1. Introduction


Levi-Civita connection is one of the most natural and effective tools for studying Riemannian manifolds [7]. In the complex case, Hsiung et al. [8] studied the general sectional curvature, the holomorphic sectional curvature, and holomorphic bisectional curvature of almost Hermitian manifolds by Levi-Civita connection and showed the relevance of above sectional curvatures. In 2012, Liu and Yang [8] gave Ricci-type curvatures and scalar curvatures of Hermitian manifolds by Levi-Civita connection (resp. Chern connection and Bismut connection) and obtained the relevance of these curvatures.

Warped product and twisted product are important methods used to construct manifold with special curvature properties in Riemann geometry and Finsler geometry. In Riemann geometry, Bishop and O’Neill [9] constructed Riemannian manifolds with negative curvature by warped product. Then, Brozos-Va’zquez et al. [10] used the warped product metrics to construct new examples of complete locally conformally flat manifolds with nonpositive curvature. After that, Leandro et al. [11] proved that an Einstein warped product manifold is a compact Riemannian manifold and its fibre is a Ricci-flat semi-Riemannian manifold.

On the other hand, warped product was extended to real Finsler geometry by the work of Asanov [12, 13]. In 2016, He and Zhong [14] generalized the warped product to complex Finsler geometry and proved that if complex Finsler manifold $\langle M_1, F_1 \rangle$ and $\langle M_2, F_2 \rangle$ are projectively flat, then the DWP-complex Finsler manifold is projectively flat if and only if the warped function $f_1$ and $f_2$ are positive constants.
curvature. Recently, Xiao et al. [16] systematically studied holomorphic curvatures of doubly twisted product complex Finsler manifolds, and they [17] gave the necessary and sufficient condition for doubly twisted product complex Finsler manifold to be locally dually flat.

Thus, it is natural and interesting to ask the following question. Let \((M_1, g)\) and \((M_2, h)\) be two Levi-Civita Ricci-flat Hermitian manifolds, whether the DWP-Hermitian manifold is also a Levi-Civita Ricci-flat Hermitian manifold. Our purpose of doing this is to study the possibility of constructing Levi-Civita Ricci-flat manifold.

The structure of this paper is as follows. In Section 2, we briefly recall some basic concepts and notations which we need in this paper. In Section 3, we derive formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of DWP-Hermitian manifolds. In Section 4, we show that if the warped function \(f_1\) and \(f_2\) are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if \((M_1, g)\) and \((M_2, h)\) are Levi-Civita Ricci-flat manifolds.

2. Preliminary

Let \((M, J, G)\) be a Hermitian manifold with \(\dim_{\mathbb{C}} M = n\); here, \(J\) is the complex structure, and \(G\) is a Hermitian metric. For a point \(p \in M\), the complexified tangent bundle \(T_p^C M = T_p M \otimes \mathbb{C}\) is decomposed as

\[
T_p^C M = T_p^{1,0} M \oplus T_p^{0,1} M,
\]

where \(T_p^{1,0} M\) and \(T_p^{0,1} M\) are the eigenspaces of \(J\) corresponding to the eigenvalues \(\sqrt{-1}\) and \(-\sqrt{-1}\), respectively.

In this paper, we set \(\partial_a = \partial / \partial z^a\) and \(\partial_a = \partial / \partial \bar{z}^a\). Let \(z = (z^1, \ldots, z^n)\) be the local holomorphic coordinates on \(M\); then, the vector fields \((\partial_1, \ldots, \partial_n)\) form a basis for \(T_p^{1,0} M\). Levi-Civita connection \(\nabla^{LC}\) on the holomorphic tangent bundle \(T_p^{1,0} M\) is defined by [18]

\[
\nabla^{LC} = \pi \nabla : \Gamma(M, T^{1,0} M) \xrightarrow{\nu} \Gamma(M, T_p M \otimes T_p M) \xrightarrow{\tau} \Gamma(M, T_p M \otimes T^{1,0} M).
\]

(2)

In local coordinate system, its connection is as follows [18]:

\[
\nabla^{LC}_{\partial_a} \bar{\partial} \bar{\partial} = \Gamma^{\nu}_{a\bar{b}} \frac{\partial}{\partial z^\nu},
\]

\[
\nabla^{LC}_{\partial_a} \bar{\partial} \partial = \Gamma^{\nu}_{a\bar{b}} \frac{\partial}{\partial \bar{z}^\nu},
\]

(3)

where the Levi-Civita connection coefficients \(\Gamma^{\nu}_{a\bar{b}}\) and \(\Gamma^{\nu}_{a\bar{b}}\) are given by [18]

\[
\Gamma^{\nu}_{a\bar{b}} = \frac{1}{2} \bar{G}^{\nu} \frac{\partial}{\partial z^a} G_{\bar{b}\bar{c}} - \frac{\partial}{\partial \bar{z}^a} G_{\bar{b}\bar{c}},
\]

(4)

\[
\Gamma^{\nu}_{a\bar{b}} = \frac{1}{2} \bar{G}^{\nu} \frac{\partial}{\partial \bar{z}^a} G_{\bar{b}\bar{c}} - \frac{\partial}{\partial z^a} G_{\bar{b}\bar{c}}.
\]

(5)

Let \(K \in \Gamma(M, \Lambda^2 T^*_p M \otimes T^{*1,0} M \otimes T^{*1,0} M)\) be the Levi-Civita curvature tensor such as

\[
K(X, Y) s = \nabla^{LC}_{\nabla^{LC}_X Y} s - \nabla^{LC}_{\nabla^{LC}_Y X} s - \nabla^{LC}_{[X,Y]} s,
\]

(6)

where \(X, Y \in T_p M, s \in T^{1,0} M\). In the local coordinate system, the coefficients of \(K\) are given by

\[
K^{\nu}_{a\bar{b} \mu} = -\left[ \frac{\partial}{\partial z^a} R^{\nu}_{\bar{b} \mu} - \frac{\partial}{\partial \bar{z}^a} R^{\nu}_{\bar{b} \mu} + \frac{\partial}{\partial z^a} R^{\nu}_{\bar{b} \mu} - \frac{\partial}{\partial \bar{z}^a} R^{\nu}_{\bar{b} \mu} \right].
\]

(7)

Definition 1 (see [6]). The first Levi-Civita Ricci curvature \(K^{(1)}\) on the Hermitian manifold \((M, J, G)\) is defined by

\[
K^{(1)} = \sqrt{-1} K^{(1)} dz^a \wedge d\bar{z}^b,
\]

(8)

where

\[
K^{(1)}_{a\bar{b}} = G^{\nu \bar{b}} K^{(1)}_{a\nu},
\]

(9)

\[
K^{(1)}_{a\nu} = G^{a\bar{b}} K^{(1)}_{\bar{b} \nu}.
\]

(10)

Levi-Civita Ricci scalar curvature \(S_{LC}\) on \(T^{1,0} M\) is given by

\[
S_{LC} = G^{a\bar{b}} K^{(1)}_{a\bar{b}}.
\]

(11)

Definition 2 (see [6]). Hermitian metric \(G\) on \(M\) is called Levi-Civita Ricci-flat if

\[
K^{(1)}(G) = 0.
\]

(12)

Let \((M_1, g)\) and \((M_2, h)\) be two Hermitian manifolds with \(\dim_{\mathbb{C}} M_1 = m\) and \(\dim_{\mathbb{C}} M_2 = n\); then, \(M_1 \times M_2\) is a Hermitian manifold with \(\dim_{\mathbb{C}} M_1 \times M_2 = m + n\).

Denote \(\pi_1 : M \rightarrow M_1\) and \(\pi_2 : M \rightarrow M_2\) the natural projections. Note that \(\pi_1(z) = z_1\) and \(\pi_2(z) = z_2\) for every \(z = (z_1, z_2) \in M\) with \(z_1 = (z^1, \ldots, z^m) \in M_1\) and \(z_2 = (z^{m+1}, \ldots, z^{m+n}) \in M_2\).

Denote \(d\pi_1 : T^{1,0}(M) \rightarrow T^{1,0} M_1, d\pi_2 : T^{1,0}(M) \rightarrow T^{1,0} M_2\) the holomorphic tangent maps induced by \(\pi_1\) and \(\pi_2\), respectively. Note that \(d\pi_1(z, v) = (z_1, v_1)\) and \(d\pi_2(z, v) = (z_2, v_2)\) for every \(v = (v_1, v_2) \in T^{1,0}_z(M)\) with \(v_1 = (v^1, \ldots, v^m) \in T^{1,0}_z M_1\) and \(v_2 = (v^{m+1}, \ldots, v^{m+n}) \in T^{1,0}_z M_2\).

Definition 3 (see [15]). Let \((M_1, g)\) and \((M_2, h)\) be two Hermitian manifolds, \(f_1 : M_1 \rightarrow (0, +\infty)\) and \(f_2 : M_2 \rightarrow (0, +\infty)\) be two positive smooth functions. The doubly warped product (abbreviated as DWP) Hermitian manifold \((f_1 M_1 \times f_2 M_2, G)\) is the product Hermitian manifold \(M = M_1 \times M_2\) endowed with the Hermitian metric \(G : M \rightarrow \mathbb{R}^+\)
defined by
\[ G(z, v) = (f_1 * \pi_1)^2(z)g(\pi_1(z), d\pi_1(v)) + (f_2 * \pi_2)^2(z)h(\pi_2(z), d\pi_2(v)), \]
(13)

for \( z = (z_1, z_2) \in M \) and \( v = (v_1, v_2) \in T^1_z M \). \( f_1 \) and \( f_2 \) are warped functions; the DWP-Hermitian manifold of \((M_1, g_1)\) and \((M_2, h)\) is denoted by \((f_1 M_1 \times f_2 M_2, G)\).

If either \( f_1 = 1 \) or \( f_2 = 1 \), then \((f_1 M_1 \times f_2 M_2, G)\) becomes a warped product of Hermitian manifolds \((M_1, g)\) and \((M_2, h)\). If \( f_1 \equiv 1 \) and \( f_2 \equiv 1 \), then \((f_1 M_1 \times f_2 M_2, G)\) becomes a product of Hermitian manifolds \((M_1, g)\) and \((M_2, h)\). If neither \( f_1 \) nor \( f_2 \) is constant, then we call \((f_1 M_1 \times f_2 M_2, G)\) a nontrivial DWP-Hermitian manifolds of \((M_1, g)\) and \((M_2, h)\).

**Notation 4.** Lowercase Greek indices such as \( \alpha, \beta, \) and \( \gamma \) will run from 1 to \( m + n \), lowercase Latin indices such as \( i, j, \) and \( k \) will run from 1 to \( m \), and lowercase Latin indices with a prime, such as \( i', j', \) and \( k' \), will run from \( m + 1 \) to \( m + n \). Quantities associated to \((M_1, g)\) and \((M_2, h)\) are denoted with upper indices 1 and 2, respectively, such as \( \Gamma^i_{jk} \) and \( \Gamma'^i_{jk} \) are Levi-Civita connection coefficients of \((M_1, g)\) and \((M_2, h)\), respectively.

Denote
\[
\begin{align*}
g_{ij} & = \frac{\partial^2 g}{\partial \nu^i \partial \nu^j}, \\
h_{i'j'} & = \frac{\partial^2 h}{\partial \nu^{i'} \partial \nu^{j'}}.
\end{align*}
\]

The fundamental tensor matrix of \( G \) is given by
\[
\begin{pmatrix}
G_{\alpha \beta} & = & \frac{\partial^2 G}{\partial \nu^\alpha \partial \nu^\beta} \\
\end{pmatrix} = \begin{pmatrix} f_2^2 g_{ij} & 0 \\
0 & f_1^2 h_{i'j'}
\end{pmatrix},
\]
(15)

and its inverse matrix \((G^\alpha_\beta)\) is given by
\[
\begin{pmatrix}
G^\alpha_\beta & = & f_2^{-2} g^{ij} \\
0 & f_1^{-2} h^{i'j'}
\end{pmatrix}.
\]
(16)

**Proposition 5.** Let \((f_1 M_1 \times f_2 M_2, G)\) be a DWP-Hermitian manifold of \((M_1, g)\) and \((M_2, h)\). Then, the Levi-Civita connection coefficients \( \Gamma_{ij}^\gamma \) associated to \( G \) are given by
\[
\Gamma_{ij}^\gamma = \Gamma_{ij}^\gamma,
\]
(17)

Proof. Substituting (15) and (16) into (4), we obtain
\[
\Gamma_{ij}^\gamma = \frac{1}{2} \frac{\partial g_{ij}}{\partial \nu^\gamma} + \frac{1}{2} \frac{\partial g_{\gamma j}}{\partial \nu^i} + \frac{1}{2} \frac{\partial g_{\gamma i}}{\partial \nu^j} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \nu^\gamma} + \frac{1}{2} \frac{\partial g_{\gamma j}}{\partial \nu^i} + \frac{1}{2} \frac{\partial g_{\gamma i}}{\partial \nu^j}.
\]
(18)

Similarly, we can obtain other equations of Proposition 5.

Plugging (15) and (16) into (5), we have the following proposition.

**Proposition 6.** Let \((f_1 M_1 \times f_2 M_2, G)\) be a DWP-Hermitian manifold of \((M_1, g)\) and \((M_2, h)\). Then, the Levi-Civita connection coefficients \( \Gamma_{ij}^\gamma \) associated to \( G \) are given by
\[
\Gamma_{ij}^\gamma = \Gamma_{ij}^\gamma,
\]
(19)
3. Levi-Civita Ricci Scalar Curvature of Doubly Warped Product Hermitian Manifolds

In this section, we derive formulae of Levi-Civita curvature, Levi-Civita Ricci curvature, and Levi-Civita Ricci scalar curvature of DWP-Hermitian manifolds.

**Proposition 7.** Let \((f_i M_j \times f_j M_j, G)\) be a DWP-Hermitian manifold of \((M_j, g)\) and \((M_j, h)\). Then, the coefficients of Levi-Civita curvature tensor \(K_{jps}^i\) are given by

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(20)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(21)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(22)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(23)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(24)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(25)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(26)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(27)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(28)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(29)

\[
K_{kji}^l = \frac{f_j}{f_i} H_{kji}^l + f_j^2 g_{ij} \frac{\partial f_j}{\partial z^j} \frac{\partial f_j}{\partial z^j} \delta_{lk},
\]

(30)

**Proof.** Using (7), we have

\[
K_{kji}^l = \left\{ \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} + \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} - \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} \right\}.
\]

(31)

Taking the formulae of Proposition 5 and Proposition 6 into (31), we obtain

\[
K_{kji}^l = \left\{ \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} + \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} - \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} \right\}
\]

(32)

Similarly, we can obtain other equations of Proposition 7.

**Proposition 8.** Let \((f_i M_j \times f_j M_j, G)\) be a DWP-Hermitian manifold of \((M_j, g)\) and \((M_j, h)\). Then,

\[
K_{kji}^l = \left\{ \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} + \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} - \frac{1}{\partial z^l} \frac{\partial f_l}{\partial z^j} \right\}
\]

(33)
Proof. According to (10), we get
\[ K_{kjp} = G_{ip}K_{kjp}^i = G_{ip}K_{kjp}^i + G_{ip}K_{kjp}^i. \]  
(34)

Substituting (20), (27), and (15) into (34), we have
\[ K_{ijp} = G_{ip} \left( \frac{f_1^2}{\delta^2} \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} g_{ij} \right) = f_1^2 K_{ijp}^i + f_1^2 f_2^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} g_{ij}. \]  
(35)

Similarly, we can obtain other equations of Proposition 8.

Proposition 9. Let \((f_1 M_1 \times f_2 M_2, G)\) be a DWP-Hermitian manifold of \((M_1, g)\) and \((M_2, h)\). Then, the coefficients of the first Levi-Civita Ricci curvature \(K_{kij}^{(1)}\) are given by
\[
\begin{align*}
K_{kij}^{(1)} &= \frac{f_1^2}{\delta^2} \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} g_{ij}, \\
K_{kij}^{(1)} &= \frac{f_1^2}{\delta^2} \frac{\partial f_2}{\partial \vartheta} h_k^j g_i^j, \\
K_{kij}^{(1)} &= 0, \\
K_{kij}^{(1)} &= 0.
\end{align*}
\]
(36)

where \(K_{kij}^{(1)}\) and \(K_{kij}^{(1)}\) are coefficients of the first Levi-Civita Ricci curvature of \(g\) and \(h\), respectively.

Proof. From (9) and (16), we get
\[ K_{kij}^{(1)} = G_{ip}K_{kjp}^{(1)} = G_{ip}K_{kjp}^{(1)} + G_{ip}K_{kjp}^{(1)}. \]  
(37)

According to (16) and the first equation of proposition 8, we have
\[ G_{ip}K_{kjp} = f_1^2 g_{ij} \left( f_1^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} g_{kjp} \right) = \left( f_1^2 + f_1^2 f_2^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} \right) g_{kjp}. \]  
(38)

Similarly, by using (16) and the third equation of proposition 8, we can get
\[ G_{ip}K_{kjp} = f_1^2 \frac{\partial f_2}{\partial \vartheta} h_k^j g_i^j. \]  
(39)

Replacing the summation index \(i'\) on the right side of (38) with \(i'\) and then taking it and (39) into (37), we can obtain
\[ K_{kij}^{(1)} = K_{kij}^{(1)} + 2f_1^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} g_{kij}. \]  
(40)

Similarly, we can obtain
\[ K_{kij}^{(1)} = \frac{2}{\delta^2} \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} h_k^j g_i^j, \]  
(41)

This completes the proof.

Theorem 10. Let \((f_1 M_1 \times f_2 M_2, G)\) be a DWP-Hermitian manifold of \((M_1, g)\) and \((M_2, h)\). Then, the Levi-Civita Ricci scalar curvature of \(G\) along a nonzero vector \(v = (v^1, v^2) \in T^1_{\vartheta}M\) is given by
\[ S_{LC}(v) = f_1^2 L_S(v_1) + f_1^2 L_S(v_2) + 2f_1^2 f_2^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} h_k^j g_i^j + 2f_1^2 f_2^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta}, \]  
(42)

where \(S_{v1}(v)\) and \(S_{v2}(v)\) are Levi-Civita Ricci scalar curvatures of \(g\) and \(h\), respectively.

Proof. According to (11), the Levi-Civita Ricci scalar curvature of \(G\) is given by
\[ S_{LC} = G_{ip}K_{kjp}^{(1)} = G_{ip}K_{kjp}^{(1)} + G_{ip}K_{kjp}^{(1)} + G_{ip}K_{kjp}^{(1)} + G_{ip}K_{kjp}^{(1)}. \]  
(43)

Combining (16) and (40), we have
\[ G_{ip}K_{kjp}^{(1)} = f_1^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta}, \]  
(44)

Similarly, we can get
\[ G_{ip}K_{kjp}^{(1)} = f_1^2 \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta} \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \vartheta}. \]  
(45)

Taking (44)–(47) into (43), we obtain (42).
Proof. If \( f_1 \) and \( f_2 \) are holomorphic functions on \( M_1 \) and \( M_2 \), respectively, i.e.,

\[
\frac{\partial f_1}{\partial z^1} = 0, \\
\frac{\partial f_2}{\partial z^2} = 0.
\]

Thus,\( 2f_1^2 f_2^2 g_i^j \frac{\partial f_1}{\partial z^1} \frac{\partial f_2}{\partial z^2} = 0, \)

\( 2f_1^2 f_2^2 h_i^j \frac{\partial f_1}{\partial z^1} \frac{\partial f_2}{\partial z^2} = 0. \)

Substituting (49) into (42), we have

\[
S_{\ell C}(v) = f_1^2 S_{g}(v_1) + f_2^2 S_{g}(v_2).
\]

4. Levi-Civita Ricci-Flat Doubly Warped Product Hermitian Manifolds

Let \((M_1, g)\) and \((M_2, h)\) be two Levi-Civita Ricci-flat Hermitian manifolds; one may want to know whether the DWP-Hermitian manifold \((f, M_1 \times f, M_2, G)\) is also a Levi-Civita Ricci-flat Hermitian manifold. We shall give an answer to this question in this section.

Theorem 12. Let \((f, M_1 \times f, M_2, G)\) be a DWP-Hermitian manifold of \((M_1, g)\) and \((M_2, h)\). If \( f_1 \) and \( f_2 \) are holomorphic functions on \( M_1 \) and \( M_2 \), respectively, then \((f, M_1 \times f, M_2, G)\) is Levi-Civita Ricci-flat if and only if \((M_1, g)\) and \((M_2, h)\) are Levi-Civita Ricci-flat.

Proof. If \( f_1 \) and \( f_2 \) are holomorphic functions on \( M_1 \) and \( M_2 \), respectively, i.e.,

\[
\frac{\partial f_1}{\partial z^1} = 0, \\
\frac{\partial f_2}{\partial z^2} = 0.
\]

Taking above equations into the first formula and second formula of (36), we get

\[
2f_1^2 \frac{\partial f_1}{\partial z^1} \frac{\partial f_2}{\partial z^2} g_i^j h_i^j = 0, \quad (52)
\]

\[
2f_2^2 \frac{\partial f_1}{\partial z^1} \frac{\partial f_2}{\partial z^2} h_i^j h_i^j = 0. \quad (53)
\]

Firstly, we assume \((f, M_1 \times f, M_2, G)\) be Levi-Civita Ricci-flat; using Definition 2 and (36), we have

\[
\begin{align*}
K_{ij}^{(1)} &= K_{ij}^{(1)} + 2f_1^2 \frac{\partial f_1}{\partial z^1} \frac{\partial f_2}{\partial z^2} g_i^j h_i^j, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} + 2f_2^2 \frac{\partial f_1}{\partial z^1} \frac{\partial f_2}{\partial z^2} h_i^j h_i^j.
\end{align*}
\]

Substituting (52) and (53) into the first formula and second formula of (54), respectively, we get

\[
\begin{align*}
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0.
\end{align*}
\]

Obviously,

\[
K_{ij}^{(1)} = K_{ij}^{(1)} = 0, \quad (56)
\]

According to Definition 2, these mean that \((M_1, g)\) and \((M_2, h)\) are Levi-Civita Ricci-flat.

Conversely, we assume \((M_1, g)\) and \((M_2, h)\) are Levi-Civita Ricci-flat; according to Definition 2, we know that

\[
K_{ij}^{(1)} = K_{ij}^{(1)} = 0, \quad (57)
\]

\[
K_{ij}^{(1)} = K_{ij}^{(1)} = 0. \quad (58)
\]

Since \( f_1 \) and \( f_2 \) are holomorphic, thus (52) and (53) are established. Then, taking (52), (53), (57), and (58) into (36), we obtain

\[
\begin{align*}
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0, \\
K_{ij}^{(1)} &= K_{ij}^{(1)} = 0.
\end{align*}
\]
By Definition 2, (59) indicates that \((f_1 M_1 \times f_2 M_2, G)\) is Levi-Civita Ricci-flat.

**Notation 13.** Theorem 12 implies that when warped functions to be holomorphic, then the DWP-Hermitian manifold is a Levi-Civita Ricci-flat Hermitian manifold if and only if its component manifolds are Levi-Civita Ricci-flat. Thus, this theorem provides us an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

**5. Conclusions**

In this paper, we derived formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of the DWP-Hermitian manifold and proved that if the warped function \(f_1\) and \(f_2\) are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if \((M_1, g)\) and \((M_2, h)\) are Levi-Civita Ricci-flat manifolds. Thus, we gave an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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