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Research Article

Central Extensions and Nijenhuis Operators of Hom-δ-Jordan Lie Triple Systems

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In this paper, the equivalence of central extensions and $H^3_{\alpha,\alpha_V}(T, V)$ is proven in the study in Hom-δ-Jordan Lie triple systems. The concepts of Nijenhuis operators of Hom-δ-Jordan Lie triple systems are given. Moreover, a trivial deformation is got.

1. Introduction

It is well known that Lie triple systems are closely related to geometry. In a symmetric space, its tangent algebra is a Lie triple system. The definitions of the semisimplicity, radicality, and solvability for Lie triple systems are discussed, and the simple Lie triple system is determined by Lister [1]. Cohomologies of Lie triple systems were obtained [2]. Kubo and Taniguchi showed that in Lie triple systems, this kind of cohomology plays an important role in the study of deformations and extensions in 2004 [3]. A generalization of Lie triple systems and δ-Jordan Lie triple systems was defined in [4] by Okubo and Kamiya, where $\delta = \pm 1$. The case of $\delta = 1$ yields the Lie triple system, and they call the other case of $\delta = -1$ a Jordan Lie triple system. Then, they obtained a method to construct simple Jordan superalgebras by certain triple systems and studied the F-type Jordan superalgebra of a Jordan Lie triple system [5]. Recently, the cohomologies, Nijenhuis operators, representations, abelian extensions, and T^*-extensions of δ-Jordan Lie triple systems were developed by Ma and Chen [6].

The theory of Hom-type algebras has been studied (see [7–16]). In 2012, Yau showed the concept of Hom-Lie triple systems [17]. Later, generalized derivations of Hom-Lie triple systems were determined [18]. The cohomologies, 1-parameter formal deformations, and central extensions of Hom-Lie triple systems were discussed [19]. In 2019, the generalization of δ-Jordan Lie triple systems and Hom-Lie triple systems, 1-parameter formal deformations, and cohomologies of Hom-δ-Jordan Lie triple systems were studied [20]. We pay our main attention to consider central extensions and Nijenhuis operators of Hom-δ-Jordan Lie triple systems.

The paper is organized as follows. In Section 2, we summarize basic concepts and construct a structure of multiplicative Hom-δ-Jordan Lie triple systems. In Section 3, the equivalence of the third cohomology group and the central extensions of a Hom-δ-Jordan Lie triple system is proven. We discuss Nijenhuis operators of Hom-δ-Jordan Lie triple systems and obtain a trivial deformation using a Nijenhuis operator in Section 4.

In this paper, the capital letter F denotes an arbitrary field.

2. Preliminaries

We start by recalling the definition of Hom-Lie triple systems.

Definition 1 [17]. A Hom-Lie triple system $(T, [\cdot, \cdot, \cdot], \alpha = (\alpha_1, \alpha_2))$ consists of an F-vector space T, a trilinear map

$[\cdot,\cdot,\cdot]: T^3 \longrightarrow T$, and linear maps $\alpha_i : T \longrightarrow T$ for $i = 1, 2$, called twisted maps, such that for all $t_1, t_2, t_3, t_4, t_5 \in T$,

$$\begin{aligned}&[t_1, t_1, t_3] = 0,\\&[t_1, t_2, t_3] + [t_2, t_3, t_1] + [t_3, t_1, t_2] = 0,\\&[\alpha_1(t_4), \alpha_2(t_5), [t_1, t_2, t_3]] = [[t_4, t_5, t_1], \alpha_1(t_2), \alpha_2(t_3)]\\&\quad + [\alpha_1(t_1), [t_4, t_5, t_2], \alpha_2(t_3)]\\&\quad + [\alpha_1(t_1), \alpha_2(t_2), [t_4, t_5, t_3]].\end{aligned} \tag{1}$$

Definition 2 [20]. A Hom-δ-Jordan Lie triple system $(T, [\cdot,\cdot,\cdot], \delta, \alpha = (\alpha_1, \alpha_2))$ consists of an F-vector space T, a trilinear map $[\cdot,\cdot,\cdot]: T^3 \longrightarrow T$, and linear maps $\alpha_i : T \longrightarrow T$ for $i = 1, 2$, called twisted maps, such that for all $t_1, t_2, t_3, t_4, t_5 \in T$,

$$[t_1, t_2, t_3] = -\delta[t_2, t_1, t_3], \tag{2}$$

$$[t_1, t_2, t_3] + [t_2, t_3, t_1] + [t_3, t_1, t_2] = 0, \tag{3}$$

$$\begin{aligned}[\alpha_1(t_4), \alpha_2(t_5), [t_1, t_2, t_3]] &= [[t_4, t_5, t_1], \alpha_1(t_2), \alpha_2(t_3)]\\&\quad + [\alpha_1(t_1), [t_4, t_5, t_2], \alpha_2(t_3)]\\&\quad + \delta[\alpha_1(t_1), \alpha_2(t_2), [t_4, t_5, t_3]].\end{aligned} \tag{4}$$

Remark 3. When $\delta = 1$, a Hom-δ-Jordan Lie triple system is a Hom-Lie triple system. A Hom-δ-Jordan Lie triple system is a δ-Jordan Lie triple system if the twisted maps α_i are both equal to the identity map. So Hom-Lie triple systems and δ-Jordan Lie triple systems are special examples of Hom-δ-Jordan Lie triple systems.

A Hom-δ-Jordan Lie triple system is said to be multiplicative if $\alpha_1 = \alpha_2 = \alpha$ and $\alpha([t_1 t_2 t_3]) = [\alpha(t_1)\alpha(t_2)\alpha(t_3)]$ and denoted by $(T, [\cdot,\cdot,\cdot], \alpha)$.

A morphism $f : (T, [\cdot,\cdot,\cdot], \alpha = (\alpha_1, \alpha_2)) \longrightarrow (T', [\cdot,\cdot,\cdot]', \alpha' = (\alpha_1', \alpha_2'))$ of the Hom-δ-Jordan Lie triple system is a linear map satisfying $f([t_1 t_2 t_3]) = [f(t_1)f(t_2)f(t_3)]'$ and $f \circ \alpha_i = \alpha_i' \circ f$ for $i = 1, 2$. An isomorphism is a bijective morphism.

Definition 4 [20]. Let $(T, [\cdot,\cdot,\cdot], \delta, \alpha)$ be multiplicative Hom-δ-Jordan Lie triple systems, V be an F-vector space, and $A \in \mathrm{End}(V)$. V is said to be a $(T, [\cdot,\cdot,\cdot], \delta, \alpha)$-module with respect to A if there exists a bilinear map $\theta : T^2 \longrightarrow \mathrm{End}(V)$, $(t_1, t_2) \mapsto \theta(t_1, t_2)$ such that for all $t_1, t_2, t_3, t_4 \in T$,

$$\theta(\alpha(t_1), \alpha(t_2)) \circ A = A \circ \theta(t_1, t_2), \tag{5}$$

$$\begin{aligned}&\theta(\alpha(t_3), \alpha(t_4))\theta(t_1, t_2) - \delta\theta(\alpha(t_2), \alpha(t_4))\theta(t_1, t_3)\\&\quad - \theta(\alpha(t_1), [t_2, t_3, t_4]) \circ A + D(\alpha(t_2), \alpha(t_3))\theta(t_1, t_4) = 0,\end{aligned} \tag{6}$$

$$\begin{aligned}&\delta\theta(\alpha(t_3), \alpha(t_4))D(t_1, t_2) - \delta D(\alpha(t_1), \alpha(t_2))\theta(t_3, t_4)\\&\quad + \theta([t_1, t_2, t_3], \alpha(t_4)) \circ A + \delta\theta(\alpha(t_3), [t_1, t_2, t_4]) \circ A = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}&\delta D(\alpha(t_3), \alpha(t_4))D(t_1, t_2) - D(\alpha(t_1), \alpha(t_2))D(t_3, t_4)\\&\quad + \delta D([t_1, t_2, t_3], \alpha(t_4)) \circ A + \delta D(\alpha(t_3), [t_1, t_2, t_4]) \circ A = 0,\end{aligned} \tag{8}$$

where $D(t_1, t_2) = \theta(t_2, t_1) - \delta\theta(t_1, t_2)$.

Then, θ is said to be the representation of $(T, [\cdot,\cdot,\cdot], \delta, \alpha)$ on V with respect to A. In the case $\theta = 0$, V is said to be the trivial $(T, [\cdot,\cdot,\cdot], \delta, \alpha)$-module with respect to A.

In the case that $\delta = 1$, i.e., Hom-δ-Jordan Lie triple systems are Hom-Lie triple systems, we can get (8) from (7) by a direct calculation. But it is not true in the other case $\delta = -1$.

Particularly, $D(t_1, t_2)(t_3) = \delta[t_1, t_2, t_3]$ and (5), (6), and (7) hold, if $V = T$, $A = \alpha$, and $\theta(t_1, t_2)(t_3) = [t_3, t_1, t_2]$. In this case, T is called the adjoint $(T, [\cdot,\cdot,\cdot], \alpha)$-module and θ is called the adjoint representation of $(T, [\cdot,\cdot,\cdot], \alpha)$ on itself with respect to α.

In the following, the semidirect product of multiplicative Hom-δ-Jordan Lie triple systems and its module for general algebras were introduced.

Proposition 5. *Assume that $(T, [\cdot,\cdot,\cdot], \delta, \alpha)$ is a multiplicative Hom-δ-Jordan Lie triple system on V with respect to A and θ is a representation of $(T, [\cdot,\cdot,\cdot], \delta, \alpha)$. Then, $T_C = T \oplus V$ has a structure of a multiplicative Hom-δ-Jordan Lie triple system.*

Proof. We define the operation $[\cdot,\cdot,\cdot]_V : T_C^3 \longrightarrow T_C$ by $[(t_1, a), (t_2, b), (t_3, c)]_V = ([t_1, t_2, t_3], \theta(t_2, t_3)(a) - \delta\theta(t_1, t_3)(b) + \delta D(t_1, t_2)(c))$ and define the twisted map $\alpha + A : T_C \longrightarrow T_C$ by

$$(\alpha + A)(t_1, a) = (\alpha(t_1), A(a)). \tag{9}$$

By $D(t_1, t_2) = \theta(t_2, t_1) - \delta\theta(t_1, t_2)$, we get

$$\begin{aligned}&[(t_1, a), (t_2, b), (t_3, c)]_V\\&\quad = ([t_1, t_2, t_3], \theta(t_2, t_3)(a) - \delta\theta(t_1, t_3)(b) + \delta D(t_1, t_2)(c))\\&\quad = -\delta([t_2, t_1, t_3], \theta(t_1, t_3)(b)\\&\qquad - \delta\theta(t_2, t_3)(a) - D(t_1, t_2)(c))\\&\quad = -\delta([t_2, t_1, t_3], \theta(t_1, t_3)(b) - \delta\theta(t_2, t_3)(a)\\&\qquad + \delta D(t_2, t_1)(c))\\&\quad = -\delta[(t_2, b), (t_1, a), (t_3, c)]_V,\end{aligned}$$

$$
\begin{aligned}
&[(t_1,a),(t_2,b),(t_3,c)]_V+[(t_2,b),(t_3,c),(t_1,a)]_V\\
&\quad+[(t_3,c),(t_1,a),(t_2,b)]_V\\
&=([t_1,t_2,t_3],\theta(t_2,t_3)(a)-\delta\theta(t_1,t_3)(b)+\delta D(t_1,t_2)(c))\\
&\quad+([t_2,t_3,t_1],\theta(t_3,t_1)(b)-\delta\theta(t_2,t_1)(c)+\delta D(t_2,t_3)(a))\\
&\quad+([t_3,t_1,t_2],\theta(t_1,t_2)(c)-\delta\theta(t_3,t_2)(a)+\delta D(t_3,t_1)(b))\\
&=(0,\theta(t_2,t_3)(a)-\delta\theta(t_3,t_2)(a)+\delta D(t_2,t_3)(a)\\
&\quad+\theta(t_3,t_1)(b)-\delta\theta(t_1,t_3)(b)+\delta D(t_3,t_1)(b)\\
&\quad+\theta(t_1,t_2)(c)-\delta\theta(t_2,t_1)(c)+\delta D(t_1,t_2)(c))\\
&=(0,0).
\end{aligned}
\tag{10}
$$

By (6), (7), and (8), we have

$$
\begin{aligned}
&\left[[(t_1,a),(t_2,b),(u,c)]_V,(\alpha+A)(v,d),(\alpha+A)(w,e)\right]_V\\
&=[([t_1,t_2,u],\theta(t_2,u)(a)-\delta\theta(t_1,u)(b)\\
&\quad+\delta D(t_1,t_2)(c)),(\alpha(v),A(d)),(\alpha(w),A(e))]_V\\
&=([[t_1,t_2,u],\alpha(v),\alpha(w)],\theta(\alpha(v),\alpha(w))(\theta(t_2,u)(a)\\
&\quad-\delta\theta(t_1,u)(b)+\delta D(t_1,t_2)(c))\\
&\quad-\delta\theta([t_1,t_2,u],\alpha(w))(A(d))\\
&\quad+\delta D([t_1,t_2,u],\alpha(v))(A(e))),\\
&\left[(\alpha+A)(u,c),[(t_1,a),(t_2,b),(v,d)]_V,(\alpha+A)(w,e)\right]_V\\
&=[(\alpha(u),A(c)),([t_1,t_2,v],\theta(t_2,v)(a)-\delta\theta(t_1,v)(b)\\
&\quad+\delta D(t_1,t_2)(d)),(\alpha(w),A(e))]_V\\
&=([\alpha(u),[t_1,t_2,v],\alpha(w)],\theta([t_1,t_2,v],\alpha(w))(A(c))\\
&\quad-\delta\theta(\alpha(u),\alpha(w))(\theta(t_2,v)(a)-\delta\theta(t_1,v)(b)\\
&\quad+\delta D(t_1,t_2)(d))+\delta D(\alpha(u),[t_1,t_2,v])(A(e))),\\
&\delta\left[(\alpha+A)(u,c),(\alpha+A)(v,d),[(t_1,a),(t_2,b),(w,e)]_V\right]_V\\
&=\delta[(\alpha(u),A(c)),(\alpha(v),A(d)),([t_1,t_2,w],\theta(t_2,w)(a)\\
&\quad-\delta\theta(t_1,w)(b)+\delta D(t_1,t_2)(e))]_V\\
&=\delta([\alpha(u),\alpha(v),[t_1,t_2,w]],\theta(\alpha(v),[t_1,t_2,w])(A(c))\\
&\quad-\delta\theta(\alpha(u),[t_1,t_2,w])(A(d))\\
&\quad+\delta D(\alpha(u),\alpha(v))(\theta(t_2,w)(a)-\delta\theta(t_1,w)(b)\\
&\quad+\delta D(t_1,t_2)(e))),\\
&\left[(\alpha+A)(t_1,a),(\alpha+A)(t_2,b),[(u,c),(v,d),(w,e)]_V\right]_V\\
&=[(\alpha(t_1),A(a)),(\alpha(t_2),A(b)),([u,v,w],\theta(v,w)(c)\\
&\quad-\delta\theta(u,w)(d)+\delta D(u,v)(e))]_V\\
&=([\alpha(t_1),\alpha(t_2),[u,v,w]],\theta(\alpha(t_2),[u,v,w])(A(a))\\
&\quad-\delta\theta(\alpha(t_1),[u,v,w])(A(b))\\
&\quad+\delta D(\alpha(t_1),\alpha(t_2))(\theta(v,w)(c)-\delta\theta(u,w)(d)\\
&\quad+\delta D(u,v)(e))).
\end{aligned}
\tag{11}
$$

□

The calculation above shows that (2), (3), and (4) hold.

By (5) and the linearity of $\alpha+A$,

$$
\begin{aligned}
&(\alpha+A)[(t_1,a),(t_2,b),(t_3,c)]_V\\
&=(\alpha+A)([t_1,t_2,t_3],\theta(t_2,t_3)(a)-\delta\theta(t_1,t_3)(b)\\
&\quad+\delta D(t_1,t_2)(c))\\
&=(\alpha([t_1,t_2,t_3]),A\circ(\theta(t_2,t_3)(a)-\delta\theta(t_1,t_3)(b)\\
&\quad+\delta D(t_1,t_2)(c)))\\
&=([\alpha(t_1),\alpha(t_2),\alpha(t_3)],\theta(\alpha(t_2),\alpha(t_3))A(a)\\
&\quad-\delta\theta(\alpha(t_1),\alpha(t_3))A(b)+\delta D(\alpha(t_1),\alpha(t_2))A(c))\\
&=[(\alpha(t_1),A(a)),(\alpha(t_2),A(b)),(\alpha(t_3),A(c))]_V\\
&=[(\alpha+A)(t_1,a),(\alpha+A)(t_2,b),(\alpha+A)(t_3,c)]_V.
\end{aligned}
\tag{12}
$$

Hence, $(U,[\cdot,\cdot,\cdot]_v,a+A)$ is a multiplicative Hom-δ-Jordan Lie triple system.

Suppose that $f:\underbrace{T\times\cdots\times T}_{n\text{ times}}\longrightarrow V$ is an n-linear map, which satisfies

$$
\begin{aligned}
&A(f(t_1,\cdots,t_n))=f(\alpha(t_1),\cdots,\alpha(t_n)),\\
&f(t_1,\cdots,x,y,t_n)=-\delta f(t_1,\cdots,y,x,t_n),\\
&f(t_1,\cdots,t_{n-3},x,y,z)+f(t_1,\cdots,t_{n-3},y,z,x)+f(t_1,\cdots,t_{n-3},z,x,y)=0,
\end{aligned}
\tag{13}
$$

where f is said to be an n-Hom-cochain on T. The set of all n-Hom-cochains is denoted by $C^n_{\alpha,A}(T,V)$, for all $n\geq 1$.

(i) If $f\in C^1_\delta(T,V)$, then

$$
\begin{aligned}
d^1_{\text{hom}}f(t_1,t_2,t_3)&=\theta(t_2,t_3)f(t_1)-\delta\theta(t_1,t_3)f(t_2)\\
&\quad+\delta D(t_1,t_2)f(t_3)-f([t_1,t_2,t_3]).
\end{aligned}
\tag{14}
$$

(ii) If $f\in C^2_\delta(T,V)$, then

$$
\begin{aligned}
&d^2_{\text{hom}}f(y,t_1,t_2,t_3)\\
&=\theta(\alpha(t_2),\alpha(t_3))f(y,t_1)-\delta\theta(\alpha(t_1),\alpha(t_3))f(y,t_2)\\
&\quad+\delta D(\alpha(t_1),\alpha(t_2))f(y,t_3)-f(\alpha(y),[t_1,t_2,t_3]).
\end{aligned}
\tag{15}
$$

(iii) If $f\in C^3_\delta(T,V)$, then

$$
\begin{aligned}
&d^3_{\text{hom}}f(t_1,t_2,t_3,t_4,t_5)\\
&=\theta(\alpha(t_4),\alpha(t_5))f(t_1,t_2,t_3)\\
&\quad-\delta\theta(\alpha(t_3),\alpha(t_5))f(t_1,t_2,t_4)\\
&\quad-\delta D(\alpha(t_1),\alpha(t_2))f(t_3,t_4,t_5)\\
&\quad+D(\alpha(t_3),\alpha(t_4))f(t_1,t_2,t_5)\\
&\quad+f([t_1,t_2,t_3],\alpha(t_4),\alpha(t_5))\\
&\quad+f(\alpha(t_3),[t_1,t_2,t_4],\alpha(t_5))\\
&\quad+\delta f(\alpha(t_3),\alpha(t_4),[t_1,t_2,t_5])\\
&\quad-f(\alpha(t_1),\alpha(t_2),[t_3,t_4,t_5]).
\end{aligned}
\tag{16}
$$

(iv) If $f \in C^4_\delta(T, V)$, then

$$\begin{aligned} d^4_{\mathrm{hom}} f(y, t_1, t_2, t_3, t_4, t_5) &= \theta(\alpha^2(t_4), \alpha^2(t_5)) f(y, t_1, t_2, t_3) \\ &\quad - \delta\theta(\alpha^2(t_3), \alpha^2(t_5)) f(y, t_1, t_2, t_4) \\ &\quad - \delta D(\alpha^2(t_1), \alpha^2(t_2)) f(y, t_3, t_4, t_5) \\ &\quad + D(\alpha^2(t_3), \alpha^2(t_4)) f(y, t_1, t_2, t_5) \\ &\quad + f(\alpha(y), [t_1, t_2, t_3], \alpha(t_4), \alpha(t_5)) \\ &\quad + f(\alpha(y), \alpha(t_3), [t_1, t_2, t_4], \alpha(t_5)) \\ &\quad + \delta f(\alpha(y), \alpha(t_3), \alpha(t_4), [t_1, t_2, t_5]) \\ &\quad - f(\alpha(y), \alpha(t_1), \alpha(t_2), [t_3, t_4, t_5]). \end{aligned} \tag{17}$$

Definition 6 [20]. For $n = 1, 2, 3, 4$, the coboundary operator $d^n_{\mathrm{hom}} : C^n_{\alpha,A}(T, V) \longrightarrow C^{n+2}_{\alpha,A}(T, V)$ is defined as follows.

The mapping $f \in C^n_{\alpha,A}(T, V)$ is said to be an n-Hom-cocycle if $d^n_{\mathrm{hom}} f = 0, n = 1, 2, 3, 4$. Denote by $Z^n_{\alpha,A}(T, V)$ the subspace spanned by n-Hom-cocycles. For $n \geq 3$, $B^n_{\alpha,A}(T, V) = d^{n-2}_{\mathrm{hom}} C^{n-2}_{\alpha,A}(T, V)$.

Since $d^{n+2}_{\mathrm{hom}} d^n_{\mathrm{hom}} = 0$, $B^n_{\alpha,A}(T, V) \subseteq Z^n_{\alpha,A}(T, V)$. Define a cohomology space:

$$H^n_{\alpha,A}(T, \mathrm{V}) = \frac{Z^n_{\alpha,A}(T, V)}{B^n_{\alpha,A}(T, V)}. \tag{18}$$

3. Central Extensions of Hom-δ-Jordan Lie Triple Systems

Let $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ be a multiplicative Hom-δ-Jordan Lie triple system and V be a trivial $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$-module with respect to α_V. Then, $(V, 0, \alpha_V)$ is an abelian multiplicative Hom-δ-Jordan Lie triple system with the trivial product. A multiplicative Hom-δ-Jordan Lie triple system $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ is said to be a central extension of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$ if the following commutative diagram holds with the exact rows of Hom-δ-Jordan Lie triple systems.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & T_C & \underset{s}{\overset{\pi}{\rightleftarrows}} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_C} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & V & \xrightarrow{\iota} & T_C & \underset{s}{\overset{\pi}{\rightleftarrows}} & T & \longrightarrow & 0 \end{array}$$

where $\alpha_C \circ \iota = \iota \circ \alpha_V, \alpha \circ \pi = \pi \circ \alpha_C$, s is a linear map satisfying $\pi s = \mathrm{id}_T$ and $\alpha_C \circ s = s \circ \alpha$, and $\iota(V) \subseteq Z(T_C) = \{x \in T_C \mid [x, T_C, T_C]_C = 0\}$. Two central extensions $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ and $(T_{C'}, [\cdot, \cdot, \cdot]_{C'}, \delta, \alpha_{C'})$ are equivalent, if the following commutative diagram holds.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & T_C & \xrightarrow{\pi} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_T} & & \\ 0 & \longrightarrow & V & \xrightarrow{\iota'} & T'_C & \xrightarrow{\pi'} & T & \longrightarrow & 0, \end{array}$$

where $\varphi : (T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C) \longrightarrow (T_{C'}, [\cdot, \cdot, \cdot]_{C'}, \delta, \alpha_{C'})$ is an isomorphism.

Theorem 7. *There is bijective mapping between $H^3_{\alpha,\alpha_V}(T, V)$ and equivalent classes of central extensions of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$.*

Proof. First, we show that there is a bijective mapping between $Z^3_{\alpha,\alpha_V}(T, V)$ and central extensions of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$. □

Suppose that $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ is a central extension of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$. Then, the following commutative diagram holds:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & T_C & \underset{s}{\overset{\pi}{\rightleftarrows}} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_C} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & V & \xrightarrow{\iota} & T_C & \underset{s}{\overset{\pi}{\rightleftarrows}} & T & \longrightarrow & 0 \end{array}$$

with $\alpha_C \circ \iota = \iota \circ \alpha_V$, $\alpha \circ \pi = \pi \circ \alpha_C$, and a linear map s satisfying $\alpha_C \circ s = s \circ \alpha$ and $\pi s = \mathrm{id}_T$.

For $t_1, t_2, t_3 \in T$, since $\pi[s(t_1), s(t_2), s(t_3)]_C - \pi s[t_1, t_2, t_3] = [\pi s(t_1), \pi s(t_2), \pi s(t_3)] - [t_1, t_2, t_3] = 0$, it follows that $[s(t_1), s(t_2), s(t_3)]_C - s[t_1, t_2, t_3] \in \operatorname{Ker} \pi = \iota(V)$. Define a trilinear map $g : T \times T \times T \longrightarrow V$ by

$$\iota g(t_1, t_2, t_3) = [s(t_1), s(t_2), s(t_3)]_C - s[t_1, t_2, t_3]. \tag{19}$$

Since ι is injective, g is well defined, and it follows from $\iota(V) \subseteq Z(T_C)$ that

$$\left[[s(t_1), s(t_2), s(t_3)]_C, u, v\right]_C = [s[t_1, t_2, t_3], u, v]_C, \forall u, v \in T_C. \tag{20}$$

Note that g satisfies $g(t_1, t_2, t_3) = -\delta g(t_2, t_1, t_3)$, $g(t_1, t_2, t_3) + g(t_2, t_3, t_1) + g(t_3, t_1, t_2) = 0$ and

$$\begin{aligned} \iota g(\alpha(t_1), \alpha(t_2), \alpha(t_3)) &= [s\alpha(t_1), s\alpha(t_2), s\alpha(t_3)]_C - s[\alpha(t_1), \alpha(t_2), \alpha(t_3)] \\ &= [\alpha_C s(t_1), \alpha_C s(t_2), \alpha_C s(t_3)]_C - \alpha_C s[t_1, t_2, t_3] \\ &= \alpha_C([s(t_1), s(t_2), s(t_3)]_C - s[t_1, t_2, t_3]) \\ &= \alpha_C \iota g(t_1, t_2, t_3) \\ &= \iota \alpha_V g(t_1, t_2, t_3). \end{aligned} \tag{21}$$

Hence, $g \in C^3_{\alpha,\alpha_V}(T, V)$. Moreover, $g \in Z^3_{\alpha,\alpha_V}(T, V)$ since

$$\begin{aligned} \iota(d^3_{\mathrm{hom}} g)(t_1, t_2, t_3, t_4, t_5) &= \iota(g([t_1, t_2, t_3], \alpha(t_4), \alpha(t_5)) + g(\alpha(t_3), [t_1, t_2, t_4], \alpha(t_5)) \\ &\quad + \delta g(\alpha(t_3), \alpha(t_4), [t_1, t_2, t_5]) - g(\alpha(t_1), \alpha(t_2), [t_3, t_4, t_5])) \\ &= [s[t_1, t_2, t_3], s\alpha(t_4), s\alpha(t_5)]_C - s[[t_1, t_2, t_3], \alpha(t_4), \alpha(t_5)] \\ &\quad + [s\alpha(t_3), s[t_1, t_2, t_4], s\alpha(t_5)]_C - s[\alpha(t_3), [t_1, t_2, t_4], \alpha(t_5)] \\ &\quad + \delta[s\alpha(t_3), s\alpha(t_4), s[t_1, t_2, t_5]]_C - \delta s[\alpha(t_3), \alpha(t_4), [t_1, t_2, t_5]] \\ &\quad - [s\alpha(t_1), s\alpha(t_2), s[t_3, t_4, t_5]]_C + s[\alpha(t_1), \alpha(t_2), [t_3, t_4, t_5]] \end{aligned}$$

$$
\begin{aligned}
&= \left[[s(t_1), s(t_2), s(t_3)]_c, \alpha_C s(t_4), \alpha_C s(t_5)\right]_C \\
&\quad + \left[\alpha_C s(t_3), [s(t_1), s(t_2), s(t_4)]_C, \alpha_C s(t_5)\right]_C \\
&\quad + \delta\left[\alpha_C s(t_3), \alpha_C s(t_4), [s(t_1), s(t_2), s(t_5)]_C\right]_C \\
&\quad - \left[\alpha_C s(t_1), \alpha_C s(t_2), [s(t_3), s(t_4), s(t_5)]_C\right]_C = 0.
\end{aligned} \tag{22}
$$

On the other hand, let $g \in Z^3_{\alpha,\alpha_V}(T, V)$ and $T_C = T \oplus V$ with

$$
\begin{aligned}
&[(t_1, a), (t_2, b), (t_3, c)]_C \\
&\quad = ([t_1, t_2, t_3], g(t_1, t_2, t_3)); \quad \alpha_C(t_1, a) = (\alpha(t_1), \alpha_V(a)).
\end{aligned} \tag{23}
$$

Thus, α_C is linear, and

$$
\begin{aligned}
&\alpha_C[(t_1, a), (t_2, b), (t_3, c)]_C \\
&\quad = \alpha_C([t_1, t_2, t_3], g(t_1, t_2, t_3)) \\
&\quad = (\alpha[t_1, t_2, t_3], \alpha_V g(t_1, t_2, t_3)) \\
&\quad = ([\alpha(t_1), \alpha(t_2), \alpha(t_3)], g(\alpha(t_1), \alpha(t_2), \alpha(t_3))) \\
&\quad = [(\alpha(t_1), \alpha_V(a)), (\alpha(t_2), \alpha_V(b)), (\alpha(t_3), \alpha_V(c))]_C \\
&\quad = [\alpha_C(t_1, a), \alpha_C(t_2, b), \alpha_C(t_3, c)]_C.
\end{aligned} \tag{24}
$$

Since that

$$
\begin{aligned}
&\left[\alpha_C(t_1, a), \alpha_C(t_2, b), [(u, c), (v, d), (w, e)]_C\right]_C \\
&\quad = [(\alpha(t_1), \alpha_V(a)), (\alpha(t_2), \alpha_V(b)), ([u, v, w], g(u, v, w))]_C \\
&\quad = ([\alpha(t_1)\alpha(t_2)[u, v, w]], g(\alpha(t_1), \alpha(t_2), [u, v, w])) \\
&\quad = ([[t_1, t_2, u]\alpha(v)\alpha(w)], g([t_1, t_2, u], \alpha(v), \alpha(w))) \\
&\qquad + ([\alpha(u)[t_1, t_2, v]\alpha(w)], g(\alpha(u), [t_1, t_2, v], \alpha(w))) \\
&\qquad + (\delta[\alpha(u)\alpha(v)[t_1, t_2, w]], \delta g(\alpha(u), \alpha(v), [t_1, t_2, w])) \\
&\quad = \left[[(t_1, a), (t_2, b), (u, c)]_C, \alpha_C(v, d), \alpha_C(w, e)\right]_C \\
&\qquad + \left[\alpha_C(u, c), [(t_1, a), (t_2, b), (v, d)]_C, \alpha_C(w, e)\right]_C \\
&\qquad + \delta\left[\alpha_C(u, c), \alpha_C(v, d), [(t_1, a), (t_2, b), (w, e)]_C\right]_C.
\end{aligned} \tag{25}
$$

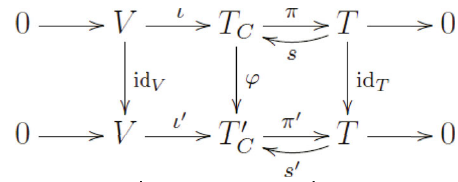
We have $(T_C, [\cdot, \cdot, \cdot]_c, \delta, \alpha_C)$ which is a multiplicative Hom-δ-Jordan Lie triple system.

Define three mappings $\iota : V \longrightarrow T_C$, $\pi : T_C \longrightarrow T$, and $s : T \longrightarrow T_C$ by $\iota(a) = (0, a)$, $\pi(t, a) = t$, and $s(t) = (t, 0)$, respectively. Then,

$$
\begin{aligned}
&\alpha_C \circ \iota(a) = \alpha_C(0, a) = (0, \alpha_V(a)) = \iota \circ \alpha_V(a), \\
&\pi \circ \alpha_C(t, a) = \pi(\alpha(t), \alpha_V(a)) = \alpha(t) = \alpha \circ \pi(t, a), \\
&\pi s = \mathrm{id}_T, \alpha_C s(t) = \alpha_C(t, 0) = (\alpha(t), 0) = s\alpha(t).
\end{aligned} \tag{26}
$$

It is clear that $\iota(V)$ is a subspace of $Z(T_C)$. Hence, $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ is a central extension of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$.

Assume that $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ and $(T'_C, [\cdot, \cdot, \cdot]'_C, \delta, \alpha'_C)$ are equivalent central extensions of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$. Then, the following commutative diagram holds:



such that $\pi = \pi' \circ \varphi$ and $\varphi \circ \iota = \iota'$ with an isomorphism φ and $\pi s = \pi' s' = id_T$. For their corresponding 3-Hom-cocycles g and g' as above, we have

$$
\begin{aligned}
&\iota g(t_1, t_2, t_3) = [s(t_1), s(t_2), s(t_3)]_C - s[t_1, t_2, t_3], \\
&\iota' g'(t_1, t_2, t_3) = \left[s'(t_1), s'(t_2), s'(t_3)\right]'_C - s'[t_1, t_2, t_3], \\
&\iota' g(t_1, t_2, t_3) = \varphi\iota g(t_1, t_2, t_3) \\
&\qquad = \varphi[s(t_1), s(t_2), s(t_3)]_C - \varphi s[t_1, t_2, t_3].
\end{aligned} \tag{27}
$$

We have $g - g' \in B^3_{\alpha,\alpha_V}(T, V)$. In fact, since

$$
\pi' s'(t_1) - \pi'\varphi s(t_1) = t_1 - \pi s(t_1) = 0, \tag{28}
$$

there exists a linear mapping $f : T \longrightarrow V$ by $\iota' f(t_1) = s'(t_1) - \varphi s(t_1)$, for all $t_1 \in T$. Then,

$$
\begin{aligned}
&\iota' f\alpha(t_1) = s'\alpha(t_1) - \varphi s\alpha(t_1) = \alpha_{C'} s'(t_1) - \varphi\alpha_C s(t_1) \\
&\qquad = \alpha_{C'} s'(t_1) - \alpha_{C'}\varphi s(t_1) = \alpha_{C'}\iota' f(t_1) = \iota'\alpha_V f(t_1),
\end{aligned} \tag{29}
$$

that is, $f \in C^1_{\alpha,\alpha_V}(T, V)$. By $s'(t_1) - \varphi s(t_1) = \iota' f(t_1) \in Z(T_{C'})$,

$$
\begin{aligned}
\left[s'(t_1), s'(t_2), s'(t_3)\right]'_C &= [\varphi s(t_1), \varphi s(t_2), \varphi s(t_3)]'_C \\
&= \varphi[s(t_1), s(t_2), s(t_3)]_C.
\end{aligned} \tag{30}
$$

Then,

$$
\iota'\left(g' - g\right)(t_1, t_2, t_3) = -\iota' f([t_1, t_2, t_3]) = \iota'\left(d^1_{\mathrm{hom}} f\right)(t_1, t_2, t_3), \tag{31}
$$

so $g' - g = d^1_{\mathrm{hom}} f \in B^3_{\alpha,\alpha_V}(T, V)$.

Suppose $g, g' \in Z^3_{\alpha,\alpha_V}(T, V)$ and $g' - g \in B^3_{\alpha,\alpha_V}(T, V)$; i.e., there is $f \in C^1_{\alpha,\alpha_V}(T, V)$ satisfying $g' - g = d^1_{\mathrm{hom}} f$. Then, $(g' - g)(t_1, t_2, t_3) = -f([t_1, t_2, t_3])$. Let $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ and $(T_{C'}, [\cdot, \cdot, \cdot]_{C'}, \delta, \alpha_C)$, which are defined as above with respect to g and g', be two central extensions of $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ by $(V, 0, \delta, \alpha_V)$. Then, $\iota(a) = (0, a) = \iota'(a)$ and $\pi(t, a)$

$= t = \pi'(t, a)$. There is a linear map:

$$\begin{aligned} \varphi : \left(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C\right) &\longrightarrow \left(T_{C'}, [\cdot, \cdot, \cdot]_{C'}, \delta, \alpha_C\right), \\ (t, a) &\mapsto (t, a - f(t)), \end{aligned} \tag{32}$$

such that $\varphi\iota(a) = \iota'(a)$ and $\pi'\varphi(t, a) = \pi'(t, a - f(t)) = t = \pi(t, a)$. The following commutative diagram holds:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & T_C & \xrightarrow{\pi} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_T} & & \\ 0 & \longrightarrow & V & \xrightarrow{\iota'} & T'_C & \xrightarrow{\pi'} & T & \longrightarrow & 0. \end{array}$$

The sufficiency of φ is an isomorphism that is proven.

If $\varphi(t, a) = \varphi(\tilde{t}, \tilde{a})$, then $(t, a - f(t)) = (\tilde{t}, \tilde{a} - f(\tilde{t}))$; that is, $t = \tilde{t}$ and $a - f(t) = \tilde{a} - f(\tilde{t})$; then, $a = \tilde{a}$; hence, φ is injective; φ is obviously surjective. Note that

$$\begin{aligned} &\varphi\alpha_C(t, a) \\ &\quad = \varphi(\alpha(t), \alpha_V(a)) = (\alpha(t), \alpha_V(a) - f\alpha(t)) \\ &\quad = (\alpha(t), \alpha_V(a) - \alpha_V f(t)) = \alpha_{C'}(t, a - f(t)) = \alpha_{C'}\varphi(t, a), \\ &\varphi[(t_1, a), (t_2, b), (t_3, c)]_C \\ &\quad = \varphi([t_1, t_2, t_3], g(t_1, t_2, t_3)) \\ &\quad = ([t_1, t_2, t_3], g(t_1, t_2, t_3) - f([t_1, t_2, t_3])) \\ &\quad = \left([t_1, t_2, t_3], g'(t_1, t_2, t_3)\right) \\ &\quad = [(t_1, a - f(t_1)), (t_2, b - f(t_2)), (t_3, c - f(t_3))]'_C \\ &\quad = [\varphi(t_1, a), \varphi(t_2, b), \varphi(t_3, c)]'_C. \end{aligned} \tag{33}$$

The equivalence of $(T_C, [\cdot, \cdot, \cdot]_C, \delta, \alpha_C)$ and $(T'_C, [\cdot, \cdot, \cdot]'_C, \delta, \alpha'_C)$ is proven.

4. Nijenhuis Operators of Hom-δ-Jordan Lie Triple Systems

In this section, the deformation of Hom-δ-Jordan Lie triple systems is studied. The notion of Nijenhuis operators of Hom-δ-Jordan Lie triple systems is introduced, and the trivial deformations of this kind of operators are shown.

Let $(T, [\cdot, \cdot, \cdot], \delta, \alpha)$ be a Hom-δ-Jordan Lie triple system and $\psi : T^3 \longrightarrow T$ be a trilinear mapping. Consider a λ-parametrized family of linear operations:

$$[t_1, t_2, t_3]_\lambda = [t_1, t_2, t_3] + \lambda\psi(t_1, t_2, t_3), \tag{34}$$

where λ is a formal variable.

We call that ψ generates a λ-parameter infinitesimal deformation of the Hom-δ-Jordan Lie triple system, if $[\cdot, \cdot, \cdot]_\lambda$ endow T with the Hom-δ-Jordan Lie triple system structure which is denoted by T_λ.

(i) ψ itself defines a Hom-δ-Jordan Lie triple system structure on T

(ii) ψ is a 3-cocycle of T

Theorem 8. *ψ generates a λ -parameter infinitesimal deformation of the Hom-δ-Jordan Lie triple system T; then, the following two conclusions hold:*

Proof.

$$\begin{aligned} [t_1, t_2, t_3]_\lambda &= [t_1, t_2, t_3] + \lambda\psi(t_1, t_2, t_3), \\ -\delta[t_2, t_1, t_3]_\lambda &= -\delta[t_2, t_1, t_3] - \delta\lambda\psi(t_2, t_1, t_3). \end{aligned} \tag{35}$$

We have

$$\psi(t_1, t_2, t_3) = -\delta\psi(t_2, t_1, t_3). \tag{36}$$

From the equality

$$\begin{aligned} 0 &= [t_1, t_2, t_3]_\lambda + [t_2, t_3, t_1]_\lambda + [t_3, t_1, t_2]_\lambda \\ &= [t_1, t_2, t_3] + [t_2, t_3, t_1] + [t_3, t_1, t_2] + \lambda(\psi(t_1, t_2, t_3) \\ &\quad + \psi(t_2, t_3, t_1) + \psi(t_3, t_1, t_2)), \end{aligned} \tag{37}$$

it follows that

$$\psi(t_1, t_2, t_3) + \psi(t_2, t_3, t_1) + \psi(t_3, t_1, t_2) = 0. \tag{38}$$

For the equality

$$\begin{aligned} \left[\alpha(t_1), \alpha(t_2), [r_1, r_2, r_3]_\lambda\right]_\lambda = {} & \left[[t_1, t_2, r_1]_\lambda, \alpha(r_2), \alpha(r_3)\right]_\lambda \\ & + \left[\alpha(r_1), [t_1, t_2, r_2]_\lambda \alpha(r_3)\right]_\lambda \\ & + \delta\left[\alpha(r_1), \alpha(r_2), [t_1, t_2, r_3]_\lambda\right]_\lambda, \end{aligned} \tag{39}$$

the left hand side is equal to

$$\begin{aligned} &[\alpha(t_1), \alpha(t_2), [r_1, r_2, r_3] + \lambda\psi(r_1, r_2, r_3)]_\lambda \\ &\quad = [\alpha(t_1), \alpha(t_2), [r_1, r_2, r_3]] + \lambda\psi(\alpha(t_1), \alpha(t_2), [r_1, r_2, r_3]) \\ &\qquad + [\alpha(t_1), \alpha(t_2), \lambda\psi(r_1, r_2, r_3)] \\ &\qquad + \lambda\psi(\alpha(t_1), \alpha(t_2), \lambda\psi(r_1, r_2, r_3)) \\ &\quad = [\alpha(t_1), \alpha(t_2), [r_1, r_2, r_3]] + \lambda(\psi(\alpha(t_1), \alpha(t_2), [r_1, r_2, r_3]) \\ &\qquad + [\alpha(t_1), \alpha(t_2), \psi(r_1, r_2, r_3)]) \\ &\qquad + \lambda^2\psi(\alpha(t_1), \alpha(t_2), \psi(r_1, r_2, r_3)), \end{aligned} \tag{40}$$

and the right hand side is equal to

$$\begin{aligned} &[[t_1, t_2, r_1] + \lambda\psi(t_1, t_2, r_1), \alpha(r_2), \alpha(r_3)]_\lambda \\ &\qquad + [\alpha(r_1), [t_1, t_2, r_2] + \lambda\psi(t_1, t_2, r_2), \alpha(r_3)]_\lambda \\ &\qquad + \delta[\alpha(r_1), \alpha(r_2), [t_1, t_2, r_3] + \lambda\psi(t_1, t_2, r_3)]_\lambda \\ &\quad = [[t_1, t_2, r_1], \alpha(r_2), \alpha(r_3)] + [\alpha(r_1), [t_1, t_2, r_2], \alpha(r_3)] \\ &\qquad + \delta[\alpha(r_1), \alpha(r_2), [t_1, t_2, r_3]] \\ &\qquad + \lambda(\psi([t_1, t_2, r_1], \alpha(r_2), \alpha(r_3)) \\ &\qquad + [\psi(t_1, t_2, r_1), \alpha(r_2), \alpha(r_3)] + \psi(\alpha(r_1), [t_1, t_2, r_2], \alpha(r_3)) \\ &\qquad + [\alpha(r_1), \psi(t_1, t_2, r_2), \alpha(r_3)] \end{aligned}$$

$$+\delta\psi(\alpha(r_1),\alpha(r_2),[t_1,t_2,r_3]) + \delta[\alpha(r_1),\alpha(r_2),\psi(t_1,t_2,r_3)]) + \lambda^2(\psi(\psi(t_1,t_2,r_1),\alpha(r_2),\alpha(r_3)) + \psi(\alpha(r_1),\psi(t_1,t_2,r_2),\alpha(r_3)) + \delta\psi(\alpha(r_1),\alpha(r_2),\psi(t_1,t_2,r_3))). \quad (41)$$

Thus, we have

$$\begin{aligned}\psi(\alpha(t_1),\alpha(t_2),[r_1,r_2,r_3]) + \delta D(\alpha(t_1),\alpha(t_2))\psi(r_1,r_2,r_3) &= \psi([t_1,t_2,r_1],\alpha(r_2),\alpha(r_3)) + \theta(\alpha(r_2),\alpha(r_3))\psi(t_1,t_2,r_1)\\ &+ \psi(\alpha(r_1),[t_1,t_2,r_2],\alpha(r_3))\\ &- \delta\theta(\alpha(r_1),\alpha(r_3)\psi(t_1,t_2,r_2))\\ &+ \delta\psi(\alpha(r_1),\alpha(r_2),[t_1,t_2,r_3])\\ &+ D(\alpha(r_1),\alpha(r_2))\psi(t_1,t_2,r_3),\end{aligned} \quad (42)$$

$$\begin{aligned}\psi(\alpha(t_1),\alpha(t_2),\psi(r_1,r_2,r_3)) &= \psi(\psi(t_1,t_2,r_1),\alpha(r_2),\alpha(r_3))\\ &+ \psi(\alpha(r_1),\psi(t_1,t_2,r_2),\alpha(r_3))\\ &+ \delta\psi(\alpha(r_1),\alpha(r_2),\psi(t_1,t_2,r_3)).\end{aligned} \quad (43)$$

Therefore, ψ defines a Hom-δ-Jordan Lie triple system structure on T by (36), (38), and (43). Furthermore, by (42), ψ is a 3-cocycle. □

A deformation is called trivial if there exists a linear map $N : T \longrightarrow T$ such that for $\varphi_\lambda = \mathrm{id} + \lambda N : T_\lambda \longrightarrow T$,

$$\varphi_\lambda[t_1,t_2,t_3]_\lambda = [\varphi_\lambda t_1,\varphi_\lambda t_2,\varphi_\lambda t_3]. \quad (44)$$

It is clear that

$$\begin{aligned}\varphi_\lambda[t_1,t_2,t_3]_\lambda &= [t_1,t_2,t_3] + \lambda\psi(t_1,t_2,t_3)\\ &+ \lambda N([t_1,t_2,t_3] + \lambda\psi(t_1,t_2,t_3))\\ &= [t_1,t_2,t_3] + \lambda(\psi(t_1,t_2,t_3) + N[t_1,t_2,t_3])\\ &+ \lambda^2 N\psi(t_1,t_2,t_3),\\ [\varphi_\lambda t_1,\varphi_\lambda t_2,\varphi_\lambda t_3] &= [t_1+\lambda Nt_1, t_2+\lambda Nt_2, t_3+\lambda Nt_3]\\ &= [t_1,t_2,t_3] + \lambda([Nt_1,t_2,t_3] + [t_1,Nt_2,t_3]\\ &+ [t_1,t_2,Nt_3]) + \lambda^2([Nt_1,Nt_2,t_3]\\ &+ [Nt_1,t_2,Nt_3] + [t_1,Nt_2,Nt_3])\\ &+ \lambda^3[Nt_1,Nt_2,Nt_3].\end{aligned} \quad (45)$$

Thus, we have

$$\begin{aligned}\psi(t_1,t_2,t_3) &= [Nt_1,t_2,t_3] + [t_1,Nt_2,t_3] + [t_1,t_2,Nt_3]\\ &- N[t_1,t_2,t_3] = \theta(t_2,t_3)N(t_1)\\ &- \delta\theta(t_1,t_3)N(t_2) + \delta D(t_1,t_2)N(t_3)\\ &- N[t_1,t_2,t_3],\end{aligned} \quad (46)$$

$$N\psi(t_1,t_2,t_3) = [Nt_1,Nt_2,t_3] + [Nt_1,t_2,Nt_3] + [t_1,Nt_2,Nt_3], \quad (47)$$

$$0 = [Nt_1,Nt_2,Nt_3]. \quad (48)$$

By the cohomologies discussed in Section 2, equation (46) can be represented in terms of 1-coboundary as $\psi = d^1_{\mathrm{hom}}N$. Furthermore, the following condition holds for N by (46) and (47):

$$\begin{aligned}N^2[t_1,t_2,t_3] &= N[Nt_1,t_2,t_3] + N[t_1,Nt_2,t_3] + N[t_1,t_2,Nt_3]\\ &- ([Nt_1,Nt_2,t_3] + [Nt_1,t_2,Nt_3] + [t_1,Nt_2,Nt_3]).\end{aligned} \quad (49)$$

In the following, we denote by

$$\psi(t_1,t_2,t_3) = [t_1,t_2,t_3]_N, \quad (50)$$

then, (47) is equivalent to 51

$$N[t_1,t_2,t_3]_N = [Nt_1,Nt_2,t_3] + [Nt_1,t_2,Nt_3] + [t_1,Nt_2,Nt_3]. \quad (51)$$

Definition 9. A linear operator $N : T \longrightarrow T$ is said to be a Nijenhuis operator if and only if (48) and (49) hold.

Theorem 10. *Let N be a Nijenhuis operator for T. Then, a deformation of T can be obtained if*

$$\begin{aligned}\psi(t_1,t_2,t_3) &= \theta(t_2,t_3)N(t_1) - \delta\theta(t_1,t_3)N(t_2)\\ &+ \delta D(t_1,t_2)N(t_3) - N[t_1,t_2,t_3].\end{aligned} \quad (52)$$

Moreover, ψ is a trivial deformation.

Proof. Clearly, $\psi = dN$ and $d\psi = d^2N = 0$. Then, ψ is a 3-cocycle of T. In the following, we show that (4) holds for ψ. By (46), (50), and (51), we have

$$\begin{aligned}\psi(\alpha(t_1),\alpha(t_2),\psi(r_1,r_2,r_3)) &= [\alpha(t_1),\alpha(t_2),[Nr_1,r_2,r_3]\\ &+ [r_1,Nr_2,r_3] + [r_1,r_2,Nr_3] - N[r_1,r_2,r_3]]_N\\ &= [\alpha(t_1),\alpha(t_2),[Nr_1,r_2,r_3]]_N + [\alpha(t_1),\alpha(t_2),[r_1,Nr_2,r_3]]_N\\ &+ [\alpha(t_1),\alpha(t_2),[r_1,r_2,Nr_3]]_N - [\alpha(t_1),\alpha(t_2),N[r_1,r_2,r_3]]_N\\ &= [N\alpha(t_1),\alpha(t_2),[Nr_1,r_2,r_3]] + [\alpha(t_1),N\alpha(t_2),[Nr_1,r_2,r_3]]\\ &+ [\alpha(t_1),\alpha(t_2),N[Nr_1,r_2,r_3]] - N[\alpha(t_1),\alpha(t_2),[Nr_1,r_2,r_3]]\\ &+ [N\alpha(t_1),\alpha(t_2),[r_1,Nr_2,r_3]] + [\alpha(t_1),N\alpha(t_2),[r_1,Nr_2,r_3]]\\ &+ [\alpha(t_1),\alpha(t_2),N[r_1,Nr_2,r_3]] - N[\alpha(t_1),\alpha(t_2),[r_1,Nr_2,r_3]]\\ &+ [N\alpha(t_1),\alpha(t_2),[r_1,r_2,Nr_3]] + [\alpha(t_1),N\alpha(t_2),[r_1,r_2,Nr_3]]\\ &+ [\alpha(t_1),\alpha(t_2),N[r_1,r_2,Nr_3]] - N[\alpha(t_1),\alpha(t_2),[r_1,r_2,Nr_3]]\\ &- [N\alpha(t_1),\alpha(t_2),N[r_1,r_2,r_3]] - [\alpha(t_1),N\alpha(t_2),N[r_1,r_2,r_3]]\\ &- \left[\alpha(t_1),\alpha(t_2),N^2[r_1,r_2,r_3]\right] + N[\alpha(t_1),\alpha(t_2),N[r_1,r_2,r_3]]\end{aligned}$$

$$
\begin{aligned}
&= [N\alpha(t_1), \alpha(t_2), [Nr_1, r_2, r_3]] + [\alpha(t_1), N\alpha(t_2), [Nr_1, r_2, r_3]]\\
&\quad - N[\alpha(t_1), \alpha(t_2), [Nr_1, r_2, r_3]] + [N\alpha(t_1), \alpha(t_2), [r_1, Nr_2, r_3]]\\
&\quad + [\alpha(t_1), N\alpha(t_2), [r_1, Nr_2, r_3]] - N[\alpha(t_1), \alpha(t_2), [r_1, Nr_2, r_3]]\\
&\quad + [N\alpha(t_1), \alpha(t_2), [r_1, r_2, Nr_3]] + [\alpha(t_1), N\alpha(t_2), [r_1, r_2, Nr_3]]\\
&\quad - N[\alpha(t_1), \alpha(t_2), [r_1, r_2, Nr_3]] - [N\alpha(t_1), \alpha(t_2), N[r_1, r_2, r_3]]\\
&\quad - [\alpha(t_1), N\alpha(t_2), N[r_1, r_2, r_3]] + N[\alpha(t_1), \alpha(t_2), N[r_1, r_2, r_3]]\\
&\quad + [\alpha(t_1), \alpha(t_2), [Nr_1, Nr_2, r_3] + [\alpha(t_1), \alpha(t_2), [Nr_1, r_2, Nr_3]\\
&\quad + [\alpha(t_1), \alpha(t_2), [r_1, Nr_2, Nr_3]].
\end{aligned}
\tag{53}
$$

Similarly, a direct computation shows that

$$
\begin{aligned}
&\psi(\psi(t_1, t_2, r_1), \alpha(r_2), \alpha(r_3))\\
&\quad = [[Nt_1, Nt_2, r_1], \alpha(r_2), \alpha(r_3)] + [[Nt_1, t_2, Nr_1], \alpha(r_2), \alpha(r_3)]\\
&\qquad + [[t_1, Nt_2, Nr_1], \alpha(r_2), \alpha(r_3)] + [[Nt_1, t_2, r_1], N\alpha(r_2), \alpha(r_3)]\\
&\qquad + [[Nt_1, t_2, r_1], \alpha(r_2), N\alpha(r_3)] - N[[Nt_1, t_2, r_1], \alpha(r_2), \alpha(r_3)]\\
&\qquad + [[t_1, Nt_2, r_1], N\alpha(r_2), \alpha(r_3)] + [[t_1, Nt_2, r_1], \alpha(r_2), N\alpha(r_3)]\\
&\qquad - N[[t_1, Nt_2, r_1], \alpha(r_2), \alpha(r_3)] + [[t_1, t_2, Nr_1], N\alpha(r_2), \alpha(r_3)]\\
&\qquad + [[t_1, t_2, Nr_1], \alpha(r_2), N\alpha(r_3)] - N[[t_1, t_2, Nr_1], \alpha(r_2), \alpha(r_3)]\\
&\qquad - [N[t_1, t_2, r_1], N\alpha(r_2), \alpha(r_3)] - [N[t_1, t_2, r_1], \alpha(r_2), N\alpha(r_3)]\\
&\qquad + N[N[t_1, t_2, r_1], \alpha(r_2), \alpha(r_3)], \psi(\alpha(r_1), \psi(t_1, t_2, r_2), \alpha(r_3))\\
&\quad = [N\alpha(r_1), [Nt_1, t_2, r_2], \alpha(r_3)] + [\alpha(r_1), [Nt_1, t_2, r_2], N\alpha(r_3)]\\
&\qquad - N[\alpha(r_1), [Nt_1, t_2, r_2], \alpha(r_3)] + [N\alpha(r_1), [t_1, Nt_2, r_2], \alpha(r_3)]\\
&\qquad + [\alpha(r_1), [t_1, Nt_2, r_2], N\alpha(r_3)] - N[\alpha(r_1), [t_1, Nt_2, r_2], \alpha(r_3)]\\
&\qquad + [N\alpha(r_1), [t_1, t_2, Nr_2], \alpha(r_3)] + [\alpha(r_1), [t_1, t_2, Nr_2], N\alpha(r_3)]\\
&\qquad - N[\alpha(r_1), [t_1, t_2, Nr_2], \alpha(r_3)] - [N\alpha(r_1), N[t_1, t_2, r_2], \alpha(r_3)]\\
&\qquad - [\alpha(r_1), N[t_1, t_2, r_2], N\alpha(r_3)] + N[\alpha(r_1), N[t_1, t_2, r_2], \alpha(r_3)]\\
&\qquad + [\alpha(r_1), [Nt_1, Nt_2, r_2], \alpha(r_3)] + [\alpha(r_1), [Nt_1, t_2, Nr_2], \alpha(r_3)]\\
&\qquad + [\alpha(r_1), [t_1, Nt_2, Nr_2], \alpha(r_3)],\\
&\delta\psi(\alpha(r_1), \alpha(r_2), \psi(t_1, t_2, r_3))\\
&\quad = \delta[N\alpha(r_1), \alpha(r_2), [Nt_1, t_2, r_3]] + \delta[\alpha(r_1), N\alpha(r_2), [Nt_1, t_2, r_3]]\\
&\qquad - \delta N[\alpha(r_1), \alpha(r_2), [Nt_1, t_2, r_3]] + \delta[N\alpha(r_1), \alpha(r_2), [t_1, Nt_2, r_3]]\\
&\qquad + \delta[\alpha(r_1), N\alpha(r_2), [t_1, Nt_2, r_3]] - \delta N[\alpha(r_1), \alpha(r_2), [t_1, Nt_2, r_3]]\\
&\qquad + \delta[N\alpha(r_1), \alpha(r_2), [t_1, t_2, Nr_3]] + \delta[\alpha(r_1), N\alpha(r_2), [t_1, t_2, Nr_3]]\\
&\qquad - \delta N[\alpha(r_1), \alpha(r_2), [t_1, t_2, Nr_3]] - \delta[N\alpha(r_1), \alpha(r_2), N[t_1, t_2, r_3]]\\
&\qquad - \delta[\alpha(r_1), N\alpha(r_2), N[t_1, t_2, r_3]] + \delta N[\alpha(r_1), \alpha(r_2), N[t_1, t_2, r_3]]\\
&\qquad + \delta[\alpha(r_1), \alpha(r_2), [Nt_1, Nt_2, r_3]] + \delta[\alpha(r_1), \alpha(r_2), [Nt_1, t_2, Nr_3]]\\
&\qquad + \delta[\alpha(r_1), \alpha(r_2), [t_1, Nt_2, Nr_3]].
\end{aligned}
\tag{54}
$$

□

Note that $N \in C^1_\alpha(T)$; by (4), (51), and Theorem 8, it follows that

$$
\begin{aligned}
&\psi(\alpha(t_1), \alpha(t_2), \psi(r_1, r_2, r_3)) - \psi(\psi(t_1, t_2, r_1), \alpha(r_2), \alpha(r_3))\\
&\quad - \psi(\alpha(r_1), \psi(t_1, t_2, r_2), \alpha(r_3))\\
&\quad - \delta\psi(\alpha(r_1), \alpha(r_2), \psi(t_1, t_2, r_3)) = 0.
\end{aligned}
\tag{55}
$$

The conclusion of Theorem 10 is proven.

Remark 11. Let N be a Nijenhuis operator. If $k, m > 0$, then

$$
\begin{aligned}
[t_1, t_2, t_3]_{N^{k+1}} &= \left([t_1, t_2, t_3]_{N^k}\right)_N,\\
\left([t_1, t_2, t_3]_{N^k}\right)_{N^m} &= \left(\left([t_1, t_2, t_3]_{N^k}\right)_N\right)_{N^{m-1}}\\
&= \left([t_1, t_2, t_3]_{N^{k+1}}\right)_{N^{m-1}} = \left([t_1, t_2, t_3]_{N^{k+2}}\right)_{N^{m-2}}\\
&= \cdots = [t_1, t_2, t_3]_{N^{k+m}}.
\end{aligned}
\tag{56}
$$

Remark 12. Let N be a Nijenhuis operator; by mathematical induction and (48), for any $k > 0$, N^k is also a Nijenhuis operator.

Data Availability

The data used to support the finding of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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