Research Article

Statistical Inference of Stress-Strength Reliability of Gompertz Distribution under Type II Censoring

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This paper develops the problem of estimating stress-strength reliability for Gompertz lifetime distribution. First, the maximum likelihood estimation (MLE) and exact and asymptotic confidence intervals for stress-strength reliability are obtained. Then, Bayes estimators under informative and noninformative prior distributions are obtained by using Lindley approximation, Monte Carlo integration, and MCMC. Bayesian credible intervals are constructed under these prior distributions. Also, simulation studies are used to illustrate these inference methods. Finally, a real dataset is analyzed to show the implementation of the proposed methodologies.

1. Introduction

The stress-strength reliability \( R = P(X > Y) \) is an assessment of the reliability of a component based on its strength \( X \) and its stress \( Y \). The idea of a stress-strength reliability was introduced by Birnbaum [1] and spread two years later by Birnbaum and McCarty [2].


Gompertz distribution which was first proposed by Gompertz [8] is one of the most widely used distributions in the fields of survival, lifetime data, mortality tables, computer, biology, sociology, and marketing [9–12]. Some recent studies on the distribution of Gompertz include the following: [13] presented a new and practical generalization of the Gompertz distribution. [14] developed acceptance sampling plans forerverent encoding in which the quality characteristic of the products follows the Topp-Leone Gompertz distribution. An application of Gamma-Gompertz distribution was proposed by [15]. [16] estimated the parameters of a new generalization of Gompertz distribution and investigated the features and application of this new model.


A brief explanation of the type II censoring is given. Let \( x_1, x_2, \ldots, x_{n_1} \) and \( y_1, y_2, \ldots, y_{n_2} \) be independent random samples from \( X \) and \( Y \) random variable, respectively. Suppose the ordered statistics of these samples are \( x_{(1)} < x_{(2)} < \cdots < x_{(n_1)} \)
and \( y_1 < y_2 < \cdots < y_r \).\( x_i \)'s and \( y_i \)'s are collected until \( r_1 \) failures and \( r_2 \) failures occur, respectively (where \( r_1 \leq n_1 \) and \( r_2 \leq n_2 \)).

The rest of the article is organized as follows. In Section 2, we introduce the Gompertz distribution. In Section 3, we obtain the MLE of stress strength reliability (\( R \)). In Section 4, we construct the exact and asymptotic confidence intervals for \( R \). In Section 5, we calculate the Bayes estimator of \( R \) by considering the conjugate informative and Jeffreys noninformative prior distributions. In Section 6, we provide Bayesian credible intervals, including equi-tailed and HPD intervals under the conjugate informative and Jeffreys noninformative prior distributions. In Section 7, the performance of these inference methods is compared by using simulation studies. Finally, Section 8 performs a real data analysis to demonstrate the application of these methods.

2. Gompertz Distribution

Let \( X \sim \text{Gompertz}(\beta_1, \gamma) \) and \( Y \sim \text{Gompertz}(\beta_2, \gamma) \) be two independent random variables. The probability density function (PDF) and cumulative distribution function (CDF) of \( X \) and \( Y \) are given:

\[
f_X(x; \beta_1, \gamma) = \beta_1 e^{\beta_1 x} e^{-e^{\beta_1 x} \gamma}; 
F_X(x; \beta_1, \gamma) = 1 - e^{\beta_1 x} e^{-e^{\beta_1 x} \gamma}, \quad x > 0, \beta_1, \gamma > 0,
\]

\[
f_Y(y; \beta_2, \gamma) = \beta_2 e^{\beta_2 y} e^{-e^{\beta_2 y} \gamma}; 
F_Y(y; \beta_2, \gamma) = 1 - e^{\beta_2 y} e^{-e^{\beta_2 y} \gamma}, \quad y > 0, \beta_2, \gamma > 0.
\]

(1)

The reliability function is calculated as follows:

\[
R = P(X > Y) = \int_{-\infty}^\infty F_Y(u)f_X(u)du = \int_{0}^\infty \left(1 - e^{\beta_1 u} e^{-e^{\beta_1 u} \gamma}\right) \beta_1 e^{\beta_1 u} e^{-e^{\beta_1 u} \gamma} du = \frac{\beta_2}{\beta_1 + \beta_2}.
\]

(2)

3. MLE of \( R \)

Let \( x_{(1)}, x_{(2)}, \cdots, x_{(n)} \) be a type II censored sample from Gompertz(\( \beta_1, \gamma \)) and \( y_{(1)}, y_{(2)}, \cdots, y_{(2)} \) be a type II censored sample from Gompertz(\( \beta_2, \gamma \)). Suppose these two samples are independent. The likelihood function is given by

\[
L(\beta_1, \beta_2, y|x) = \frac{n_1 n_2}{(n_1 - r_1)! (n_2 - r_2)!} \prod_{i=1}^{n_1} f_x(x_{(i)}; \beta_1, \gamma) \left[ S_y(y_{(i)}) \right]^{r_1} \\
\times \prod_{j=1}^{n_2} f_Y(y_{(j)}; \beta_2, \gamma) [S_x(x_{(j)})]^{r_2} \\
= \frac{n_1 n_2}{(n_1 - r_1)! (n_2 - r_2)!} \beta_1^r e^{-e^{\beta_1 r_1} \gamma} \sum_{i=1}^{n_1} x_{(i)} e^{-e^{\beta_1 x_{(i)} - \gamma x_{(i)}}} \\
\times \beta_2^r e^{-e^{\beta_2 r_2} \gamma} \sum_{j=1}^{n_2} y_{(j)} e^{-e^{\beta_2 y_{(j)} - \gamma y_{(j)}}} \\
= \frac{n_1 n_2}{(n_1 - r_1)! (n_2 - r_2)!} \beta_1^r e^{-e^{\beta_1 r_1} \gamma} \sum_{i=1}^{n_1} x_{(i)} + \sum_{j=1}^{n_2} y_{(j)} \\
\times \beta_2^r e^{-e^{\beta_2 r_2} \gamma} e^{-e^{\beta_2 r_2} \gamma}.
\]

(3)

where

\[
b_1^* = \frac{1}{\gamma} \sum_{i=1}^{r_1} e^{\beta_1 x_{(i)}} + (n_1 - r_1) e^{\beta_1 r_1} - n_1, \quad (4)
\]

\[
b_2^* = \frac{1}{\gamma} \sum_{j=1}^{r_2} e^{\beta_2 y_{(j)}} + (n_2 - r_2) e^{\beta_2 r_2} - n_2. \quad (5)
\]

Then, the log-likelihood function is

\[
l(\beta_1, \beta_2, y|x, y) = \log (n_1! n_2!) - \log \left( (n_1 - r_1)! (n_2 - r_2)! \right) \\
+ \beta_1 \sum_{i=1}^{r_1} e^{\beta_1 x_{(i)}} + (n_1 - r_1) e^{\beta_1 r_1} - n_1 \gamma \\
+ \beta_2 \sum_{j=1}^{r_2} e^{\beta_2 y_{(j)}} + (n_2 - r_2) e^{\beta_2 r_2} - n_2 \gamma.
\]

(6)

To obtain the MLE of parameters \( \beta_1, \beta_2 \) and \( \gamma \), it is sufficient to derive the log-likelihood function with respect to parameters \( \beta_1, \beta_2 \) and \( \gamma \) and equal them to zero:

\[
\frac{\partial l}{\partial \beta_1} = -\frac{r_1}{\gamma} - \frac{1}{\gamma} \sum_{i=1}^{r_1} e^{\beta_1 x_{(i)}} + (n_1 - r_1) e^{\beta_1 r_1} - n_1 = 0, \quad (7)
\]

\[
\frac{\partial l}{\partial \beta_2} = -\frac{r_2}{\gamma} - \frac{1}{\gamma} \sum_{j=1}^{r_2} e^{\beta_2 y_{(j)}} + (n_2 - r_2) e^{\beta_2 r_2} - n_2 = 0, \quad (8)
\]

\[
\frac{\partial l}{\partial \gamma} = -\frac{r_1}{\gamma} + \gamma \sum_{i=1}^{r_1} e^{\beta_1 x_{(i)}} + (n_1 - r_1) e^{\beta_1 r_1} - n_1 \\
\times \frac{\gamma}{\gamma} \sum_{j=1}^{r_2} e^{\beta_2 y_{(j)}} + (n_2 - r_2) e^{\beta_2 r_2} - n_2 \gamma = 0. \quad (9)
\]

From Equations (7) and (8), we get

\[
\hat{\beta}_1 = \frac{\gamma r_1}{\left[ \sum_{i=1}^{r_1} e^{\beta_1 x_{(i)}} + (n_1 - r_1) e^{\beta_1 r_1} - n_1 \right]} = \frac{r_1}{b_1^*}, \quad (10)
\]

\[
\hat{\beta}_2 = \frac{\gamma r_2}{\left[ \sum_{j=1}^{r_2} e^{\beta_2 y_{(j)}} + (n_2 - r_2) e^{\beta_2 r_2} - n_2 \right]} = \frac{r_2}{b_2^*}. \quad (11)
\]

Now, by substituting (10) and (11) into (9), the MLE of parameter \( \gamma \) is obtained. Then, to get \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \), we
substitute $\hat{y}$ into Equations (10) and (11). Therefore, the MLE of $R$ is

$$\hat{R} = \frac{\hat{\beta}_2}{\beta_1 + \hat{\beta}_2}.$$  (12)

4. Confidence Interval of $R$

In this section, the exact and asymptotic confidence intervals for $R$ are calculated.

4.1. Exact Confidence Interval. Let $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ be a type II censored sample from Gompertz($\hat{\beta}_1, \hat{\gamma}$). Consider $W_i = \hat{\beta}_1/y(e^{x_{(i)}} - 1)$, $i = 1, 2, \ldots, r_1$, where $w_1 \leq w_2 \leq \ldots \leq w_{r_1}$ is a type II censored dependent sample from the standard exponential distribution (SED). Now, apply the following conversion:

$$W_i' = (n_i - i + 1)(W_i - W_{i-1}).$$  (13)

It can be concluded that $W_1', W_2', \ldots, W_{r_1}' \sim \text{ind SED}$. Therefore,

$$2 \sum_{i=2}^{r_1} W_i' \sim \chi^2(2(r_1-1)).$$  (14)

Similarly, suppose $y_{(1)}, y_{(2)}, \ldots, y_{(r)}$ be a type II censored sample from Gompertz($\hat{\beta}_2, \hat{\gamma}$). Define $M_j = \hat{\beta}_2/y(e^{y_{(j)}} - 1)$, $j = 1, 2, \ldots, r_2$, where $m_1 \leq m_2 \leq \ldots \leq m_{r_2}$ is a type II censored dependent sample from the SED. Now, apply the following transformation:

$$M_j' = (n_j - j + 1)(M_j - M_{j-1}).$$  (15)

It results that $M_1', M_2', \ldots, M_{r_2}' \sim \text{ind SED}$. So,

$$2 \sum_{j=2}^{r_2} M_j' \sim \chi^2(2(r_2-1)).$$  (16)

Based on the independence of $\sum_{i=2}^{r_1} W_i'$ and $\sum_{j=2}^{r_2} M_j'$ can be written

$$F_0 = \frac{(r_2 - 1)\sum_{i=2}^{r_1} W_i'}{(r_1 - 1)\sum_{j=2}^{r_2} M_j'} \sim F(2(r_1-1), 2(r_2-1)).$$  (17)

Then, confidence interval for $R$ is

$$\left(1 + \frac{Q_1}{Q_2}, \frac{1}{1 + \frac{Q_2}{Q_1}F_{1,2}(2(r_1-1), 2(r_2-1))}\right)^{-1},$$

where

$$Q_1 = \frac{r_2 - 1}{\gamma} \sum_{i=2}^{r_1} (n_i - i + 1)(e^{y_{(i)}} - e^{x_{(i-1)}}),$$

$$Q_2 = \frac{r_1 - 1}{\gamma} \sum_{j=2}^{r_2} (n_j - j + 1)(e^{y_{(j)}} - e^{x_{(j-1)}}).$$  (19)

4.2. Asymptotic Confidence Interval. In this section, the asymptotic confidence interval for $R$ is calculated using Wald statistics. Based on Wald statistics, we have

$$W = \frac{(1/r_1 + 1/r_2)^{-1/2}(\hat{R} - R)}{\hat{\eta}} \sim N(0, 1),$$  (20)

where $\hat{\eta} = \text{Var}(R) = \hat{\beta}_1\hat{\beta}_2(\hat{\beta}_1 + \hat{\beta}_2)^2$.

Theorem 1. Let $r_1 \longrightarrow \infty$ and $r_2 \longrightarrow \infty$, then

$$\left[2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right]^{1/2} \left(\hat{\beta}_1 - \beta_1, \hat{\beta}_2 - \beta_2\right) \sim N(0, \kappa^2),$$  (21)

where $\kappa^2 = 2(\hat{\beta}_1\hat{\beta}_2)^2/(\beta_1 + \beta_2)^2$.

Proof. Given that the maximum likelihood estimator is asymptotically normal [21], when $r_1 \longrightarrow \infty$ and $r_2 \longrightarrow \infty$, then

$$\left(\sqrt{r_1} \left(\hat{\beta}_1 - \beta_1\right), \sqrt{r_2} \left(\hat{\beta}_2 - \beta_2\right)\right) \sim N(0, \omega),$$  (22)

where

$$\omega = \begin{bmatrix} \beta_1^2 & 0 \\ 0 & \beta_2^2 \end{bmatrix}. $$  (23)

Define $h(\hat{\beta}_1, \hat{\beta}_2) = \hat{\beta}_1/\beta_1 + \hat{\beta}_2$. According to Taylor expansion $h(\hat{\beta}_1, \hat{\beta}_2)$ around $\beta_1$ and $\beta_2$, we have

$$\hat{R} = h(\hat{\beta}_1, \hat{\beta}_2) = h(\beta_1, \beta_2) + \nabla h(\beta_1, \beta_2)^T \left[\begin{array}{c} \hat{\beta}_1 - \beta_1 \\ \hat{\beta}_2 - \beta_2 \end{array}\right],$$

$$= R + \left[\begin{array}{c} -\beta_2 \\ (\beta_1 + \beta_2)^2 \end{array}\right] \left[\begin{array}{c} \hat{\beta}_1 - \beta_1 \\ \hat{\beta}_2 - \beta_2 \end{array}\right] + O_{\text{as}},$$  (24)

where the remaining sentences in the following relation apply:

$$\kappa_1 = O_p\left(\left(\frac{\hat{\beta}_1 - \beta_1}{\beta_1} + \frac{\hat{\beta}_2 - \beta_2}{\beta_2}\right)^2\right).$$  (25)
Based on (22) and (24), when \( r_1 \to \infty \) and \( r_2 \to \infty \), then

\[
E \left[ \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \right] = 0, \tag{26}
\]

\[
\text{Var} \left[ \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \right] = 2 \left( \frac{1}{r_1} + \frac{1}{r_2} \right)^{-1} \text{Var}(\bar{R} - R) = 2 \left( \frac{1}{r_1} + \frac{1}{r_2} \right)^{-1} S^2, \tag{27}
\]

where

\[
S^2 = \text{Var} \left( \begin{pmatrix}
-\beta_2 \\
(\beta_1 + \beta_2)^2
\end{pmatrix}
\right) \cdot \begin{pmatrix}
\hat{\beta}_1 - \beta_1 \\
\hat{\beta}_2 - \beta_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\beta_2 \\
(\beta_1 + \beta_2)^2
\end{pmatrix} \cdot \begin{pmatrix}
\beta_1 \\
(\beta_1 + \beta_2)^2
\end{pmatrix} \cdot \text{Var} \left[ \begin{pmatrix}
\hat{\beta}_1 - \beta_1 \\
\hat{\beta}_2 - \beta_2
\end{pmatrix}
\right]
\]

\[
= \begin{pmatrix}
-\beta_2 \\
(\beta_1 + \beta_2)^2
\end{pmatrix} \cdot \begin{pmatrix}
\beta_1 \\
(\beta_1 + \beta_2)^2
\end{pmatrix} \cdot \begin{pmatrix}
\beta_1 \\
0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\beta_1 \\
0
\end{pmatrix} \cdot \begin{pmatrix}
\beta_1 \\
0
\end{pmatrix} = \beta_1^2.
\]

\[
\text{Var} \left[ \left( \frac{1}{r_1} + \frac{1}{r_2} \right)^{-1/2} (\bar{R} - R) \right] = \kappa^2. \tag{29}
\]

Equation (26) holds because according to the property of MLE \( \hat{R} \to R \).

**Corollary 2.** A 100(1 - \( \xi \))% asymptotic confidence interval for \( R \) is

\[
\left[ \hat{R} - \eta \frac{\sqrt{2}}{\hat{R}} \right] \to N(0, 1). \tag{30}
\]

where \( \hat{R} = \hat{\beta}_1 \hat{\beta}_2 \hat{\beta}_1^2 \).

**Proof.** Define \( \eta = (\hat{\beta}_1 \hat{\beta}_2)^2 / (\hat{\beta}_1 + \hat{\beta}_2)^4 \) and \( \eta^2 = (\hat{\beta}_1 \hat{\beta}_2)^2 / (\hat{\beta}_1 + \hat{\beta}_2)^4 \). According to the asymptotic property of the MLE, \( \hat{\eta}/\eta \) tends to 1 in probability. On the other hand, according to Theorem 1,

\[
\frac{(1/r_1 + 1/r_2)^{-1/2} (\bar{R} - R)}{\eta} \to N(0, 1). \tag{31}
\]

Therefore, according to the Slutsky theorem, we have

\[
\frac{(1/r_1 + 1/r_2)^{-1/2} (\bar{R} - R)}{\eta/\eta} \to N(0, 1). \tag{32}
\]

Thus,

\[
P \left( \frac{(1/r_1 + 1/r_2)^{-1/2} (\bar{R} - R)}{\eta} \right) \leq R \leq \frac{(1/r_1 + 1/r_2)^{-1/2} (\bar{R} - R)}{\eta} \right) = 1 - \xi. \tag{33}
\]

\[
\square
\]

### 5. Bayesian Estimation of \( R \)

Bayes [22] and Laplace [23] found that the uncertainty about the parameters of a model, which we represent with \( \Theta \), could be modeled on \( \Theta \) through a probability distribution such as \( \pi(\Theta) \), called the prior distribution. With this approach, the inference is based on the conditional distribution \( \Theta | x \), \( \pi(\Theta | x) \). This conditional distribution is called the posterior distribution. In this section, Bayesian estimation is obtained by using the conjugate informative and Jeffreys noninformative prior distributions.

#### 5.1. Conjugate Informative Prior

Let \( \beta_1 \sim \text{Gamma}(\alpha_1, \eta_1) \) and \( \beta_2 \sim \text{Gamma}(\alpha_2, \eta_2) \) and are independent. The PDF of these priors is as follows:

\[
\pi(\beta_1) = \frac{\eta_1^{\alpha_1}}{I(\alpha_1)} \beta_1^{\alpha_1 - 1} e^{-\eta_1 \beta_1}, \alpha_1, \eta_1 > 0, \tag{34}
\]

\[
\pi(\beta_2) = \frac{\eta_2^{\alpha_2}}{I(\alpha_2)} \beta_2^{\alpha_2 - 1} e^{-\eta_2 \beta_2}, \alpha_2, \eta_2 > 0.
\]
where $\alpha_1, \eta_1, \alpha_2$ and $\eta_2$ are the hyperparameters. Then, the joint prior possibility distribution is

$$
\pi(\beta_1, \beta_2) = \pi(\beta_1) \pi(\beta_2) = \frac{\eta_1^{\alpha_1}}{\Gamma(\alpha_1)} \beta_1^{\alpha_1-1} e^{-\eta_1 \beta_1} \cdot \frac{\eta_2^{\alpha_2}}{\Gamma(\alpha_2)} \beta_2^{\alpha_2-1} e^{-\eta_2 \beta_2}.
$$

(35)

The posterior probability distribution is calculated as follows:

$$
\pi(\beta_1, \beta_2, \gamma | x, y) = \frac{L(\beta_1, \beta_2; x, y) \pi(\beta_1, \beta_2)}{\int_0^\infty \int_0^\infty L(\beta_1, \beta_2; x, y) \pi(\beta_1, \beta_2) d\beta_1 d\beta_2} \propto \beta_1^{\alpha_1+r_1-1} e^{-\beta_1(\eta_1+b_1')} \beta_2^{\alpha_2+r_2-1} e^{-\beta_2(\eta_2+b_2')}.
$$

(36)

where $b_1'$ and $b_2'$ were given in (4) and (5), respectively. For $i = 1, 2$, we can write

$$
\beta_i | x, y \sim \text{Gamma}(\alpha_i^*, \eta_i^*),
$$

(37)

where $\alpha_i^* = \alpha_i + r_i$ and $\eta_i^* = \eta_i + b_i'$. [24] proposed $\alpha_1 = \eta_1 = \alpha_2 = \eta_2 = 0.001$, and Robert [25] suggested an empirical Bayesian approach to determining the values of hyperparameters. According to the approach presented by Robert, to get $\alpha_1$ and $\eta_1$, we maximize the following function:

$$
m(x | \alpha_1, \eta_1) = \int_0^\infty \int_0^\infty f(x | \alpha_1, \eta_1) \pi(\beta_1 | \alpha_1, \eta_1) \pi(\beta_2 | \alpha_2, \eta_2) d\beta_1 d\beta_2
$$

$$
= \frac{n_1 \eta_1^{\alpha_1} e^{\sum_i (x_i)} \Gamma(\alpha_1)}{\Gamma(\alpha_1) (n_1 - r_1)!} \int_0^\infty \beta_1^{\alpha_1+r_1-1} e^{-\beta_1} d\beta_1
$$

$$
= \frac{n_1 \eta_1^{\alpha_1} \Gamma(\alpha_1+r_1) e^{\sum_i (x_i)}}{\Gamma(\alpha_1) (n_1 - r_1)! (\eta_1 + r_1)}.
$$

(38)

We have

$$
M(x | \alpha_1, \eta_1) = \log [m(x | \alpha_1, \eta_1)]
$$

$$
= \log \left[ \frac{n_1!}{(n_1 - r_1)!} \right] + \alpha_1 \log \eta_1 + \log \Gamma(\alpha_1 + r_1)
$$

$$
- \log \Gamma(\alpha_1) - (\alpha_1 + r_1) \log (\eta_1 + r_1')
$$

$$
+ y \sum_{i=1}^{r_1} x_i.
$$

(39)

$\alpha_1$ and $\eta_1$ are obtained by solving the following equations:

$$
\frac{\partial M}{\partial \alpha_1} = \log \eta_1 + \psi(r_1 + \alpha_1) - \psi(\alpha_1) - \log (\eta_1 + r_1') = 0,
$$

(40)

where $\psi$ shows the digamma function. By solving Equation (41), $\eta_1$ is

$$
\eta_1 = \frac{\alpha_1 b_1'}{r_1}.
$$

(42)

By substituting (42) into (40), we get

$$
0 = \log \left( \frac{\eta_1}{\eta_1 + b_1'} \right) + \psi(r_1 + \alpha_1) - \psi(\alpha_1)
$$

$$
= \log \left( \frac{\alpha_1}{r_1 + \alpha_1} \right) + \psi(r_1 + \alpha_1) - \psi(\alpha_1)
$$

$$
= \log \left( \frac{\alpha_1}{r_1 + \alpha_1} \right) + \sum_{k=0}^{r_1-1} \frac{1}{\alpha_1 + k}.
$$

(43)

Similarly, this method can be repeated to calculate $\alpha_2$ and $\eta_2$. So, $\eta_2$ is as follows:

$$
\eta_2 = \frac{\alpha_2 b_2'}{r_2}.
$$

(44)

Also, $\alpha_2$ is obtained by solving the following equation:

$$
0 = \log \left( \frac{\alpha_2}{r_2 + \alpha_2} \right) + \sum_{k=0}^{r_2-1} \frac{1}{\alpha_2 + k}.
$$

(45)

Abravesh et al. [4] showed that Equations (43) and (45) have no answer and set $\alpha_1 = \alpha_2 = 1$ to solve this problem. Then, $\eta_1 = b_1'/r_1$ and $\eta_2 = b_2'/r_2$.

5.2 Posterior Distribution of $R$. To calculate the posterior distribution of $R$, we have the following transformations:

$$
R = \frac{\beta_1}{\beta_1 + \beta_2},
$$

(46)

$$
V = \beta_2.
$$

(47)
Transformations (46) and (47) are equivalent to \( \beta_1 = V(1 - R/R) \) and \( \beta_2 = V \). The posterior distribution of \( R \) and \( V \) can be calculated by the following formula:

\[
\pi(r, v|x, y) = |J| \cdot \pi \left( v \left( \frac{1 - r}{r} \right), v|x, y \right).
\]

(48)

In the above formula, \( J \) is called Jacobin and is calculated as follows:

\[
|J| = \left| \text{det} \begin{bmatrix} \frac{\partial \beta_1}{\partial v} & \frac{\partial \beta_2}{\partial v} \\ \frac{\partial \beta_1}{\partial r} & \frac{\partial \beta_2}{\partial r} \end{bmatrix} \right| = \text{det} \left| \begin{bmatrix} \frac{1 - r}{r} & 1 \\ \frac{v}{r^2} & 0 \end{bmatrix} \right| = \frac{v}{r^2}.
\]

(49)

The marginal distribution of \( R \) is calculated from the joint distribution in (48) as

\[
\pi_i(r, v|x, y) = \int_0^\infty \pi_i(r, v|x, y) \, du
\]

\[
= \int_0^\infty \frac{v^{\eta_1} \eta_2^{\eta_2} \tau_1^{\gamma_1-1} \gamma_2^{\gamma_2-1} e^{-\gamma_2 (1 - r)} e^{\gamma_2 (1 - r) \gamma_2} du}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \cdot \frac{(1 - r)^{\alpha_1 - 1} \alpha_2 \alpha_2}{(1 - r)^2} \Gamma(\alpha_1)(1 - r) + \eta_2^{\gamma_2} \gamma_2, 0 \leq r \leq 1.
\]

(50)

5.3. Jeffreys Noninformative Prior. In this section, by using Jeffreys noninformative prior [26], the Bayesian estimation of \( R \) is obtained. The Jeffreys prior is as follows:

\[
\pi_j(\beta_1, \beta_2) \propto \sqrt{\text{det} [I(\beta_1, \beta_2)]},
\]

(51)

where

\[
I(\beta_1, \beta_2) = \begin{bmatrix}
E \left( \frac{\partial^2 I}{\partial \beta_1^2} \right) & E \left( \frac{\partial^2 I}{\partial \beta_1 \partial \beta_2} \right) \\
E \left( \frac{\partial^2 I}{\partial \beta_2 \partial \beta_1} \right) & E \left( \frac{\partial^2 I}{\partial \beta_2^2} \right)
\end{bmatrix}.
\]

(52)

Considering Jeffreys prior \( \pi_j \propto 1/\beta_1 \beta_2 \), the marginal posterior distribution is given by

\[
\beta_i | x, y \sim \Gamma \left( \alpha_1, \beta_1 \right), i = 1, 2.
\]

(53)

Similar to the process in Subsection 5.2, the marginal posterior distribution can be obtained:

\[
\pi_j(r|x, y) = \frac{\tilde{b}_1 \tilde{b}_2 \Gamma(r_1) \Gamma(r_2)}{\Gamma(r_1) \Gamma(r_2)} \cdot \frac{(1 - r)^{\alpha_1 - 1}}{r^2 \left[ \tilde{b}_1 (1 - r) + \tilde{b}_2 \right] r_1^{r_1} r_2^{r_2}}, 0 \leq r \leq 1.
\]

(54)

5.4. Lindley Approximation. Lindley [27] proposed a method for approximating the ratio of integrals. The Lindley approximation of \( R \) can be calculated using the following formula:

\[
E(R|\text{data}) = \int \frac{R(\tau) e^{\mu(1/R) \tau} \, d\tau}{\int e^{\mu(1/R) \tau} \, d\tau} \approx R(\hat{\tau}) + \frac{1}{2} \left( \sum_{i,j} \rho_{ij} + 2 \rho_{ii} \Pi_i \right) \sigma_{ij} + \sum_{i,j,k} l_{ijk} \sigma_{ij} \sigma_{kl},
\]

(55)

where \( \Sigma = [\sigma_{ij}] = [-l_{ij}(\hat{\tau})]^{-1}, \hat{\tau} \) is MLE of \( \tau \) and

\[
\rho_i = \frac{\partial R}{\partial \tau} \bigg|_{\tau = \hat{\tau}}, \rho_{ij} = \frac{\partial^2 R}{\partial \tau^2} \bigg|_{\tau = \hat{\tau}}, \Pi_i = \frac{\partial \Pi}{\partial \tau} \bigg|_{\tau = \hat{\tau}}, l_{ijk} = \frac{\partial^3 \Pi}{\partial \tau^3} \bigg|_{\tau = \hat{\tau}}.
\]

(56)

Here, \( l \) is log-likelihood function and \( \Pi \) is log-prior distribution.

5.4.1. Informative Prior. Based on the prior distribution (35) and \( R = \beta_2 / \beta_1 + \beta_2 \), we obtain

\[
\Pi = \log \pi(\beta_1, \beta_2) = \frac{\alpha_1 \log \eta_1 - \log \Gamma(\alpha_1) + (\alpha_1 - 1) \log \beta_1 - \eta_1 \beta_1 + \alpha_2 \log \eta_2 - \log \Gamma(\alpha_2) + (\alpha_2 - 1) \log \beta_2 - \eta_2 \beta_2}.
\]

(57)

So,

\[
\Pi_1 = \frac{\partial \Pi}{\partial \beta_1} = \frac{\alpha_1 - 1}{\beta_1} - \eta_1,
\]

\[
\Pi_2 = \frac{\partial \Pi}{\partial \beta_2} = \frac{\alpha_2 - 1}{\beta_2} - \eta_2,
\]

\[
\rho_1 = \frac{\partial R}{\partial \beta_1} = \frac{-\beta_2}{(\beta_1 + \beta_2)^2},
\]

\[
\rho_2 = \frac{\partial R}{\partial \beta_2} = \frac{\beta_1}{(\beta_1 + \beta_2)^2},
\]

\[
\rho_{11} = \frac{\partial^2 R}{\partial \beta_1^2} = \frac{2\beta_2}{(\beta_1 + \beta_2)^3},
\]

\[
\rho_{22} = \frac{\partial^2 R}{\partial \beta_2^2} = \frac{-2\beta_1}{(\beta_1 + \beta_2)^3},
\]

\[
\rho_{12} = \frac{\partial^2 R}{\partial \beta_1 \partial \beta_2} = \frac{\beta_2 - \beta_1}{(\beta_1 + \beta_2)^3}.
\]

(58)

Inverse of the Hessian matrix is given by

\[
\Sigma = \begin{bmatrix} \rho_1^2 & 0 \\ 0 & \rho_2^2 \end{bmatrix}.
\]

(59)
1. Generate a MCMC sample \( \{R_i, i = 1, 2, \cdots, k\} \) from \( \pi(R|x, y) \);
2. Sort \( \{R_i, i = 1, 2, \cdots, l\} \) and suppose \( R_{(1)} \leq R_{(2)} \leq \cdots \leq R_{(l)} \);
3. Consider \( C_j = (R_j, R_{j(l-1)}) \), \( j = 1, 2, \cdots, l - \lfloor 1 - \xi \rfloor \);
4. Consider \( W_j = R_{(j)} - R_j \);
5. Select \( j' \) such that \( W_{j'} = \min \{ W_j, j = 1, 2, \cdots, l - \lfloor 1 - \xi \rfloor \} \);
6. Introduce \( C_{j'} \) as a 100(1 - \( \xi \))% HPD interval for \( R \).

Algorithm 1: Chen-Shao algorithm for \( R \).

Finally, the Lindley approximation of the Bayes estimator of \( R \) is

\[
R_{\text{bayes}} = R(\bar{\beta}_1, \bar{\beta}_2) + \frac{1}{2} \left( \sum_{i=1}^{k} (\rho_i + 2 \rho_i \Pi_i) \sigma_{ij} + \sum_{j=1}^{N} l_{ij} \rho_i \sigma_{ij} \right) \]

\[
= R(\bar{\beta}_1, \bar{\beta}_2) + \frac{1}{2} \left( \sum_{i=1}^{k} (\rho_i + 2 \rho_i \Pi_i) \sigma_{ij} + \sum_{j=1}^{N} l_{ij} \rho_i \sigma_{ij} \right) \]

\[
+ \frac{\bar{\beta}_1^2}{\bar{\tau}_1} \left( \frac{-\bar{\beta}_1}{\bar{\beta}_1 + \bar{\beta}_2} + \frac{\bar{\beta}_1}{\bar{\beta}_1 + \bar{\beta}_2} \right) + \frac{\bar{\beta}_2^2}{\bar{\tau}_2} \left( \frac{\bar{\beta}_2}{\bar{\beta}_1 + \bar{\beta}_2} \right) \]

\[
+ \frac{\bar{\beta}_1 \bar{\beta}_2}{\bar{\beta}_1 + \bar{\beta}_2} \left( \frac{1}{\bar{\tau}_2} - \frac{1}{\bar{\tau}_1} \right). \tag{60} \]

5.4.2. Noninformative Prior. Under Jeffreys prior \((\pi, /\beta_1, \beta_2) = 1 / \beta_1 \beta_2 \), we have

\[
\Pi = - \log \beta_1 - \log \beta_2, \tag{62} \]

\[
\Pi_1 = \frac{1}{\beta_1}, \Pi_2 = \frac{1}{\beta_2}. \]

Therefore, the Bayes estimator of \( R \) using the Lindley approximation is

\[
R_{\text{bayes}} = R(\bar{\beta}_1, \bar{\beta}_2) + \frac{1}{2} \left( \sum_{i=1}^{k} (\rho_i + 2 \rho_i \Pi_i) \sigma_{ij} + \sum_{j=1}^{N} l_{ij} \rho_i \sigma_{ij} \right) \]

\[
= R(\bar{\beta}_1, \bar{\beta}_2) + \frac{1}{2} \left( \sum_{i=1}^{k} (\rho_i + 2 \rho_i \Pi_i) \sigma_{ij} + \sum_{j=1}^{N} l_{ij} \rho_i \sigma_{ij} \right) \]

\[
+ \frac{\bar{\beta}_1^2}{\bar{\tau}_1} \left( \frac{-\bar{\beta}_1}{\bar{\beta}_1 + \bar{\beta}_2} + \frac{\bar{\beta}_1}{\bar{\beta}_1 + \bar{\beta}_2} \right) + \frac{\bar{\beta}_2^2}{\bar{\tau}_2} \left( \frac{\bar{\beta}_2}{\bar{\beta}_1 + \bar{\beta}_2} \right) \]

\[
+ \frac{\bar{\beta}_1 \bar{\beta}_2}{\bar{\beta}_1 + \bar{\beta}_2} \left( \frac{1}{\bar{\tau}_2} - \frac{1}{\bar{\tau}_1} \right). \tag{63} \]

5.5. Monte Carlo Integration. The Monte Carlo integration method was introduced by Metropolis and Ulam [28] and Neumann [29]. Let \( \theta_1, \theta_2, \cdots, \theta_k \) be a random sample from posterior density \( \pi(\theta|\text{observations}) \). In this case, according to the strong law of large numbers, for large \( k \), an approximation for the expected values of posterior is equal to

\[
E(\xi(\theta)|\text{observation}) = \frac{1}{k} \sum_{i=1}^{k} \xi(\theta_i). \tag{64} \]

This method was very simple and does not involve complicated calculations. The only problem this method may have is generating a random sample of posterior density.

Now, the Bayes estimator of \( R \) is obtained using this method. Let \( \{\beta^{(1)}_1, \beta^{(2)}_2, \cdots, \beta^{(N)}_1 \} \) be the random sample from \( \pi(\beta|x, y) \), \( i = 1, 2 \), then for large \( N \),

\[
E(R|x, y) \approx \frac{1}{N} \sum_{j=1}^{N} R(\beta^{(j)}_1, \beta^{(j)}_2) = \frac{1}{N} \sum_{j=1}^{N} \left( \beta^{(j)}_2 \right). \tag{65} \]

5.5.1. Informative Prior. To calculate the Bayes estimator of \( R \) under the conjugate prior (35), we assume \( \beta^{(1)}_1, \beta^{(2)}_1, \cdots, \beta^{(N)}_1 \sim \text{id} \text{Gamma}(\alpha_i, r_i, \eta_i + b_i), i = 1, 2 \). So,

\[
R_{MC}^I = \frac{1}{N} \sum_{j=1}^{N} R(\beta^{(j)}_1, \beta^{(j)}_1) = \frac{1}{N} \sum_{j=1}^{N} \left( \beta^{(j)}_2 \right). \tag{66} \]

5.5.2. Noninformative Prior. Under the Jeffreys prior, we consider \( \beta^{(1)}_1, \beta^{(2)}_1, \cdots, \beta^{(N)}_1 \sim \text{id} \text{Gamma}(r_i, b_i), i = 1, 2 \). Then,

\[
R_{MC}^I = \frac{1}{N} \sum_{j=1}^{N} R(\beta^{(j)}_1, \beta^{(j)}_1) = \frac{1}{N} \sum_{j=1}^{N} \left( \beta^{(j)}_2 \right). \tag{67} \]

5.6. MCMC. To solve the stated problem of the Monte Carlo integration method, a more general method is used to generate approximate random variables from the posterior distribution, called the Markov chain Monte Carlo (MCMC) method [30]. The Metropolis-Hastings algorithm is used to create Markov chains with a given distribution. The application of
the MCMC method, the following integral is approximated:

$$
E(R|x, y) = \int_0^1 \pi_0(r|x, y) \, dr.
$$

(68)

### 5.6.1 Informative Prior

Let $r_1, r_2, \ldots, r_J$ be an ergodic MCMC sample from $\pi(r|x, y)$, we have

$$
E(R|x, y) = \frac{1}{J} \sum_{j=1}^{J} r_j.
$$

(69)

Table 1: The bias and MSE values (MSE in parentheses) of estimators for $\beta_1 = 1, \beta_2 = 1, \gamma = 1, R = 1/2$.

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>Prior</th>
<th>MLE</th>
<th>Lindley</th>
<th>MC</th>
<th>MCMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>Conjugate</td>
<td>0.011339 (0.015528)</td>
<td>0.010107 (0.012152)</td>
<td>0.010295 (0.012573)</td>
<td>0.006669 (0.006495)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Jeffreys</td>
<td>—</td>
<td>0.010822 (0.014271)</td>
<td>0.011231 (0.014295)</td>
<td>0.007375 (0.007236)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: The bias and MSE values (MSE in parentheses) of estimators for $\beta_1 = 1, \beta_2 = 2, \gamma = 1, R = 2/3$.

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>Prior</th>
<th>MLE</th>
<th>Lindley</th>
<th>MC</th>
<th>MCMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>Conjugate</td>
<td>0.009500 (0.012304)</td>
<td>0.015630 (0.010022)</td>
<td>0.012485 (0.010080)</td>
<td>-0.053973 (0.008464)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Jeffreys</td>
<td>—</td>
<td>0.002976 (0.011512)</td>
<td>0.003521 (0.011480)</td>
<td>-0.045037 (0.008308)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

this algorithm in mechanical physics was first developed by Metropolis et al. [30]. A few years later, Hastings and Keith generalized the algorithm in more statistical detail [31]. Using the MCMC method, the following integral is approximated:

$$
E(R|x, y) = \frac{1}{J} \sum_{j=1}^{J} r_j.
$$

(69)

5.6.1 Informative Prior. Considering the conjugate prior
Similarly, using the Jeﬀreys prior and the posterior distribution (50) for \( \beta \) and \( \gamma \), by generating an ergodic sample \( r_I \) from \( \pi_r \) using the Metropolis-Hastings algorithm, the R Bayesian estimator is as follows:

\[
E(R|\pi, x) = \frac{1}{T} \sum_{t=1}^{T} r_{jt}.
\]

5.6.2. Noninformative Prior. Similarly, using the Jeﬀreys prior and the posterior distribution (54) for \( \beta_1 \) and \( \beta_2 \), we generate an ergodic sample \( r'_1, r'_2, \ldots, r'_T \) from \( \pi_r \), using the Metropolis-Hastings algorithm; the R Bayesian estimator is given by

\[
E(R|x, y) = \frac{1}{T} \sum_{t=1}^{T} r'_{jt}.
\]

6. Bayesian Credible Interval

In fact, the Bayesian view oﬀers conﬁdence interval that are more realistic than its classic counterpart. We start this section with two deﬁnitions.

**Deﬁnition 3.** Set \( C_x \) is called a \( \xi \)-credible region whenever

\[
P_\pi(\theta \in C_x|\pi) \geq 1 - \xi,
\]

where \( P_\pi \) is the \( \theta \) posterior probability function of condition \( x \).

**Deﬁnition 4.** The \( \xi \)-credible region \( C_x \) is called a region with the highest posterior density (HPD) whenever it can be written as follows:

\[
C_x(\nu) = \{\theta : \pi(\theta|x) \geq \nu\},
\]

where \( \nu \) is the largest ﬁxed number that applies to

\[
P_\pi(\theta \in C_x(\nu)) \geq 1 - \xi.
\]

Although the 100(1 − \( \xi \))% HPD interval is an optimal answer among the \( \xi \)-credible intervals, in some cases, it is not easy to calculate directly, and approximate methods must be used to obtain it [32]. It is usually easier to calculate approximate intervals with equal tails than HPD interval.

### Table 3: The bias and MSE values (MSE in parentheses) of estimators for \( \beta_1 = 2, \beta_2 = 1, \gamma = 1, R = 1/3 \).

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>Prior</th>
<th>MLE</th>
<th>Lindley</th>
<th>MC</th>
<th>MCMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>Conjugate</td>
<td>-0.006236 (0.013546)</td>
<td>0.017747 (0.011277)</td>
<td>0.014489 (0.011353)</td>
<td>0.055189 (0.009427)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>Jeffreys</td>
<td>—</td>
<td>0.000054 (0.012776)</td>
<td>-0.000131 (0.012800)</td>
<td>0.046938 (0.009554)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>Conjugate</td>
<td>-0.020955 (0.020210)</td>
<td>0.001998 (0.015037)</td>
<td>0.002221 (0.015536)</td>
<td>0.049048 (0.010972)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>Conjugate</td>
<td>-0.036934 (0.018266)</td>
<td>-0.006422 (0.014205)</td>
<td>-0.010386 (0.014360)</td>
<td>0.039886 (0.010667)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1: Select \( n_1, n_2, \beta_1, \beta_2, r_1, r_2, \) and \( y \) values;
2: Generate \( (\tau_1, \tau_2, \ldots, \tau_n) \) and \( (\delta_1, \delta_2, \ldots, \delta_n) \) from the SED;
3: Consider \( x_i = (1/\gamma)(\gamma/\beta_i) x + 1, i = 1, 2, \ldots, n_1 \) and \( y_i = (1/\gamma)(\gamma/\beta_i) y + 1, i = 1, 2, \ldots, n_2 \);
4: Sort \( x, y \) and suppose \( x_{(1)} < x_{(2)} < \cdots < x_{(n_1)} \) and \( y_{(1)} < y_{(2)} < \cdots < y_{(n_2)} \);
5: Report \( (x_{(1)}, x_{(2)}, \ldots, x_{(n_1)}) \) and \( (y_{(1)}, y_{(2)}, \ldots, y_{(n_2)}) \) as the type II censored samples from Gompertz(\( \beta_1, \gamma \)) and Gompertz(\( \beta_2, \gamma \)), respectively.

**Algorithm 2:** The type II censored sample generation algorithm.
Chen and Shao [34] have proposed an algorithm to construct an approximate HPD interval. We obtain confidence intervals of equal tails and HPD.

### 6.1. An Equi-Tailed Bayesian Credible Interval

In this subsection, confidence intervals with equal tails are calculated under the conjugate and Jeffreys prior distributions.

#### Table 4: L and CP of confidence interval for \( R = 1/2, \beta_1 = 1, \beta_2 = 1 \) and \( \gamma = 1 \).

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>Asymptotic</th>
<th>Exact</th>
<th>Credible equi-tailed Conjugate</th>
<th>Credible equi-tailed Jeffreys</th>
<th>HPD Conjugate</th>
<th>HPD Jeffreys</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>0.410623</td>
<td>0.417916</td>
<td>0.328047</td>
<td>0.337885</td>
<td>0.358535</td>
<td>0.400056</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0.435411</td>
<td>0.447157</td>
<td>0.342757</td>
<td>0.354050</td>
<td>0.406417</td>
<td>0.423652</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>0.443006</td>
<td>0.454225</td>
<td>0.347341</td>
<td>0.358796</td>
<td>0.412829</td>
<td>0.431075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>0.344899</td>
<td>0.352722</td>
<td>0.287869</td>
<td>0.295543</td>
<td>0.327492</td>
<td>0.338830</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>0.35745</td>
<td>0.369685</td>
<td>0.297746</td>
<td>0.306382</td>
<td>0.340650</td>
<td>0.353603</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>27</td>
<td>0.373987</td>
<td>0.386498</td>
<td>0.306679</td>
<td>0.316532</td>
<td>0.353156</td>
<td>0.367765</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Table 5: L and CP of confidence interval for \( R = 2/3, \beta_1 = 1, \beta_2 = 2 \) and \( \gamma = 1 \).

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>Asymptotic</th>
<th>Exact</th>
<th>Credible equi-tailed Conjugate</th>
<th>Credible equi-tailed Jeffreys</th>
<th>HPD Conjugate</th>
<th>HPD Jeffreys</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>0.358133</td>
<td>0.371438</td>
<td>0.311195</td>
<td>0.317441</td>
<td>0.349515</td>
<td>0.355055</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0.370774</td>
<td>0.390539</td>
<td>0.324441</td>
<td>0.332062</td>
<td>0.365106</td>
<td>0.370252</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>0.383027</td>
<td>0.403275</td>
<td>0.329960</td>
<td>0.338316</td>
<td>0.374478</td>
<td>0.380763</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>0.248407</td>
<td>0.249667</td>
<td>0.223203</td>
<td>0.226709</td>
<td>0.242077</td>
<td>0.245737</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>0.257244</td>
<td>0.258682</td>
<td>0.230090</td>
<td>0.233250</td>
<td>0.250227</td>
<td>0.254282</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>27</td>
<td>0.261087</td>
<td>0.262638</td>
<td>0.233356</td>
<td>0.236327</td>
<td>0.253800</td>
<td>0.258014</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[33]. Chen and Shao [34] have proposed an algorithm to construct an approximate HPD interval. We obtain confidence intervals of equal tails and HPD.

6.1. An Equi-Tailed Bayesian Credible Interval. In this subsection, confidence intervals with equal tails are calculated under the conjugate and Jeffreys prior distributions.
6.1.1. Informative Prior. We use the prior distribution (35).

The posterior distributions of $\beta_1$ and $\beta_2$ are as follows:

$$
\beta_i | \mathbf{x}, \mathbf{y} \sim \text{Gamma}(\alpha_i^*, \eta_i^*), \; i = 1, 2.
$$

(75)

Therefore, one can conclude

$$
2\eta_i^* \beta_i | \mathbf{x}, \mathbf{y} \sim \chi^2(2\alpha_i^*), \forall i = 1, 2.
$$

(76)

Given that the posterior distributions of $\beta_1$ and $\beta_2$ are independent, we get

$$
\frac{\eta_i^* \alpha_i^*}{\alpha_i^* \eta_i^*} \beta_i | \mathbf{x}, \mathbf{y} \sim F(2\alpha_i^*, 2\alpha_i^*). \quad (77)
$$

Thus, a $100(1 - \xi)$% Bayesian credible interval with equal tails for $R$ under the conjugate prior is

$$
P \left[ \frac{1}{1 + \alpha_i^* \eta_i^* \beta_i^* F_{1 - \xi/2}(2\alpha_i^*, 2\alpha_i^*)} < R < \frac{1}{1 + \alpha_i^* \eta_i^* \beta_i^* F_{\xi/2}(2\alpha_i^*, 2\alpha_i^*)} \right] = 1 - \xi.
$$

(78)

6.1.2. Noninformative Prior. Similarly, considering the Jeffreys prior, the posterior distributions of $\beta_1$ and $\beta_2$ are

$$
\beta_i | \mathbf{x}, \mathbf{y} \sim \text{ind} \gamma(\alpha_i, \beta_i), \beta_j | \mathbf{x}, \mathbf{y} \sim \text{ind} \gamma(\alpha_j, \beta_j), \quad i, j = 1, 2.
$$

respectively. So, we have

$$
2b_i \beta_i | \mathbf{x}, \mathbf{y} \sim \chi^2(2\beta_i), \quad i = 1, 2.
$$

(79)
As mentioned earlier, it is difficult to obtain HPD interval for $R$. Hence, for $R$ under the Jeffreys prior is

$$b_1^\prime r_2 \frac{b_1^\prime | x, y}{r_1 b_2^\prime} \sim \text{F}(2r_1, 2r_2).$$  

(80)

A 100$(1 - \xi)$% Bayesian credible interval with equal tails for $R$ under the Jeffreys prior is

$$P \left[ \frac{1}{1 + r_1 b_1^\prime b_2^\prime F_{1, F_{1,2}}(2r_1, 2r_2)} < R < \frac{1}{1 + r_1 b_1^\prime b_2^\prime F_{1,2}(2r_1, 2r_2)} \right] = 1 - \xi.$$  

(81)

6.2. HPD Interval. As mentioned earlier, it is difficult to obtain the HPD interval directly. Therefore, in this subsection, the Chen-Shao algorithm [34] is used to calculate the approximate HPD interval for $R$. This algorithm is expressed as follows.

In Step 3, $C_i$'s are 100$(1 - \xi)$% credible intervals for $R$. To obtain the HPD interval under the conjugate and Jeffreys priors, it is enough to substitute the posterior distributions (50) and (54) in Step 1 of Algorithm 1.

7. Simulation Study

In this section, we use simulation to compare estimators and confidence intervals of $R$. Therefore, for different sample sizes, different numbers of type II censorship, and different $R$ values with 1000 repetitions, bias and mean square error (MSE) values of $R$ estimators are calculated. For the conjugate informative prior distribution, hyperparameters $\alpha_1 = \alpha_2 = 1$, $\eta_1 = b_1/r_1$ and $\eta_2 = b_2/r_2$ are considered. Tables 1–3 show biases and MSEs of point estimators with $R = 1/2$, $R = 2/3$, and $R = 1/3$, respectively. Using Algorithm 2, type II censored samples are generated from two independent Gompertz distributions.

The results of the proposed methods for point estimation are summarized in Tables 1–3. Based on these tables, the following results can be achieved:

(i) The MCMC method has the lowest MSE
(ii) For small sample size, the Bayesian method performs better than the MLE method
(iii) The MSE of Bayesian estimators under conjugate and Jeffreys priors is not significantly different
(iv) The MSE of all estimators decreases significantly with increasing sample size

8. Application

The sample lifetime of a steel particular type under two different pressures of 35.5 (X) and 35 (Y) is reported in Table 7. This data contains 20 observations in each sample. This data has been studied by Kimber [35]. To verify that the data have a Gompertz distribution, we perform the Kolmogorov-Smirnov test. Based on the test statistics and $P$ value, it is concluded that $X$ has a Gompertz distribution with parameters $\beta_1 = 0.0013$ and $\gamma = 0.0027(D = 0.2, P$-value = 0.832) and $Y$ has a Gompertz distribution with parameters $\beta_2 = 0.00093$ and $\gamma = 0.0027(D = 0.35, P$-value = 0.1745). We consider $r_1 = r_2 = 2$. In Bayesian estimation, the values of the hyperparameters are $\alpha_1 = \alpha_2 = 1$, $\eta_1 = b_1/r_1$ and $\eta_2 = b_2/r_2$. Now, we apply the proposed methods for point estimation and confidence interval estimation to this data. The results are summarized in Tables 8 and 9. Based on the value of $R = P(X > Y) = 0.417$, it can be concluded that the lifetime of steel under pressure 35 is greater than the lifetime of steel under pressure 35.5.

9. Conclusion

This paper proposed a classical and Bayesian inference for stress-strength reliability of Gompertz distribution under type II censoring. First, the MLE of $R$ was obtained. Then,
exact and asymptotic confidence intervals for $R$ were presented. In addition, Bayesian estimators of $R$ obtained using Lindley approximation, Monte Carlo, and MCMC under conjugate informative and Jeffrey noninformative priors were discussed. Also, Bayesian credible intervals with equitailed and HPD intervals under conjugate and Jeffrey prior distributions were obtained. The proposed methods were compared with simulation studies. Finally, the application of these methods was examined with a real data.

**Data Availability**

This paper has no associated data.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


