}\theta(\xi) = -\int_{\Gamma_1\cup\Gamma_2} E\frac{h}{k}(\theta - \theta_\infty) ds + \int_{\Gamma_3} E\frac{\partial \theta}{\partial n} ds - \int_{\Gamma_1\cup\Gamma_2\cup\Gamma_4} \theta\frac{\partial E}{\partial n} ds. \tag{61}$$

The solution of equations (60) and (61) in $H^1(\Omega)$ is equal to the variational solution, because it is unique in $H^1(\Omega)$. The solution of the problem in equations (60) and (61) can be approximated numerically by dividing the border into small parts, as shown by the previous results.

4. Conclusion

In this study, we present a two-dimensional heat transfer problem utilizing a boundary integral equation with specific initial and boundary conditions, and we discuss how a variational solution to a mixed boundary value problem can be obtained even though a classical solution does not exist. Also, we have transformed the boundary value problem for the steady-state heat transfer problem into boundary integral equation and the solution of boundary integral equation for the mixed boundary value problem by using Green's identity and fundamental solution. The boundary integral equation for the problem guided by the Laplace operator has a unique solution that is similar to the variational solution in $H^1(\Omega)$. As a result, a numerical approximation of the variational solution for the boundary integral problem can be obtained. Furthermore, the approach used in this study can be used for three-dimensional heat transfer problems as well as other elliptic problems.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the preparation of this research paper.

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