Research Article

Characterizations and Entropy Measures of the Exponentiated Generalized Frechet Geometric Distribution

Naz Saud,¹ Aqsa Rafique,¹ Muhammad Ijaz,² Naila Amjad,¹ Mahmoud El-Morshedy ,^{3,4} and Syed Habib Shah⁵

¹Lahore College for Women University, Lahore, Pakistan

²The University of Haripur, Haripur 22630, Pakistan

³Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

⁴Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

⁵Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, KPK, Pakistan

Correspondence should be addressed to Mahmoud El-Morshedy; mah_elmorshedy@mans.edu.eg

Received 22 October 2021; Revised 7 December 2021; Accepted 28 December 2021; Published 4 February 2022

Academic Editor: Bin Wang

Copyright © 2022 Naz Saud et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Some characterizations and entropy measures of the Exponentiated Generalized Fréchet Geometric (EGFG) distribution are studied in this paper. Firstly, characterizations of the EGFG distribution based on five different approaches are discussed. The submodels for the EGFG distribution with their characterization expressions formed on the ratio of two truncated moments are also presented. Secondly, four different entropy measures are considered and expressed analytically via the incomplete gamma function. The behavior of all these entropy measures is discussed by performing a numerical study.

1. Introduction

Characterization is a distributional property of statistics that determines the associated stochastic model. In designing a stochastic model for a particular modeling problem, it is crucial to validate whether the specified probability distribution fulfills the underlying conditions by its characterization before a specified probability model is used to fit data applications. Thus, it will rely on the characterization of the chosen distribution. It plays an essential role in different fields of mathematical and statistical sciences.

Although in several statistical applications, an increase in the number of parameters gives a more appropriate model, and a smaller number of parameters (without affecting the model suitability) is mathematically more attractive in characterization complications. Over the years, many researchers have studied the different techniques of characterization of continuous probability distributions such as Glänzel [1]; Glänzel [2]; Hamedani [3]; Hamedani [4];

Hamedani [5]; Hamedani and Ahsanullah [6]; Hamedani [7]; Bhatti et al. [8]; and Rafique and Saud [9]. Consequently, numerous characterization techniques have been studied in this paper.

Entropy is used to measure the amount of uncertainty, randomness, and disorder in the system. These essential mathematical techniques are used to evaluate the uncertainty of the stochastic variable. To be more specific, the distribution of a random variable is related to some level of uncertainty, which is quantified by entropy.

In any stochastic process, the probability distribution varies over time; hence, it is clear that the entropy or uncertainty of a probability distribution changes over time as well. It becomes knowing how the uncertainty changes over time. The application of entropy measures is useful in many branches of science and disciplines. Several examples from different areas where different entropy concepts were used, e.g., topological entropy, Kolmogorov–Sinai entropy, topological order, can be found in the following references

proposed by Caneco et al. [10], Rocha and Aleixo [11], and Rocha and Carvalho [12].

The essential references in this regard are briefly presented below. Shannon [13] introduced the concept of entropy. Cho et al. [14] estimated the entropy of a Weibull distribution under generalized progressive hybrid censoring. The comparison of different entropy measures was discussed by Dey et al. [15]. Basit et al. [16] studied the entropy measures for weighted and truncated weighted exponential distributions. The entropy measures for the uncertainty quantification of stochastic processes were investigated by Namdari and Li [17], while Ijaz et al. [18] compared different

entropy measures for Lomax distribution using relative loss approach.

This present work deals with the characterization and entropy measures of the univariate continuous distribution named as “Exponentiated Generalized Fréchet Geometric (EGFG) distribution.” The EGFG distribution was proposed by Rafique and Saud [19]. They studied the mathematical properties of the suggested model and also provided its application study.

The probability density function (*pdf*) and cumulative distribution function (*cdf*) of the Exponentiated Generalized Fréchet Geometric distribution with parameters η , κ , ϕ , δ , and $\tau \in (0, 1)$ are defined, respectively, as follows:

$$f(y; \underline{\theta}) = \frac{(1-\tau)\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^2}, \quad y \geq 0, \eta, \kappa, \phi, \delta > 0, \tau \in (0, 1), \quad (1)$$

and

$$F(y; \underline{\theta}) = 1 - \frac{(1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}. \quad (2)$$

$\underline{\theta} \in \mathbb{R}^5$ given by $\underline{\theta} = \{\eta, \kappa, \phi, \delta, \tau\} \in \mathbb{R}^5$ [19].

2. Structural Properties

According to Rafique and Saud [19], the hazard rate, reverse hazard rate, and survival function of the EGFG are defined, respectively, by

$$h(y) = \frac{\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}, \quad (3)$$

$$r(y) = \frac{(1-\tau)\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\left[1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right] - (1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}, \quad (4)$$

$$S(y) = \frac{(1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}. \quad (5)$$

The mills ratio and elasticity function of any probability distribution can be defined as

$$m(y) = \frac{1 - F(y)}{f(y)}, \quad (6)$$

$$e(y) = yr(y).$$

[20] The mills ratio and elasticity function of the EGFG distribution are given by

$$m(y) = \frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}}, \quad (7)$$

$$e(y) = \frac{(1 - \tau) \kappa \delta \phi \eta^\phi y^{-\phi} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\left[1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right] - (1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}. \quad (8)$$

3. Materials and Methods

3.1. Characterization via Ratio of Two Truncated Moments. The EGFG distribution is characterized through ratio of two truncated moments. Glänzel [21] stated that this characterization is stable in the sense of weak convergence.

Proposition 1. Suppose $Y: \Omega \longrightarrow (0, \infty)$ be a continuous random variable with probability density function defined in equation (1) and let

$$k_1(y) = 1, \quad (9)$$

$$k_2(y) = k_1(y) \left\{ \frac{(1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right\}, \quad y > 0, \quad (10)$$

and then, the function Σ which is presented in theorem (Glänzel [1, 2]) has the following expression:

$$\Sigma(y) = \frac{1}{2} \left\{ \frac{(1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right\}, \quad y > 0. \quad (11)$$

Proof. Suppose Y has pdf in equation (1), then

$$(1 - F(y))E(k_1|Y \geq y) = \left\{ \frac{(1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right\}, \quad y > 0, \quad (12)$$

$$(1 - F(y))E(k_2|Y \geq y) = \frac{1}{2} \left\{ \frac{(1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right\}^2,$$

and so

$$\Sigma(y) = \frac{(1-F(y))E(k_2|Y \geq y)}{(1-F(y))E(k_1|Y \geq y)} = \frac{1}{2} \left\{ \frac{(1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)} \right\}, \quad (13)$$

$$\Sigma'(y) = \frac{1}{2} \frac{(1-\tau)\kappa\delta\phi\eta^\phi y^{-\phi} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^2}. \quad (14)$$

As

$$\Sigma(y)k_1(y) - k_2(y) = -\frac{k_1(y)}{2} \left\{ \frac{(1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)} \right\} \neq 0, \quad \forall y, \quad (15)$$

the proof is as follows:

Conversely, if Σ is defined as above, then

$$s'(y) = \frac{\Sigma'(y)k_1(y)}{\Sigma(y)k_1(y) - k_2(y)} = \frac{\Sigma'(y)k_1(y)}{-(1/2)k_2(y)},$$

$$s'(y) = \frac{\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}, \quad (16)$$

and hence,

$$s(y) = -\ln \left\{ \frac{(1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)} \right\}, \quad (17)$$

and

$$e^{-s(y)} = \frac{(1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}, \quad y > 0. \quad (18)$$

Therefore, in the light of theorem, given in Glänzel [1] and Glänzel [21], Y has pdf in equation (1).

Corollary 1. Suppose that the continuous random $Y: \Omega \longrightarrow (0, \infty)$ and let $k_1(y)$ as given in proposition equation (1). The pdf of Y in equation (1) provided functions $k_2(y)$ and Σ defined in the theorem (Glänzel [1]). The theorem is based on truncated moments specified in the ratio form.

$$\frac{\Sigma'(y)k_1(y)}{\Sigma(y)k_1(y) - k_2(y)} = \frac{\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}. \quad (19)$$

Remark 1. The solution of the equation in corollary equation (1) is

$$\Sigma(y) = \frac{1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{(1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)} \int \left\{ \frac{-g(y)(1 - \tau)\kappa\delta\phi\eta^\phi y^{-\phi} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^2} \right\} dy + D, \quad (20)$$

where D is the constant.

3.1.1. Special Cases of the Exponentiated Generalized Fréchet Geometric Distribution. The expressions of $k_1(y)$, $k_2(y)$, and $\Sigma(y)$ to characterize the submodels of the EGFG distribution are given as follows:

(1) Generalized Fréchet Geometric Distribution. Putting $\kappa = 1$, suppose that

$$k_1(y) = 1,$$

$$k_2(y) = k_1(y) \frac{(1 - \tau) \left(1 - e^{-\delta(\eta/y)^\phi} \right)}{1 - \tau \left(1 - e^{-\delta(\eta/y)^\phi} \right)}, \quad y \geq 0, \quad (21)$$

so that Y follows GFG distribution if and only if the function Σ obtained in Proposition 1 has the following expression:

$$\Sigma(y) = \frac{1}{2} \frac{(1 - \tau) \left(1 - e^{-\delta(\eta/y)^\phi} \right)}{1 - \tau \left(1 - e^{-\delta(\eta/y)^\phi} \right)}. \quad (22)$$

(2) Fréchet Geometric Distribution. For $\kappa = \delta = 1$, let

$$k_1(y) = 1,$$

$$k_2(y) = k_1(y) \frac{(1 - \tau) \left(1 - e^{-(\eta/y)^\phi} \right)}{1 - \tau \left(1 - e^{-(\eta/y)^\phi} \right)}, \quad y \geq 0, \quad (23)$$

and if Σ obtained in Proposition 1 has the expression, then Y follows Fréchet Geometric distribution as follows:

$$\Sigma(y) = \frac{1}{2} \frac{(1 - \tau) \left(1 - e^{-(\eta/y)^\phi} \right)}{1 - \tau \left(1 - e^{-(\eta/y)^\phi} \right)}. \quad (24)$$

(3) Inverse Rayleigh Geometric Distribution. With $\kappa = 1$, $\delta = 1$, and $\phi = 2$, we get

$$k_1(y) = 1,$$

$$k_2(y) = k_1(y) \frac{(1 - \tau) \left(1 - e^{-(\eta/y)^2} \right)}{1 - \tau \left(1 - e^{-(\eta/y)^2} \right)}, \quad y \geq 0, \quad (25)$$

and then, r.v. Y follows IRG distribution if the function Σ has the following expression:

$$\Sigma(y) = \frac{1}{2} \frac{(1 - \tau) \left(1 - e^{-(\eta/y)^2} \right)}{1 - \tau \left(1 - e^{-(\eta/y)^2} \right)}. \quad (26)$$

(4) Inverse Exponential Distribution. For $\kappa = \delta = \phi = 1$, we have

$$k_1(y) = 1,$$

$$k_2(y) = k_1(y) \frac{(1 - \tau) \left(1 - e^{-(\eta/y)^1} \right)}{1 - \tau \left(1 - e^{-(\eta/y)^1} \right)}, \quad y \geq 0, \quad (27)$$

and

$$\Sigma(y) = \frac{1}{2} \frac{(1 - \tau) \left(1 - e^{-(\eta/y)^1} \right)}{1 - \tau \left(1 - e^{-(\eta/y)^1} \right)}. \quad (28)$$

3.2. Characterization of EGFG under Hazard Rate Function. In this subsection, the characterization of the EGFG distribution is obtained by hazard rate function.

Definition 1. Suppose the absolutely continuous random variable “ Y ” with $pdf f(y)$ and $cdf F(y)$. The hazard function to F is symbolized as h_F and is obtained by

$$h_F(y) = \frac{f(y)}{1 - F(y)}, \quad y \in \text{Supp } F, \quad (29)$$

where F represents twice “differentiable distribution function.” The hazard function satisfies the following differential equation:

$$\frac{h'_F(y)}{h_F(y)} - h_F(y) = \frac{f'(y)}{f(y)} = \sigma(y), \quad (30)$$

where $\sigma(y)$ is an appropriate integrable function. The purpose here is to define a differential equation that has a

simple form as possible and a nontrivial form as in equation (30). However, in some continuous distributions, this may be impossible.

Proposition 2. *Let the random variable $Y: \Omega \in (0, \infty)$ follows the EGFG distribution. The density function of “ Y ” is equation (1) if and only if its hazard rate function $h_F(y)$ satisfies the following differential equation:*

$$\begin{aligned} h'_F(y) - (-\phi - 1)y^{-1}h_F(y) &= \tau^2 \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \\ &\frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \left[(\tau\delta)^{-1} (\delta - 1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\ &\frac{\phi \eta^\phi y^{-(\phi+1)} - (\kappa - 1) \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1}}{\tau \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]} \\ &\left. 2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} + \frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right]. \end{aligned} \quad (31)$$

Proof. If the r.v Y has the hrf given in equation (3), then

$$\begin{aligned} h'_F(y) &= \tau^2 \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \\ &\frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \left[(\tau\delta)^{-1} (\delta - 1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\ &\frac{- (\phi + 1) y^{-1} + \phi \eta^\phi y^{-(\phi+1)} - (\kappa - 1) \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1}}{\tau \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]} \\ &\left. - 2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} + \frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right], \\ h'_F(y) - (-\phi - 1)y^{-1}h_F(y) &= \tau^2 \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \end{aligned}$$

$$\begin{aligned}
& \frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \left[(\tau\delta)^{-1} (\delta - 1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
& \left. \frac{\phi\eta^\phi y^{-(\phi+1)} - (\kappa - 1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1}}{\tau \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]} \right. \\
& \left. - 2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} + \frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right], \\
\frac{d}{dy} [y^{\phi+1} h_F(y)] &= \frac{d}{dy} \left[\frac{\kappa\delta\phi\eta^\phi e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)} \right], \tag{32}
\end{aligned}$$

and simplification results in equation (3).

3.3. Characterization of EGFG under Reverse Hazard Rate Function. In this subsection, the EGFG distribution is characterized via the reverse hazard rate function.

Definition 2. Suppose the absolutely continuous random variable “Y” with *pdf* equation (1) and *cdf* equation (2). The reverse hazard rate function to F is symbolized as r_F and is given by

$$r_F(y) = \frac{f(y)}{F(y)}, \quad y \in \text{Supp}F, \tag{33}$$

where F represents the twice “differentiable distribution function.” The reverse hazard function satisfies the following differential equation:

$$\frac{f'(y)}{f(y)} = \frac{r_F'(y)}{r_F(y)} + r_F(y). \tag{34}$$

Proposition 3. Let the random variable $Y: \Omega \in (0, \infty)$ follows the EGFG distribution. The density function of Y is equation (1) if and only if its reverse hazard function $r_F(y)$ satisfies the following differential equation:

$$\begin{aligned}
r_F'(y) - (-\phi - 1)y^{-1}r_F(y) &= \tau^2 \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \\
& \tau(1 - \tau) - \frac{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \left[(\tau\delta)^{-1} (\delta - 1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
& \left. + \frac{\phi\eta^\phi y^{-(\phi+1)} - (\kappa - 1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1}}{\tau \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]} \right]
\end{aligned}$$

$$\begin{aligned}
& -2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-1} \\
& \frac{\left(\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-2} / \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)}{\tau(1-\tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)} \quad (35)
\end{aligned}$$

Proof. If the r.v Y has density function in equation (1), then surely equation (35) holds.

$$\begin{aligned}
r'_F(y) &= \tau^2 \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1} \right]^2 \\
& \tau(1-\tau) - \frac{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)} \left[(\tau\delta)^{-1} (\delta-1) \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{-\delta} + \right. \\
& \frac{-(\phi+1)y^{-1} + \phi\eta^\phi y^{-(\phi+1)} - (\kappa-1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{-1}}{\tau \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1} \right]} \\
& \left. -2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-1} \right. \\
& \left. \frac{\left(\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-2} / \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)}{\tau(1-\tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)} \right]
\end{aligned}$$

$$\begin{aligned}
r'_F(y) - (-\phi-1)y^{-1}r_F(y) &= \tau^2 \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1} \right]^2 \\
& \tau(1-\tau) - \frac{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)} \left[(\tau^{-1}\delta)(\delta-1) \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{-\delta} + \right. \\
& \frac{+\phi\eta^\phi y^{-(\phi+1)} - (\kappa-1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{-1}}{\tau \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1} \right]}
\end{aligned}$$

$$\begin{aligned}
& -2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-1} \\
& \left(\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-2} / \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right) \Bigg] \\
& - \frac{\tau(1-\tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)}{\tau(1-\tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)}, \\
\frac{d}{dy} [y^{(\phi+1)} r(y)] = & \\
\frac{d}{dy} \left[\frac{(1-\tau)\kappa\delta\phi\eta^\phi e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1} \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-1}}{\left[1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right] - (1-\tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)} \right], & (36)
\end{aligned}$$

and simplification results in equation (4).

3.4. Characterization of EGFG under Mills Ratio. In this subsection, the EGFG distribution is characterized via mills ratio.

Definition 3. Suppose the absolutely continuous random variable “ Y ” with $pdf f(y)$ and $cdf F(y)$. The mills ratio to F is symbolized as m_F and is obtained by

$$m_F(y) = \frac{1 - F(y)}{f(y)}, \quad y \in \text{Supp}F, \quad (37)$$

where F represents the twice “differentiable distribution function.” The mills ratio satisfies the following differential equation:

$$\frac{f'(y)}{f(y)} = \sigma(y) = - \left[\frac{m'_F(y)}{m_F(y)} + \frac{1}{m_F(y)} \right], \quad (38)$$

where $\sigma(y)$ is an appropriate integrable function.

Proposition 4. Let the random variable $Y: \Omega \in (0, \infty)$ follows the EGFG distribution. The density function of Y is equation (1) if and only if its mills ratio $m_F(y)$ defined in equation (7) satisfies the following differential equation:

$$\begin{aligned}
y^{-(\phi+1)} m'(y) - (\phi+1) y^{-(\phi+2)} m(y) = & -y^{-(\phi+1)} \frac{\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-1}} \\
& \left[-2 \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-\left(\frac{\eta}{y}\right)^\phi} \right)^\kappa \right]^\delta \right) \right)^{-1} + (\tau^{-1}\delta)(\delta-1) \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{-\delta} + \right. \\
& \left. \frac{\phi\eta^\phi y^{-(\phi+1)} - (\kappa-1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{-1}}{\tau \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi} \right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi} \right)^\kappa \right]^{\delta-1} \right]} \right] y^{-(\phi+1)}. & (39)
\end{aligned}$$

Proof. If the r.v Y has density function equation (1), then surely equation (7) holds.

$$\begin{aligned}
 m'(y) &= -\frac{\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\left(1 - \tau\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}} \\
 &\quad \left[-2\left(1 - \tau\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} + (\tau^{-1}\delta)(\delta - 1)\left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
 &\quad \left. -(\phi + 1)y^{-1} + \phi\eta^\phi y^{-(\phi+1)} - (\kappa - 1)\phi\eta^\phi y^{-(\phi+1)}e^{-(\eta/y)^\phi}\left(1 - e^{-(\eta/y)^\phi}\right)^{-1}\right] \\
 &\quad \left[\tau\left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)}e^{-(\eta/y)^\phi}\left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1}\left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}\right]\right]^{-1} - 1, \\
 y^{-(\phi+1)}m'(y) - (\phi + 1)y^{-(\phi+2)}m(y) &= -y^{-(\phi+1)}\frac{\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\left(1 - \tau\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}} \\
 &\quad \left[-2\left(1 - \tau\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} + (\tau^{-1}\delta)(\delta - 1)\left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
 &\quad \left. \phi\eta^\phi y^{-(\phi+1)} - (\kappa - 1)\phi\eta^\phi y^{-(\phi+1)}e^{-(\eta/y)^\phi}\left(1 - e^{-(\eta/y)^\phi}\right)^{-1}\right] \\
 &\quad \left[\tau\left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)}e^{-(\eta/y)^\phi}\left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1}\left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}\right]\right]^{-1} - y^{-(\phi+1)}, \\
 \frac{d}{dy}[m_F(y)y^{-(\phi+1)}] &= \frac{d}{dy}\left[\frac{\left(1 - \tau\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)\left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\kappa\delta\phi\eta^\phi e^{-(\eta/y)^\phi}\left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1}\left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}}\right],
 \end{aligned} \tag{40}$$

and simplification results in equation (7). $\square$

3.5. Characterization of EGFG under Elasticity Function. In this subsection, the EGFG distribution is characterized through the elasticity function.

Definition 4. Suppose the absolutely continuous random variable “ Y ” with $pdf f(y)$ and $cdf F(y)$. The elasticity function to F is symbolized as e_F and is obtained by

$$e_F(y) = \frac{yf'(y)}{F(y)}, \quad y \in \text{Supp}F, \tag{41}$$

where F represents the twice “differentiable distribution function.” The elasticity function satisfies the following equation:

$$\frac{f'(y)}{f(y)} = \frac{e_F'(y)}{e_F(y)} + \frac{e_F(y)}{Y} - \frac{1}{Y}. \tag{42}$$

Proposition 5. Let the random variable $Y: \Omega \in (0, \infty)$ follows the EGFG distribution. The density function of Y is equation (1) if and only if its elasticity function $e_F(y)$ satisfies the following differential equation:

$$\begin{aligned}
e'_F(y) - y^{-1}e_F(y) &= y\tau^2 \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \\
&\quad - \frac{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \left[\left(\tau^{-1}\delta\right)(\delta-1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
&\quad \left. \frac{-(\phi+1)y^{-1} + \phi\eta^\phi y^{-(\phi+1)} - (\kappa-1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1}}{\tau \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]} \right. \\
&\quad \left. - 2y \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} \right. \\
&\quad \left. - y \frac{\left(\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-2} / \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)}{\tau(1-\tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \right].
\end{aligned} \tag{43}$$

Proof. If the r.v Y has density function equation (1), then surely equation (43) holds.

$$\begin{aligned}
e'_F(y) &= y\tau^2 \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \\
&\quad - \frac{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \left[\left(\tau^{-1}\delta\right)(\delta-1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
&\quad \left. \frac{-(\phi+1)y^{-1} + \phi\eta^\phi y^{-(\phi+1)} - (\kappa-1)\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1}}{\tau \left[\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]} \right. \\
&\quad \left. - 2y \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} \right. \\
&\quad \left. - y \frac{\left(\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-2} / \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)}{\tau(1-\tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \right]_+ \\
&\quad + (1-\tau)\kappa\delta\phi\eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1}
\end{aligned}$$

$$\begin{aligned}
& \frac{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\left[1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right) - (1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right]}, \\
e_F'(y) - y^{-1} e_F(y) &= y \tau^2 \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]^2 \\
& \quad - \frac{\tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)}{\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \left[(\tau^{-1} \delta) (\delta - 1) \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{-\delta} + \right. \\
& \quad \left. - (\phi + 1) y^{-1} + \phi \eta^\phi y^{-(\phi+1)} - (\kappa - 1) \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{-1} \right. \\
& \quad \left. - \frac{\tau \left[\kappa \delta \phi \eta^\phi y^{-(\phi+1)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \right]}{\left[1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right) - (1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right]} \right. \\
& \quad \left. - 2y \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1} \right. \\
& \quad \left. - y \frac{\left(\left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-2} / \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)}{\tau (1 - \tau) - \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)} \right]. \tag{44}
\end{aligned}$$

Now, the result is as follows: conversely if equation (43) holds, then

$$e_F(y) = \frac{(1 - \tau) \kappa \delta \phi \eta^\phi y^{-(\phi)} e^{-(\eta/y)^\phi} \left(1 - e^{-(\eta/y)^\phi}\right)^{\kappa-1} \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^{\delta-1} \left(1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right)^{-1}}{\left[1 - \tau \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right) - (1 - \tau) \left(1 - \left[1 - \left(1 - e^{-(\eta/y)^\phi}\right)^\kappa\right]^\delta\right)\right]}, \tag{45}$$

which is the elasticity function of the EGFG distribution.

4. Entropy of the Exponentiated Generalized Fréchet Geometric Distribution

Entropy measures have a strength of prediction. Different studies of time series may influence the predictive capacity of entropy measures. According to Liang et al. [22], time series analysis affects many entropy methods. Different entropy methods for measuring the complexity of time series were studied by Chen et al. [23]. According to Yin and Shang [24], entropy is a useful tool for examining time series because it does not include any constraints on the probability distribution. Solís-Montufar et al. [25] stated that several entropy measures can be used for complex time series. Its value may be positive, negative, or zero. A higher value shows randomness, while zero shows the certainty of information.

4.1. An Integral Result. The following results indicate that some integrals, including the density of the EGFG distribution, may be defined in terms of the incomplete gamma function.

Proposition 6. Let $\omega > 0$, $f(y, \underline{\theta})$, be specified by equation (1) and

$$I_\omega(\underline{\theta}) = \int_0^\infty f(y; \underline{\theta})^\omega dy. \tag{46}$$

Then, $I_\omega(\underline{\theta})$ exists if and only if $\min(\underline{\theta}) > \max(1 - (1/\omega), 0)$, and it is defined as

$$\begin{aligned}
I_\omega(\underline{\theta}) &= \frac{1}{\phi} \left((\omega + k)^{(1/\phi)} \eta \right)^{-\omega(\phi+1)-1} Y_{mkji} \Gamma \\
& \quad \left(\frac{1}{\phi} \{ \omega(\phi + 1) - 1 \} \right), \tag{47}
\end{aligned}$$

where $Y_{mkji} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (s+b-1 \ b-1) \tau^m ((1-\tau)\kappa\delta\phi\eta^\phi)^\omega (-1)^{i+j+k} \binom{\kappa(\omega+j)-\omega}{k} (\delta(i+\omega)-\omega \ j) \binom{m}{i}$.

Proof. Owing to equation (1), we have

$$I_\omega(\underline{\theta}) = \int_0^\infty \frac{((1-\tau)\kappa\delta\phi\eta^\phi)^\omega y^{-\omega(\phi+1)} e^{-\omega(\eta/y)^\phi} (1 - e^{-(\eta/y)^\phi})^{\omega(\kappa-1)} \left[1 - (1 - e^{-(\eta/y)^\phi})^\kappa\right]^{\omega(\delta-1)}}{\left(1 - \tau \left(1 - \left[1 - (1 - e^{-(\eta/y)^\phi})^\kappa\right]^\delta\right)\right)^{-2\omega}} dy. \quad (48)$$

For any positive real number b and for $|w| < 1$, we have the binomial and generalized binomial expansion:

$$(1-w)^{b-1} = \sum_{s=0}^{\infty} \binom{b-1}{s} (-1)^s w^s, \quad (49)$$

$$(1-w)^{-b} = \sum_{s=0}^{\infty} \binom{s+b-1}{b-1} w^s. \quad (50)$$

Applying equations (50) and (49), we get

$$\begin{aligned} I_\omega(\underline{\theta}) &= Y_{mkji} \int_0^\infty y^{-\omega(\phi+1)} e^{-(\omega+k)(\eta/y)^\phi} dy, \\ &= \frac{1}{\phi} ((\omega+k)^{(1/\phi)} \eta)^{-\omega(\phi+1)-1} Y_{mkji} \Gamma \\ &\quad \left(\frac{1}{\phi} \{ \omega(\phi+1) - 1 \} \right). \end{aligned} \quad (51)$$

In this study, the interest of Proposition 6 is that $I_\omega(\underline{\theta})$ is the major ingredient in the definitions of various entropy measures of the EGFG distribution as derived in the next part.

4.2. Various Entropy Measures. In this subsection, some important entropy measures are studied using different

techniques. The literature contains many entropies that are defined in Table 1 for the general distribution with $pdf f(k; \theta)$. Suppose that $\omega > 0$ and $\omega \neq 0$ as basic assumptions in this general case.

From Table 2, we see that the integral $\int_{-\infty}^{\infty} f(k; \theta)^\omega dk$ is central to obtain the considered entropy measures. Now, we study the entropy measures of the EGFG distribution based on Proposition 6.

4.2.1. Rényi Entropy. Based on Table 2, equation (1), and Proposition 6, the Rényi entropy of the EGFG distribution can be defined as

$$\begin{aligned} R_\omega(\theta) &= \frac{1}{1-\omega} \log[I_\omega(\theta)], \\ &= \frac{1}{1-\omega} \log \\ &\quad \left[\frac{1}{\phi} ((\omega+k)^{(1/\phi)} \eta)^{-\omega(\phi+1)+1} Y_{mkji} \Gamma \left(\frac{1}{\phi} \{ \omega(\phi+1) - 1 \} \right) \right]. \end{aligned} \quad (52)$$

4.2.2. Havrda and Charvát Entropy. From Table 2, equation (1), and Proposition 6, the Havrda and Charvát entropy of the EGFG distribution is specified by

$$\begin{aligned} HC_\omega(\theta) &= \frac{1}{2^{1-\omega}-1} [I_\omega(\theta) - 1], \\ &= \frac{1}{2^{1-\omega}-1} \left[\frac{1}{\phi} ((\omega+k)^{(1/\phi)} \eta)^{-\omega(\phi+1)+1} Y_{mkji} \Gamma \left(\frac{1}{\phi} \{ \omega(\phi+1) - 1 \} \right) - 1 \right]. \end{aligned} \quad (53)$$

4.2.3. Arimoto Entropy. Again, from Table 2, equation (1), and Proposition 6, the Arimoto entropy of the EGFG distribution is expressed by

TABLE 1: Submodels from the exponentiated generalized frchet geometric (EGFG) distribution.

Sr.No.	γ	ϕ	κ	δ	τ	Models	$p \, df$
1	$\eta > 0$	$\phi > 0$	$\kappa = 1$	$\delta > 0$	$0 < \tau < 1$	GFG	$\frac{(1-\tau)\delta\phi\eta^\phi y^{-(\phi+1)}e^{-\delta(\eta/y)^\phi}}{(1-\tau(1-e^{-\delta(\eta/y)^\phi}))^2}$
2	$\eta > 0$	$\phi > 0$	$\kappa = 1$	$\delta = 1$	$0 < \tau < 1$	FG	$\frac{(1-\tau)\phi\eta^\phi y^{-(\phi+1)}e^{-(\eta/y)^\phi}}{(1-\tau(1-e^{-(\eta/y)^\phi}))^2}$
3	$\eta > 0$	$\phi = 2$	$\kappa = 1$	$\delta = 1$	$0 < \tau < 1$	IRG	$\frac{2(1-\tau)\eta^2 y^{-(2+1)}e^{-(\eta/y)^2}}{(1-\tau(1-e^{-(\eta/y)^2}))^2}$
4	$\eta > 0$	$\phi = 1$	$\kappa = 1$	$\delta = 1$	$0 < \tau < 1$	IEG	$\frac{(1-\tau)\delta\eta y^{-2}e^{-\delta(\eta/y)}}{(1-\tau(1-e^{-\delta(\eta/y)}))^2}$

TABLE 2: Important entropy measures of a distribution with $pdf f(k; \theta)$ at ω .

Name of the entropy	Reference	Notation	Expression
R�nyi	R�nyi [26]	$R_\omega(\theta)$	$(1/1-\omega)\log[\int_{-\infty}^{\infty} f(k; \theta)^\omega dk]$
Havrda and Charv�t	Havrda and Charv�t [27]	$HC_\omega(\theta)$	$(1/2^{1-\omega}-1)[\int_{-\infty}^{\infty} f(k; \theta)^\omega dk - 1]$
Arimoto	Arimoto [28]	$A_\omega(\theta)$	$(\omega/1-\omega)\{[\int_{-\infty}^{\infty} f(k; \theta)^\omega dk]^{1/\omega} - 1\}$
Tsallis	Tsallis [29]	$T_\omega(\theta)$	$(1/\omega-1)[1 - \int_{-\infty}^{\infty} f(k; \theta)^\omega dk]$

TABLE 3: Numerical values of the entropy measures of the EGFG distribution $\eta = 2$, $\delta = 3$, $\tau = 0.8$, and $\phi = 2$.

η	κ	δ	τ	ϕ	ω	$R_\omega(\theta)$	$HC_\omega(\theta)$	$A_\omega(\theta)$	$T_\omega(\theta)$
2	3	3	0.8	2	0.5	0.391	1.374	1.462	1.138
	2.9					0.406	1.441	1.549	1.193
	2.5					0.477	1.762	1.997	1.462
2	3	3	0.8	2	1.5	0.120	0.442	0.265	0.259
	2.9					0.132	0.480	0.288	0.281
	2.5					0.182	0.646	0.391	0.378
2	3	3	0.8	2	2	0.077	0.326	0.170	0.163
	2.9					0.088	0.368	0.193	0.184
	2.5					0.137	0.542	0.292	0.271

$$\begin{aligned}
A_\omega(\theta) &= \frac{\omega}{1-\omega} [I_\omega(\theta)^{1/\omega} - 1], \\
&= \frac{\omega}{1-\omega} \left[\left\{ \frac{1}{\phi} ((\omega+k)^{(1/\phi)} \eta)^{-\omega(\phi+1)+1} Y_{mkji} \Gamma\left(\frac{1}{\phi} \{\omega(\phi+1)-1\}\right) \right\}^{1/\omega} - 1 \right].
\end{aligned} \tag{54}$$

4.2.4. *Tsallis Entropy.* Based on Table 2, equation (1), and Proposition 6, the Tsallis entropy of the EGFG distribution can be presented as

$$\begin{aligned}
T_\omega(\theta) &= \frac{1}{\omega-1} [1 - I_\omega(\theta)], \\
&= \frac{1}{\omega-1} \left[1 - \frac{1}{\phi} ((\omega+k)^{(1/\phi)} \eta)^{-\omega(\phi+1)+1} Y_{mkji} \Gamma\left(\frac{1}{\phi} \{\omega(\phi+1)-1\}\right) \right].
\end{aligned} \tag{55}$$

TABLE 4: Numerical values of the entropy measures of the EGFG distribution $\eta = 2.5$, $\kappa = 3$, $\tau = 0.9$, and $\phi = 2.5$.

η	κ	δ	τ	ϕ	ω	$R_\omega(\theta)$	$HC_\omega(\theta)$	$A_\omega(\theta)$	$T_\omega(\theta)$
2.5	3	1.2	0.9	2.5	0.5	0.284	0.932	0.921	0.772
		2.2				0.304	1.012	1.014	0.838
		3.2				0.316	1.058	1.068	0.876
2.5	3	1.2	0.9	2.5	1.5	0.024	0.095	0.056	0.055
		2.2				0.043	0.164	0.097	0.096
		3.2				0.052	0.200	0.118	0.117
2.5	3	1.2	0.9	2.5	2.5	-0.046	-0.265	-0.109	-0.114
		2.2				-0.028	-0.154	-0.065	-0.067
		3.2				-0.018	-0.100	-0.042	-0.043

TABLE 5: Numerical values of the entropy measures of the EGFG distribution $\eta = 3$, $\kappa = 3$, $\delta = 1.2$, and $\phi = 2$.

η	κ	δ	τ	ϕ	ω	$R_\omega(\theta)$	$HC_\omega(\theta)$	$A_\omega(\theta)$	$T_\omega(\theta)$
3	3	1.2	0.5	2	0.5	0.620	2.514	3.166	2.082
			0.6			0.595	2.375	2.935	1.967
			0.8			0.516	1.958	2.279	1.622
3	3	1.2	0.5	2	1.5	0.376	1.199	0.752	0.702
			0.6			0.345	1.119	0.698	0.655
			0.8			0.248	0.847	0.519	0.496
3	3	1.2	0.5	2	2.5	0.310	1.016	0.580	0.483
			0.6			0.277	0.954	0.531	0.411
			0.8			0.176	0.706	0.360	0.304

TABLE 6: Numerical values of the entropy measures of the EGFG distribution $\eta = 3.5$, $\delta = 3.9$, $\delta = 1.5$, and $\tau = 0.7$.

η	κ	δ	τ	ϕ	ω	$R_\omega(\theta)$	$HC_\omega(\theta)$	$A_\omega(\theta)$	$T_\omega(\theta)$
3.5	3.9	1.5	0.7	1.2	0.5	0.724	3.145	4.302	2.605
				2.2		0.511	1.935	2.245	1.603
				3.2		0.374	1.298	1.364	1.075
3.5	3.9	1.5	0.7	1.2	1.5	0.418	1.305	0.824	0.764
				2.2		0.289	0.968	0.598	0.567
				3.2		0.176	0.626	0.379	0.367
3.5	3.9	1.5	0.7	1.2	2.5	0.342	1.073	0.628	0.462
				2.2		0.225	0.837	0.446	0.361
				3.2		0.116	0.510	0.246	0.220

TABLE 7: Entropies of the EGFG for different parameter combinations.

η	κ	δ	τ	ϕ	ω	$R_\omega(\theta)$	$HC_\omega(\theta)$	$A_\omega(\theta)$	$T_\omega(\theta)$
2.5	3	3.2	0.85	2.5	1.6	0.093	0.354	0.205	0.201
2.2	2.9	3	0.8	2.3	1.5	0.123	0.451	0.270	0.264
1.3	2.5	2.7	0.5	2	1.49	0.124	0.454	0.273	0.267
1.25	2.2	2.5	0.45	1.8	1.45	0.217	0.752	0.463	0.448
1.2	2.1	1.7	0.3	1.5	0.5	0.652	2.699	3.485	2.236
1.0	1.9	1.5	0.1	1.3	0.5	0.798	3.638	5.284	3.013
1.07	1.9	1.5	0.09	1.3	0.5	0.8	3.653	5.313	3.026
0.8	1.7	1.5	0.09	1.3	0.5	0.818	3.779	5.581	3.131
0.7	1.55	1.35	0.087	1.25	0.47	0.987	5.259	10.614	4.405

4.3. Numerical Results. In this section, the idea of following Al-Babtain et al.'s [30] numerical values for the four different entropy measures of the EGFG distribution is calculated. The findings of all considered entropy measures are presented in Tables 3–7.

Tables 3–7 show the numerical values of the entropy measures for the different sets of the parameters. It can be noticed that,

- (i) Rényi, Havrda and Charvát, Arimoto, and Tsallis entropy measures are increasing when κ and ω are increasing.
- (ii) Rényi, Havrda and Charvát, Arimoto, and Tsallis entropy measures are increasing when δ and ω are increasing.
- (iii) Rényi, Havrda and Charvát, Arimoto, and Tsallis entropy measures are decreasing when τ and ω are increasing.
- (iv) Rényi, Havrda and Charvát, Arimoto, and Tsallis entropy measures are decreasing when ϕ and ω are increasing.
- (v) Rényi, Havrda and Charvát, Arimoto, and Tsallis entropy measures are increasing when all parameter values are decreasing.

5. Conclusion

In this article, the characterization results for the Exponentiated Generalized Fréchet Geometric distribution through five approaches are presented. In the previous section, the analytical expressions of four entropy measures for the EGFG distribution are investigated. All entropy measures are compared numerically by considering the various combination of the parametric values. We hope these expressions of entropies' measures by various techniques may be helpful in uncertainty measures of a random variable of related fields.

Data Availability

The data is generated by using the idea of Al-Babtain et al. [30].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] W. Glänzel, "A characterization theorem based on truncated moments and its application to some distribution families," in *Mathematical Statistics and Probability Theory*, pp. 75–84, Springer, Berlin, Germany, 1987.
- [2] W. Glänzel, "A characterization of the normal distribution," *Studia Scientiarum Mathematicarum Hungarica*, vol. 23, no. 1, pp. 89–91, 1988.
- [3] G. Hamedani, "Characterizations of cauchy, normal, and uniform distributions," *Studia Scientiarum Mathematicarum Hungarica*, vol. 28, no. 3, pp. 243–248, 1993.
- [4] G. Hamedani, "Characterizations of univariate continuous distributions II," *Studia Scientiarum Mathematicarum Hungarica*, vol. 39, no. 3–4, pp. 407–424, 2002.
- [5] G. G. Hamedani and G. G. Hamedani, "Characterizations of univariate continuous distributions III," *Studia Scientiarum Mathematicarum Hungarica*, vol. 43, no. 3, pp. 361–385, 2006.
- [6] G. Hamedani and M. Ahsanullah, "Characterizations of weibull geometric distribution," *Journal of Statistical Theory and Applications*, vol. 10, no. 4, pp. 581–590, 2011.
- [7] D. G. G. Hamedani, "Characterizations of transmuted complementary weibull geometric distribution," *Pakistan Journal of Statistics and Operation Research*, vol. 11, no. 2, pp. 153–157, 2015.
- [8] F. A. Bhatti, G. G. Hamedani, W. Sheng, and M. Ahmad, "On extended quadratic hazard rate distribution: development, properties, characterizations and applications," *Stochastics and Quality Control*, vol. 33, no. 1, pp. 45–60, 2018.
- [9] A. Rafique and N. Saud, "Some characterizations of unimodal, reserve j-shaped distribution," in *Proceedings of the 18th International Conference on Statistical Sciences (ICCS-18)*, pp. 291–340, Lahore, Pakistan, December 2019.
- [10] A. Caneco, J. L. Rocha, and C. Grácio, "Topological entropy in the synchronization of piecewise linear and monotone maps: coupled duffing oscillators," *International Journal of Bifurcation and Chaos*, vol. 19, no. 11, pp. 3855–3868, 2009.
- [11] J. L. Rocha and S. M. Aleixo, "An extension of gompertzian growth dynamics: weibull and fréchet models," *Mathematical Biosciences and Engineering*, vol. 10, no. 2, p. 379, 2013.
- [12] J. L. Rocha and S. Carvalho, "Information theory, synchronization and topological order in complete dynamical networks of discontinuous maps," *Mathematics and Computers in Simulation*, vol. 182, pp. 340–352, 2021.
- [13] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, 1948.
- [14] Y. Cho, H. Sun, and K. Lee, "Estimating the entropy of a weibull distribution under generalized progressive hybrid censoring," *Entropy*, vol. 17, no. 1, pp. 102–122, 2015.
- [15] S. Dey, S. S. Maiti, and M. Ahmad, "Comparison of different entropy measures," *Pakistan Journal of Statistics (PJS)*, vol. 32, no. 2, pp. 97–108, 2016.
- [16] A. Basit, A. Riaz, Z. Iqbal, and M. Ahmad, "On comparison of entropy measures for weighted and truncated weighted exponential distributions," *Advances and Applications in Statistics*, vol. 50, no. 6, pp. 477–495, 2017.
- [17] A. Namdari and Z. Li, "A review of entropy measures for uncertainty quantification of stochastic processes," *Advances in Mechanical Engineering*, vol. 11, no. 6, Article ID 1687814019857350, 2019.
- [18] M. Ijaz, S. Naji AL-Aziz, S. Muhammad Asim, J. G. Dar, and A.-A. H. Abd El-Bagoury, "Comparison of different entropy measures using the relative loss approach for the lomax distribution," *Information Sciences Letters*, vol. 10, no. 3, p. 18, 2021.
- [19] A. Rafique and N. Saud, "Exponentiated generalized fréchet geometric distribution," in *15th Islamic Countries Conference on Statistical Sciences*, pp. 189–198, ICCS-15), 2019.
- [20] F. A. Bhatti and M. Ahmad, "On a new family of kies burr iii distribution: development, properties, characterizations, and applications," *Scientia Iranica*, vol. 27, no. 5, pp. 2555–2571, 2020.
- [21] W. Glänzel, "Some consequences of a characterization theorem based on truncated moments," *Statistics*, vol. 21, no. 4, pp. 613–618, 1990.

- [22] Z. Liang, Y. Wang, X. Sun et al., “Eeg entropy measures in anesthesia,” *Frontiers in Computational Neuroscience*, vol. 9, p. 16, 2015.
- [23] Z. Chen, Y. Li, H. Liang, and J. Yu, “Improved permutation entropy for measuring complexity of time series under noisy condition,” *Complexity*, vol. 2019, Article ID 1403829, 12 pages, 2019.
- [24] Y. Yin and P. Shang, “Weighted permutation entropy based on different symbolic approaches for financial time series,” *Physica A: Statistical Mechanics and Its Applications*, vol. 443, pp. 137–148, 2016.
- [25] E. E. Solís-Montufar, G. Gálvez-Coyt, and A. Muñoz-Diosdado, “Entropy analysis of rr-time series from stress tests,” *Frontiers in Physiology*, vol. 11, p. 981, 2020.
- [26] A. Rényi, “On measures of entropy and information,” in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pp. 547–561, University of California Press, Berkeley, CA, USA, July 1960.
- [27] J. Havrda and F. Charvát, “Quantification method of classification processes: concept of structural α -entropy,” *Kybernetika*, vol. 3, no. 1, pp. 30–35, 1967.
- [28] S. Arimoto, “Information-theoretical considerations on estimation problems,” *Information and Control*, vol. 19, no. 3, pp. 181–194, 1971.
- [29] C. Tsallis, “Possible generalization of boltzmann-gibbs statistics,” *Journal of Statistical Physics*, vol. 52, no. 1, pp. 479–487, 1988.
- [30] A. A. Al-Babtain, I. Elbatal, C. Chesneau, and M. Elgarhy, “Estimation of different types of entropies for the kumaraswamy distribution,” *PLoS One*, vol. 16, no. 3, Article ID e0249027, 2021.