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Research Article

Conformally Flat Pseudoprojective Symmetric Spacetimes in $f(R, \mathcal{G})$ Gravity

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Sufficient conditions on a pseudoprojective symmetric spacetime $(\mathrm{PPS})_n$ whose Ricci tensor is of Codazzi type to be either a perfect fluid or Einstein spacetime are given. Also, it is shown that a $(\mathrm{PPS})_n$ is Einstein if its Ricci tensor is cyclic parallel. Next, we illustrate that a conformally flat $(\mathrm{PPS})_n$ spacetime is of constant curvature. Finally, we investigate conformally flat $(\mathrm{PPS})_4$ spacetimes and conformally flat $(\mathrm{PPS})_4$ perfect fluids in $f(R, \mathcal{G})$ theory of gravity, and amongst many results, it is proved that the isotropic pressure and the energy density of conformally flat perfect fluid $(\mathrm{PPS})_4$ spacetimes are constants and such perfect fluid behaves like a cosmological constant. Further, in this setting, we consider some energy conditions.

1. Introduction

The notion of a pseudoprojective symmetric manifold, briefly denoted by $(\mathrm{PPS})_n$, was first introduced and studied in 1989 by Chaki and Saha [1]. Such a manifold is a nonflat pseudo-Riemannian manifold whose projective curvature tensor [2]

$$\mathcal{P}_{hijk} = R_{hijk} - \frac{1}{n-1}\left[g_{hk}R_{ij} - g_{hj}R_{ik}\right], \tag{1}$$

satisfies the condition

$$\nabla_l \mathcal{P}_{hijk} = 2\lambda_l \mathcal{P}_{hijk} + \lambda_h \mathcal{P}_{lijk} + \lambda_i \mathcal{P}_{hljk} + \lambda_j \mathcal{P}_{hilk} + \lambda_k \mathcal{P}_{hijl}, \tag{2}$$

where R_{hijk} is the Riemann curvature tensor, R_{ij} is the Ricci tensor, λ_l is a nonzero 1-form, and ∇ denotes the covariant differentiation with respect to the metric g. In [1], it was proved that a $(\mathrm{PPS})_n$ manifold is of constant scalar curvature, that is,

$$\nabla_l R = 0, \tag{3}$$

and λ^l is an eigenvector of the Ricci tensor and the corresponding eigenvalue is R/n, that is,

$$\lambda^l R_{lk} = \frac{R}{n}\lambda_k. \tag{4}$$

Also, it was shown that if a $(\mathrm{PPS})_n$ manifold admits a unit parallel vector field, then it is reduced to a pseudosymmetric manifold [3].

An n-dimensional Lorentzian manifold M is said to be a pseudoprojective symmetric spacetime if its projective curvature tensor $\mathcal{P}$ agrees with (2). A Lorentzian manifold M

is said to be perfect fluid if its Ricci tensor satisfies

$$R_{ij} = \alpha g_{ij} + \beta u_i u_j, \tag{5}$$

where α and β are scalar fields and $u_i u_i = -1$, that is, u_i is a time-like velocity vector field [4, 5]. In differential geometry, a manifold satisfying the foregoing relation of the Ricci tensor is called a quasi-Einstein manifold without any restrictions on the velocity vector field u_i [6, 7]. Throughout this paper, let λ^l be a unit timelike vector field.

The standard theory of gravity follows from Einstein's field equations (EFE) [8, 9].

$$R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij}^{(m)}, \tag{6}$$

where R, κ, and $T_{ij}^{(m)}$ are the scalar curvature tensor and the Newtonian gravitational constant, and $T_{ik}^{(m)}$ is the energy-momentum tensor describing the ordinary matter. These equations correlate the geometry of a spacetime with its matter content. That is, the geometry of a spacetime determines the matter content of the spacetime conversely. Many modifications of EFE have been introduced and studied on a large scale (see references [10–12] for examples of the modified gravity theories). Amongst these modified theories, there was one known under the name $f(R, \mathcal{G})$ gravity theory [13], which is obtained by replacing the scalar curvature R with a function $f(R, \mathcal{G})$ of the scalar curvature R and Gauss-Bonnet scalar $\mathcal{G}$.

$$\mathcal{G} = R_{hijk}R^{hijk} - 4R^{hk}R_{hk} + R^2, \tag{7}$$

in the gravitational action term

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R, \mathcal{G}) + S^{\text{matter}}, \tag{8}$$

with S^{matter} being the action term of the standard matter fields. The $f(R, \mathcal{G})$ field equations are given by

$$\begin{aligned}\kappa T_{ik}^{(m)} = &-\frac{1}{2}f g_{ik} + f_{\mathcal{G}}\left(R_{ik}R - 4R_{hilk}R^{hl} + 2R_{ihmn}R_k^{hmn} - 4R_i^h R_{hk}\right)\\ &+ \left(2g_{ik}R\nabla^2 + 4R_i^h\nabla_h\nabla_k + 4R_k^h\nabla_h\nabla_i - 2R\nabla_i\nabla_k - 4R_{ik}\nabla^2\right)f_{\mathcal{G}}\\ &+ \left(4R_{lihk}\nabla^l\nabla^h - 4g_{ik}R_{hl}\nabla^h\nabla^l\right)f_{\mathcal{G}} + \left(R_{ik} - \nabla_i\nabla_k + g_{ik}\nabla^2\right)f_R,\end{aligned} \tag{9}$$

where $T_{ik}^{(m)}$ results from S^{matter} and $f_{\mathcal{G}} = \partial_{\mathcal{G}}f(R, \mathcal{G})$, $f_R = \partial_R f(R, \mathcal{G})$ [14].

In this paper, we investigate $(\text{PPS})_n$ spacetimes whose Ricci tensor is of Codazzi type or cyclic parallel. Next, a conformally flat $(\text{PPS})_n$ spacetime is studied. After that, certain investigations of conformally flat $(\text{PPS})_4$ spacetimes in $f(R, \mathcal{G})$ modified gravity theory are carried out. Finally, we study conformally flat $(\text{PPS})_4$ perfect fluid spacetimes in $f(R, \mathcal{G})$ gravity.

2. On a $(\text{PPS})_n$ Spacetime Whose Ricci Tensor Is of Codazzi Type or Cyclic Parallel

In this section, a $(\text{PPS})_n$ spacetime whose Ricci tensor is of Codazzi type or cyclic parallel is considered. The Ricci tensor R_{ij} is called of Codazzi type if [15, 16]

$$\nabla_l R_{hk} = \nabla_k R_{hl}, \tag{10}$$

whereas R_{ij} is called cyclic parallel if

$$\nabla_k R_{hl} + \nabla_l R_{hk} + \nabla_h R_{lk} = 0. \tag{11}$$

Transvecting equation (1) with g^{ij}, one gets

$$\mathcal{P}_{hk} = \frac{1}{n-1}[nR_{hk} - g_{hk}R], \tag{12}$$

where $\mathcal{P}_{hk} = g^{ij}\mathcal{P}_{hijk}$.

Contracting equation (2) with g^{ij}, we obtain

$$\nabla_l\mathcal{P}_{hk} = 2\lambda_l\mathcal{P}_{hk} + \lambda_h\mathcal{P}_{lk} + \lambda^j\mathcal{P}_{hljk} + \lambda^i\mathcal{P}_{hilk} + \lambda_k\mathcal{P}_{hl}. \tag{13}$$

Using (1) and (12) in (13), we have

$$\begin{aligned}\nabla_l R_{hk} = &\frac{2\lambda_l}{n}[nR_{hk} - g_{hk}R] + \frac{\lambda_h}{n}[nR_{lk} - g_{lk}R]\\ &+ \frac{n-1}{n}\left[\lambda^j R_{hljk} + \lambda^i R_{hilk}\right] - \frac{1}{n}\left[g_{hk}\lambda^j R_{lj} - \lambda_h R_{lk}\right]\\ &- \frac{1}{n}\left[g_{hk}\lambda^i R_{il} - g_{hl}\lambda^i R_{ik}\right] + \frac{\lambda_k}{n}[nR_{hl} - g_{hl}R] + \frac{1}{n}g_{hk}\nabla_l R.\end{aligned} \tag{14}$$

With the help of equations (3) and (4), one finds

$$\begin{aligned}\nabla_l R_{hk} = &\left(\frac{-2n-2}{n^2}\right)R\lambda_l g_{hk} + \left(\frac{1-n}{n^2}\right)R\lambda_k g_{hl}\\ &- \frac{1}{n}R\lambda_h g_{lk} + \lambda_k R_{hl} + 2\lambda_l R_{hk} + \left(\frac{n+1}{n}\right)\lambda_h R_{lk}\\ &+ \left(\frac{n-1}{n}\right)\left[\lambda^j R_{hljk} + \lambda^i R_{hilk}\right].\end{aligned} \tag{15}$$

First, suppose that the Ricci tensor of $(\text{PPS})_n$ spacetime is of Codazzi type; thus, we have

$$\nabla_l R_{hk} - \nabla_k R_{hl} = 0. \tag{16}$$

The use of equation (15) in (16) implies that

$$\begin{aligned}0 = &\left(\frac{-n-3}{n^2}\right)R\lambda_l g_{hk} + \left(\frac{n+3}{n^2}\right)R\lambda_k g_{hl} - \lambda_k R_{hl} + \lambda_l R_{hk}\\ &+ \left(\frac{n-1}{n}\right)\left(\lambda^j R_{hljk} - \lambda^j R_{hkjl} + 2\lambda^i R_{hilk}\right).\end{aligned} \tag{17}$$

It is to be noted that the Riemann curvature tensor has the following properties:

$$R_{hljk} + R_{hklj} + R_{hjkl} = 0,\qquad R_{hkjl} = -R_{hklj}. \tag{18}$$

The use of the above properties of the Riemann curvature tensor in equation (17) implies

$$0 = \left(\frac{-n-3}{n^2}\right) R\lambda_l g_{hk} + \left(\frac{n+3}{n^2}\right) R\lambda_k g_{hl} - \lambda_k R_{hl} + \lambda_l R_{hk} - 3\left(\frac{n-1}{n}\right)\lambda^j R_{hjkl}. \tag{19}$$

Contracting with λ^l and using (4), we have

$$R_{hk} = \left(\frac{n+3}{n^2}\right) R g_{hk} + \left(\frac{3}{n^2}\right) R\lambda_k\lambda_h - 3\left(\frac{n-1}{n}\right)\lambda^l\lambda^j R_{hjkl}. \tag{20}$$

We thus can state the following theorem:

Theorem 1. *Let M be a $(PPS)_n$ spacetime whose Ricci tensor is of Codazzi type; then, the Ricci tensor of M is given by (20).*

Suppose that $\lambda^l\lambda^j R_{hjkl} = 0$, then (20) becomes

$$R_{hk} = \left(\frac{n+3}{n^2}\right) R g_{hk} + \left(\frac{3}{n^2}\right) R\lambda_k\lambda_h, \tag{21}$$

which means that a $(\mathrm{PPS})_n$ spacetime is perfect fluid.

Corollary 2. *Let M be a $(PPS)_n$ spacetime whose Ricci tensor is of Codazzi type. Then, M is perfect fluid if $\lambda^l\lambda^j R_{hjkl} = 0$.*

The conformal curvature tensor is given by [17].

$$C_{hjkl} = R_{hjkl} - \frac{1}{n-2}\left\{g_{hl}R_{jk} + g_{jk}R_{hl} - g_{hk}R_{jl} - g_{jl}R_{hk}\right\} + \frac{R}{(n-1)(n-2)}\left\{g_{hl}g_{jk} - g_{hk}g_{jl}\right\}. \tag{22}$$

A contraction with $\lambda^l\lambda^j$ implies

$$\lambda^j\lambda^l R_{hjkl} = \lambda^j\lambda^l C_{hjkl} + \frac{R\lambda_h\lambda_k}{n(n-1)} - \frac{R g_{hk}}{n(n-1)(n-2)} + \frac{R_{hk}}{n-2}. \tag{23}$$

Equations (20) and (23) are combined to give

$$R_{hk} = \frac{R}{n} g_{hk} - \frac{3(n-1)(n-2)}{(n^2+n-3)}\lambda^j\lambda^l C_{hjkl}, \tag{24}$$

where $C_{hk} = \lambda^j\lambda^l C_{hjkl}$ is the contracted Weyl tensor. Hence, we can state the following theorem:

Theorem 3. *Let M be a $(PPS)_n$ spacetime whose Ricci tensor is of Codazzi type; then, the Ricci tensor of M is of the form (24).*

In particular case, if $C_{hk} = 0$, then equation (24) is reduced to be in the following form:

$$R_{hk} = \frac{R}{n} g_{hk}, \tag{25}$$

which means a $(\mathrm{PPS})_n$ spacetime is Einstein.

Corollary 4. *Let M be a $(PPS)_n$ spacetime whose Ricci tensor is of Codazzi type. Then, M is Einstein if the contracted Weyl tensor vanishes.*

Assume that M has cyclic parallel Ricci tensor, that is, the Ricci tensor agrees with (11). Then, using (15) in (11) infers

$$0 = -\frac{4}{n} R\lambda_k g_{hl} - \left(\frac{4n+1}{n^2}\right) R\lambda_l g_{hk} - \left(\frac{4n+2}{n^2}\right)\lambda_h g_{lk} R + \left(\frac{4n+1}{n}\right)\lambda_l R_{hk} + 4\lambda_k R_{hl} + \left(\frac{4n+2}{n}\right)\lambda_h R_{lk}. \tag{26}$$

Contracting with λ^l and using equation (4), we obtain

$$R_{hk} = \frac{R}{n} g_{hk}, \tag{27}$$

which means a $(\mathrm{PPS})_n$ spacetime whose Ricci tensor obeys (4) is Einstein. Hence, we motivate to state the following theorem:

Theorem 5. *Let M be a $(PPS)_n$ spacetime whose Ricci tensor is cyclic parallel; then M is an Einstein spacetime.*

3. Conformally Flat $(\mathrm{PPS})_n$ Spacetimes

The divergence of the conformal curvature is expressed as [18]

$$\nabla_h \mathscr{C}^h_{ijk} = \frac{n-3}{n-2}\left[\left(\nabla_k R_{ij} - \nabla_j R_{ik}\right) - \frac{1}{2(n-1)}\left(g_{ij}\nabla_k R - g_{ik}\nabla_j R\right)\right]. \tag{28}$$

A spacetime M is called conformally flat if the conformal curvature tensor vanishes, that is, $C_{ijkl} = 0$. It is well-known that if $C_{ijkl} = 0$, then $\nabla_h \mathscr{C}^h_{ijk} = 0$. And consequently, the following equations hold

$$R_{hijk} = \frac{1}{n-2}\left[g_{hk}R_{ij} + g_{ij}R_{hk} - g_{hj}R_{ik} - g_{ik}R_{hj}\right] - \frac{R}{(n-1)(n-2)}\left[g_{hk}g_{ij} - g_{hj}g_{ik}\right], \tag{29}$$

$$\nabla_k R_{hl} - \nabla_l R_{hk} = \frac{1}{2(n-1)} \left(g_{ij} \nabla_k R - g_{ik} \nabla_j R \right). \tag{30}$$

Since in $(\mathrm{PPS})_n$ spacetime the scalar curvature is constant, then equation (30) implies that

$$\nabla_k R_{hl} = \nabla_l R_{hk}, \tag{31}$$

which shows that the Ricci tensor is of Codazzi type [19]. We thus can conclude the following theorem:

Theorem 6. *Let M be a $(PPS)_n$ spacetime with a divergence-free conformal curvature tensor; then, the Ricci tensor of M is of Codazzi type.*

In view of Theorem 1, we can state the following corollary:

Corollary 7. *Let M be a $(PPS)_n$ spacetime with a divergence-free conformal curvature tensor; then, the Ricci tensor of M is given by*

$$R_{hk} = \left(\frac{n+3}{n^2} \right) R g_{hk} + \left(\frac{3}{n^2} \right) R \lambda_k \lambda_l - 3 \left(\frac{n-1}{n} \right) \lambda^l \lambda^j R_{hjkl}. \tag{32}$$

From equation (29), we can get

$$\lambda^j R_{hljk} = -\frac{R}{n(n-1)(n-2)} g_{hk} \lambda_l + \frac{R}{n(n-1)(n-2)} \lambda_h g_{lk} + \frac{1}{(n-2)} \lambda_l R_{hk} - \frac{1}{(n-2)} \lambda_h R_{lk}, \tag{33}$$

$$\lambda^i R_{hilk} = -\frac{R}{n(n-1)(n-2)} g_{hk} \lambda_l + \frac{R}{n(n-1)(n-2)} \lambda_k g_{hl} + \frac{1}{(n-2)} \lambda_l R_{hk} - \frac{1}{(n-2)} \lambda_k R_{hl}. \tag{34}$$

Using (33) and (34) in (15), one obtains

$$\begin{aligned} \nabla_l R_{hk} = &-2 \left(\frac{n^2-n-1}{n^2(n-2)} \right) R \lambda_l g_{hk} - \left(\frac{n^2-3n+1}{n^2(n-2)} \right) R \lambda_k g_{hl} \\ &- \left(\frac{n^2-2n-1}{n(n-2)} \right) R \lambda_h g_{lk} + \left(\frac{n^2-3n+1}{n(n-2)} \right) \lambda_k R_{hl} \\ &+ \left(\frac{2n^2-2n-2}{n(n-2)} \right) \lambda_l R_{hk} + \left(\frac{n^2-2n-1}{n(n-2)} \right) \lambda_h R_{lk}. \end{aligned} \tag{35}$$

It follows that

$$\begin{aligned} \nabla_l R_{hk} - \nabla_k R_{hl} = &-\left(\frac{n^2+n-3}{n^2(n-2)} \right) R \lambda_l g_{hk} \\ &+ \left(\frac{n^2+n-3}{n^2(n-2)} \right) R \lambda_k g_{hl} - \left(\frac{n^2+n-3}{n(n-2)} \right) \lambda_k R_{hl} \\ &+ \left(\frac{n^2+n-3}{n(n-2)} \right) \lambda_l R_{hk}. \end{aligned} \tag{36}$$

In a conformally flat $(\mathrm{PPS})_n$ spacetime, the Ricci tensor is of Codazzi type; therefore,

$$\begin{aligned} 0 = &-\left(\frac{n^2+n-3}{n^2(n-2)} \right) R \lambda_l g_{hk} + \left(\frac{n^2+n-3}{n^2(n-2)} \right) R \lambda_k g_{hl} \\ &- \left(\frac{n^2+n-3}{n(n-2)} \right) \lambda_k R_{hl} + \left(\frac{n^2+n-3}{n(n-2)} \right) \lambda_l R_{hk}. \end{aligned} \tag{37}$$

Contracting with λ^l and using equation (4), we get

$$R_{hk} = \frac{R}{n} g_{hk}, \tag{38}$$

which illustrates that a conformally flat $(\mathrm{PPS})_n$ spacetime is Einstein.

Theorem 8. *A conformally flat $(PPS)_n$ spacetime is Einstein.*

The use of (38) in (1) implies that

$$\mathscr{P}_{hijk} = R_{hijk} - \frac{R}{n(n-1)} \left[g_{hk} g_{ij} - g_{hj} g_{ik} \right]. \tag{39}$$

Then, from (39) in (29), one infers

$$\begin{aligned} \mathscr{P}_{ijkl} = &\frac{1}{n-2} \left\{ g_{il} R_{jk} + g_{jk} R_{il} - g_{ik} R_{jl} - g_{jl} R_{ik} \right\} \\ &- \frac{2R}{n(n-2)} \left[g_{il} g_{jk} - g_{jk} g_{jl} \right]. \end{aligned} \tag{40}$$

Hence, from (38), we get

$$\mathscr{P}_{ijkl} = 0. \tag{41}$$

From (38) and (41) in (1), we have

$$R_{hijk} = \frac{R}{n(n-1)} \left[g_{hk} g_{ij} - g_{hj} g_{ik} \right], \tag{42}$$

which means that a conformally flat $(\mathrm{PPS})_n$ spacetime is of constant curvature.

In consequence of the above, we can state the following theorem:

Theorem 9. *A conformally flat $(PPS)_n$ spacetime is projectively flat and of constant curvature.*

4. Conformally Flat $(\mathrm{PPS})_4$ Spacetimes in $f(R, \mathscr{G})$ Gravity

In this section, conformally flat $(\mathrm{PPS})_4$ spacetimes in f(R, $\mathscr{G}$) theory of gravity are investigated. For $n = 4$, equation (38) becomes

$$R_{hk} = \frac{R}{4} g_{hk}. \tag{43}$$

It follows that

$$R^{hk} = \frac{R}{4} g^{hk}. \tag{44}$$

Multiplying equations (43) and (44), one gets

$$R_{hk} R^{hk} = \frac{R^2}{4}. \tag{45}$$

From equation (29), it follows that

$$R^{hijk} = \frac{1}{2}\left[g^{hk}R^{ij} + g^{ij}R^{hk} - g^{hj}R^{ik} - g^{ik}R^{hj}\right] - \frac{R}{6}\left[g^{hk}g^{ij} - g^{hj}g^{ik}\right]. \tag{46}$$

Multiplying equations (29) and (46), we obtain

$$R_{hijk} R^{hijk} = 2R^{hk} R_{hk} - \frac{1}{3} R^2. \tag{47}$$

With the help of equation (47), the Gauss-Bonnet topological invariant is

$$\mathscr{G} = -2R^{hk} R_{hk} + \frac{2}{3} R^2. \tag{48}$$

The use of equation (45) implies that

$$\mathscr{G} = \frac{1}{6} R^2. \tag{49}$$

Thus, we can state the following theorem:

Theorem 10. *The Gauss-Bonnet scalar in a conformally flat $(PPS)_4$ spacetime is expressed as*

$$\mathscr{G} = \frac{1}{6} R^2. \tag{50}$$

In a conformally flat spacetime, equation (9) can be rewritten as

$$R_{ij} - \frac{R}{2} g_{ij} = \kappa\left(T_{ij}^{(m)} + T_{ij}^{\mathrm{curv}}\right) = \kappa T_{ij}^{\mathrm{eff}}, \tag{51}$$

where T_{ij}^{curv} arises from the geometry of the spacetime. The tensor T_{ij}^{curv} is given as [20]

$$\begin{aligned}\kappa T_{ij}^{\mathrm{curv}} = {} & \left(\nabla_i \nabla_j - g_{ij}\nabla^2\right) f_R + 2R\left(\nabla_i \nabla_j - g_{ij}\nabla^2\right) f_{\mathscr{G}} \\ & - 4\left(R_i^m \nabla_m \nabla_j + R_j^m \nabla_m \nabla_i\right) f_{\mathscr{G}} \\ & + 4\left(R_{ij}\nabla^2 + g_{ij} R_{mn} \nabla^n \nabla^m - R_{nimj} \nabla^n \nabla^m\right) f_{\mathscr{G}} \\ & - \frac{1}{2} g_{ij}\left(R f_R + \mathscr{G} f_{\mathscr{G}} - f\right) + \left(1 - f_R\right)\left(R_{ij} - \frac{R}{2} g_{ij}\right).\end{aligned} \tag{52}$$

Since in a conformally flat $(\mathrm{PPS})_4$ spacetime the scalar curvature is constant, the previous equation reduces

$$\kappa T_{ij}^{\mathrm{curv}} = -\frac{1}{2} g_{ij}\left(R f_R + \mathscr{G} \mathrm{f}_{\mathscr{G}} - f\right) + \left(1 - f_R\right)\left(R_{ij} - \frac{R}{2} g_{ij}\right). \tag{53}$$

Utilizing equations (38) and (49) in equation (53), we get

$$\kappa T_{ij}^{\mathrm{curv}} = \left(\frac{f}{2} - \frac{R}{4} f_R - \frac{R^2}{12} f_{\mathscr{G}} - \frac{R}{4}\right) g_{ij}. \tag{54}$$

The use of (43) and (54) in (51) implies that

$$\kappa T_{ij}^{(m)} = \left(\frac{R}{4} f_R + \frac{R^2}{12} f_{\mathscr{G}} - \frac{f}{2}\right) g_{ij}. \tag{55}$$

The vector filed ξ is called Killing if

$$\mathscr{L}_\xi g_{ij} = 0, \tag{56}$$

whereas ξ is called conformal Killing if

$$\mathscr{L}_\xi g_{ij} = 2\varphi g_{ij}, \tag{57}$$

where $\mathscr{L}_\xi$ is the Lie derivative with respect to the vector filed ξ and φ is a scalar function [21, 22].

A spacetime M is said to admit a matter collineation with respect to a vector field ξ if the Lie derivative of the energy-momentum tensor T_{ij} with respect to ξ satisfies

$$\mathscr{L}_\xi T_{ij} = 0, \tag{58}$$

while it is said that the energy-momentum tensor T_{ij} has the Lie inheritance property along the flow lines of the vector field ξ if the Lie derivative of T_{ij} with respect to ξ satisfies [21, 22]

$$\mathscr{L}_\xi T_{ij} = 2\varphi T_{ij}. \tag{59}$$

Applying the Lie derivative on both sides of (55), one gets

$$\kappa \mathscr{L}_\xi T_{ij}^{(m)} = \left(\frac{R}{4} f_R + \frac{R^2}{12} f_{\mathscr{G}} - \frac{f}{2}\right) \mathscr{L}_\xi g_{ij}. \tag{60}$$

If the vector field ξ is Killing on a conformally flat $(\mathrm{PPS})_4$ spacetime M, hence equation (60) implies that

$$\mathscr{L}_\xi T_{ij}^{(m)} = 0. \tag{61}$$

In the contrast, if a conformally flat $(\mathrm{PPS})_4$ spacetime M admits matter collineation with respect to ξ, it follows from equation (60) that

$$\mathscr{L}_\xi g_{ij} = 0. \tag{62}$$

Hence, we can state the following theorem:

Theorem 11. *Let M be a conformally flat $(PPS)_4$ spacetime obeying $f(R,\mathscr{G})$ gravity theory; then, the vector field ξ is Killing if and only if M admits matter collineation with respect to ξ.*

Assume that the vector field ξ is conformal Killing; then, after using (57) in (60) utilizing (55), we acquire that

$$\mathscr{L}_\xi T_{ij}^{(m)} = 2\varphi T_{ij}^{(m)}. \tag{63}$$

Conversely, suppose that the energy-momentum tensor T_{ij} has the Lie inheritance property along the flow lines of ξ, thus making use of (59) in (60) after that using (55)), we infer that

$$\mathscr{L}_\xi g_{ij} = 2\varphi g_{ij}. \tag{64}$$

Thus, we can state the following theorem:

Theorem 12. *Let M be a conformally flat $(PPS)_4$ spacetime obeying $f(R,\mathscr{G})$ gravity theory; then, M has a conformal Killing vector filed ξ if only if the energy-momentum tensor T_{ij} has the Lie inheritance property along ξ.*

5. Conformally Flat $(\mathrm{PPS})_4$ Perfect Fluid Spacetimes in $f(R,\mathscr{G})$ Gravity

This section is mainly organized to study conformally flat $(\mathrm{PPS})_4$ perfect fluid spacetimes in $f(R,\mathscr{G})$ modified gravity theory. For a perfect fluid spacetime, the energy-momentum tensor is given as

$$T_{ij}^{(m)} = \left[p^{(m)} + \sigma^{(m)}\right]\lambda_i\lambda_j + p^{(m)} g_{ij}, \tag{65}$$

$$T_{ij}^{\mathrm{eff}} = \left[\mathrm{p}^{\mathrm{eff}} + \sigma^{\mathrm{eff}}\right]\lambda_i\lambda_j + p^{\mathrm{eff}} g_{ij}, \tag{66}$$

where $p^{(m)}$ and $\sigma^{(m)}$ are the isotropic pressure and the energy density of the ordinary matter, whereas p^{eff} and σ^{eff} are the effective isotropic pressure and the effective energy density of the effective matter.

In view of (55) and (65), we have

$$\left(\frac{R}{4}f_R + \frac{R^2}{12}f_{\mathscr{G}} - \frac{f}{2} - \kappa p^{(m)}\right) g_{ij} = \kappa\left(p^{(m)} + \sigma^{(m)}\right)\lambda_i\lambda_j. \tag{67}$$

Contracting twice with λ^i and g^{ij}, one finds

$$\sigma^{(m)} = \frac{1}{\kappa}\left(\frac{f}{2} - \frac{R}{4}f_R - \frac{R^2}{12}f_{\mathscr{G}}\right), \tag{68}$$

$$3\kappa p^{(m)} - \kappa\sigma^{(m)} = 4\left(\frac{R}{4}f_R + \frac{R^2}{12}f_{\mathscr{G}} - \frac{f}{2}\right). \tag{69}$$

Utilizing (68) in (69), it arises

$$p^{(m)} = -\frac{1}{\kappa}\left(\frac{f}{2} - \frac{R}{4}f_R - \frac{R^2}{12}f_{\mathscr{G}}\right). \tag{70}$$

We thus motivate to state the following theorem:

Theorem 13. *In a conformally flat perfect fluid $(PPS)_4$ spacetime obeying $f(R,\mathscr{G})$ gravity, the isotropic pressure $p^{(m)}$ and the energy density $\sigma^{(m)}$ are constants. Moreover, they are given by (68) and (70).*

The combination of (68) and (70) gives

$$p^{(m)} + \sigma^{(m)} = 0, \tag{71}$$

which means that the spacetime represents inflation and fluid behaves as a cosmological constant [23].

Theorem 14. *Let M be a conformally flat perfect fluid $(PPS)_4$ spacetime obeying $f(R,\mathscr{G})$ gravity; then, M represents inflation and fluid behaves as a cosmological constant.*

Using (54), (65), and (66) in (51), one infers

$$\begin{aligned}\left[p^{\mathrm{eff}} + \sigma^{\mathrm{eff}}\right]\lambda_i\lambda_j + p^{\mathrm{eff}} g_{ij} &= \left[p^{(m)} + \sigma^{(m)}\right]\lambda_i\lambda_j + p^{(m)} g_{ij} \\ &\quad + \frac{1}{\kappa}\left(\frac{f}{2} - \frac{R}{4}f_R - \frac{R^2}{12}f_{\mathscr{G}} - \frac{R}{4}\right) g_{ij}.\end{aligned} \tag{72}$$

Making a comparison of both sides, we obtain

$$\begin{aligned}p^{\mathrm{eff}} &= p^{(m)} + \frac{1}{\kappa}\left(\frac{f}{2} - \frac{R}{4}f_R - \frac{R^2}{12}f_{\mathscr{G}} - \frac{R}{4}\right), \\ \sigma^{\mathrm{eff}} &= \sigma^{(m)} - \frac{1}{\kappa}\left(\frac{f}{2} - \frac{R}{4}f_R - \frac{R^2}{12}f_{\mathscr{G}} - \frac{R}{4}\right).\end{aligned} \tag{73}$$

The use of (68) and (70) implies that

$$p^{\mathrm{eff}} = -\frac{R}{4\kappa}, \tag{74}$$

$$\sigma^{\mathrm{eff}} = \frac{R}{4\kappa}. \tag{75}$$

In the context of $f(R, \mathcal{G})$ modified gravity, let us now deduce some energy conditions of a perfect fluid type effective matter. The energy conditions are obtained as follows [24, 25]:

(1) *Null Energy Condition (NEC).* $p^{\text{eff}} + \sigma^{\text{eff}} \geq 0$.

(2) *Weak Energy Condition (WEC).* $\sigma^{\text{eff}} \geq 0$ and $p^{\text{eff}} + \sigma^{\text{eff}} \geq 0$.

(3) *Dominant Energy Condition (DEC).* $\sigma^{\text{eff}} \geq 0$ and $p^{\text{eff}} \pm \sigma^{\text{eff}} \geq 0$.

(4) *Strong Energy Condition (SEC).* $\sigma^{\text{eff}} + 3p^{\text{eff}} \geq 0$ and $p^{\text{eff}} + \sigma^{\text{eff}} \geq 0$.

In view of (74) and (22), the energy conditions are always satisfied if $R \succ 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] M. C. Chaki and S. K. Saha, “On pseudo-projective symmetric manifolds,” *Bulletin of the Institute of Mathematics Academia Sinica*, vol. 17, pp. 59–65, 1989.

[2] K. Yano and S. Bochner, *Curvature and Betti Numbers, Annals of Mathematics Studies*, Princeton University Press, 1953.

[3] M. C. Chaki, “On pseudo symmetric manifolds,” *Analele stiintifice ale Universitatii “Al.I. Cuza” din Iasi*, vol. 33, pp. 53–58, 1987.

[4] B.-Y. Chen, “Totally umbilical submanifolds,” *Soochow Journal of Mathematics*, vol. 5, pp. 9–37, 1979.

[5] C. A. Mantica, L. G. Molinari, Y. J. Suh, and S. Shenawy, “Perfect-fluid, generalized Robertson-Walker space-times, and Gray’s decomposition,” *Journal of Mathematical Physics*, vol. 60, no. 5, article 052506, 2019.

[6] M. C. Chaki and R. K. Maity, “On quasi Einstein manifold,” *Publicationes Mathematicae Debrecen*, vol. 57, pp. 297–306, 2000.

[7] U. C. De and G. C. Ghosh, “On quasi Einstein manifolds,” *Periodica Mathematica Hungarica*, vol. 48, no. 1/2, pp. 223–231, 2004.

[8] U. C. De and H. M. Abu-Donia, “On generalized projective P-curvature tensor,” *Journal of Geometry and Physics*, vol. 159, p. 103952, 2021.

[9] B. O’Neill, *Semi-Riemannian Geometry, Pure and Applied Mathematics*, Academic Press, Inc., New York, 1983.

[10] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, “Is cosmic speed-up due to new gravitational physics?,” *Physical Review D*, vol. 70, no. 4, article 043528, 2004.

[11] H. A. Buchdahl, “Non-linear Lagrangians and cosmological theory,” *Monthly Notices of the Royal Astronomical Society*, vol. 150, no. 1, pp. 1–8, 1970.

[12] S. Nojiri and S. D. Odintsov, “Modified gravity with negative and positive powers of the curvature,” *Physical Review D*, vol. 68, article 123512, 2003.

[13] A. Y. Kamenshchik, U. Moschella, and V. Pasquier, “LambdaCDM epoch reconstruction from F (R, G) and modified Gauss-Bonnet gravities,” *Physics Letters B*, vol. 511, 2001.

[14] A. De, T.-H. Loo, R. Solanki, and P. K. Sahoo, “How a conformally flat impacts Gauss-Bonnet gravity?,” *Fortschritte der Physik*, vol. 69, no. 10, article 2100088, 2021.

[15] A. Gray, “Einstein-like manifolds which are not Einstein,” *Geometriae Dedicata*, vol. 7, no. 3, pp. 259–280, 1978.

[16] C. A. Mantica and S. Shenawy, “Einstein-like warped product manifolds,” *International Journal of Geometric Methods in Modern Physics*, vol. 14, no. 11, p. 1750166, 2017.

[17] M. M. Postnikov, “Geometry VI: Riemannian Geometry,” *Springer Science & Business Media*, vol. 91, 2013.

[18] L. P. Eisenhart, *Riemannian Geometry*, Princeton university press, 1997.

[19] U. C. De, S. Shenawy, and B. Ünal, “Concircular curvature on warped product manifolds and applications,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 5, pp. 3395–3409, 2020.

[20] K. Atazadeh and F. Darabi, “Energy conditions in *f(R,G)* f (R , G) gravity,” *General Relativity and Gravitation*, vol. 46, no. 2, p. 1664, 2014.

[21] H. M. Abu-Donia, S. Shenawy, and A. A. Syied, “The *W*-curvature tensor on relativistic space-times,” *Kyungpook Mathematical Journal*, vol. 60, pp. 185–195, 2020.

[22] S. Mallick and U. C. De, “Spacetimes admitting W2-curvature tensor,” *International Journal of Geometric Methods in Modern Physics*, vol. 11, no. 4, article 1450030, 2014.

[23] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein’s Field Equations*, Cambridge university press, 2009.

[24] A. De, T. Loo, S. Arora, and P. K. Sahoo, “Energy conditions for a $(WRS)_4$ spacetime in F (R)-gravity,” *The European Physical Journal Plus*, vol. 136, no. 2, 2021.

[25] A. De, T. Loo, R. Solanki, and P. K. Sahoo, “A conformally flat generalized Ricci recurrent spacetime in F (R)-gravity,” *Physica Scripta*, vol. 96, no. 8, article 085001, 2021.