Research Article

A Study on Vague Graph Structures with an Application

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Fuzzy graph (FG) models take on the presence being ubiquitous in environmental and fabricated structures by human, specifically the vibrant processes in physical, biological, and social systems. Owing to the unpredictable and indiscriminate data which are intrinsic in real-life, problems being often ambiguous, so it is very challenging for an expert to exemplify those problems through applying an FG. Vague graph structure (VGS), belonging to FGs family, has good capabilities when facing with problems that cannot be expressed by FGs. VGSs have a wide range of applications in the field of psychological sciences as well as the identification of individuals based on oncological behaviors. Therefore, in this paper, we apply the concept of vague sets (VSs) to GS. We define certain notions, VGS, strong vague graph structure (SVGS), and vague β_i -cycle and describe these notions by several examples. Likewise, we introduce ψ -complement, self-complement (SC), strong self-complement (SSC), and totally strong self-complement (TSSC) in VGS and investigate some of their properties. Finally, an application of VGS is presented.

1. Introduction

The FG concept serves as one of the most dominant and extensively employed tools for multiple real-word problem representations, modeling, and analysis. To specify the objects and the relations between them, the graph vertices or nodes and edges or arcs are applied, respectively. Graphs have long been used to describe objects and the relationships between them. Many of the issues and phenomena around us are associated with complexities and ambiguities that make it difficult to express certainty. These difficulties were alleviated by the introduction of fuzzy sets by Zadeh [1]. A GS, proposed by Sampathkumar [2], refers to the generalization of undirected graph being relatively beneficial in investigating some structure including graphs, signed graphs, and graphs in which every edge is labeled or colored. A GS facilitates studying the different relations and the equivalent edges simultaneously. The FS focuses on the membership degree of an object in a particular set. The existence of a single degree for a true membership could not resolve the ambiguity on uncertain issues, so the need for a degree of membership was felt. Afterward, to overcome the

existing ambiguities, Gau and Buehrer [3] introduced false-membership degrees and defined a VS as the sum of degrees not greater than 1. Kaufmann [4] represented FGs based on Zadeh's fuzzy relation [5, 6]. Rosenfeld [7] described the structure of FGs obtaining analogs of several graph theoretical concepts. Harinath and Lavanya [8] studied new concepts in fuzzy graph structures. Bhattacharya [9] gave some remarks on FGs. Several concepts on FGs were introduced by Mordeson and Nair [10]. Dinesh [11] investigated the notion of a FGS and studied some related properties. Ghorai et al. [12, 13] presented fuzzy tolerance graphs and planarity in VGs. Ramakrishna [14] defined VGs. Kosari et al. [15–18] investigated domination set, equitable domination set, and new concepts of domination in vague graphs and vague incidence graphs. Pal and Rashmanlou [19, 20] have given several concepts in FGs. Akram et al. [21, 22] introduced certain fuzzy graph structures. Sunitha and Vijayakumar [23] presented some properties of complement on FGs. Muhiuddin et al. [24–26] investigated new results in cubic graphs.

A VGS is concerned with the generalized structure of an FG that expresses more exactness, adaptability, and

compatibility to a system when synchronized with systems operating on FGs. Correspondingly, a VGS is capable of focusing on determining the uncertainly combined with the inconsistent and indeterminate information of any real-world problem, in which FGs may not lead to adequate results. Hence, in this paper, we define the certain notions including VGS, SVGS, and vague β_i -cycle and describe these notions by several examples. We study ψ -complement, SC, and SSC in VGSs and investigate some of their properties. Finally, an application of VGS has been presented.

2. Preliminaries

A GS $G^* = (V, R_1, R_2, \dots, R_k)$, consists of a nonempty set V together with relations $R_1, R_2, \dots, R_k$ on V , which are mutually disjoint so that each R_i is irreflexive and symmetric. If $(a, b) \in R_i$ for some i , $1 \leq i \leq k$, we name it an R_i -edge and write it as " ab ."

A GS $G^* = (V, R_1, R_2, \dots, R_k)$ is CGS, if,

- (i) every edge R_i , $1 \leq i \leq k$ appears at least once in G^*
- (ii) among every pair of nodes ab in V , ab is an E_i -edge for some i , $1 \leq i \leq k$

Definition 1 (see [1]). A fuzzy subset σ on a set X is a map $\sigma: X \rightarrow [0, 1]$. A fuzzy binary relation on X is a fuzzy subset σ on $X \times X$. By a fuzzy relation, we mean a fuzzy binary relation given by $\sigma: X \times X \rightarrow [0, 1]$.

Definition 2 (see [11]). Let $G^* = (V, R_1, R_2, \dots, R_k)$ be a GS and let $\nu, \lambda_1, \lambda_2, \dots, \lambda_k$ be the fuzzy subset of $V, R_1, R_2, \dots, R_k$, respectively, so that $0 \leq \lambda_i(xy) \leq \sigma(x) \wedge \sigma(y)$, $\forall x, y \in V$, $i = 1, 2, \dots, k$. Then, $G = (\nu, \lambda_1, \lambda_2, \dots, \lambda_k)$ is an FGS of G^* .

Definition 3 (see [11]). Let $G = (\nu, \lambda_1, \lambda_2, \dots, \lambda_k)$ be an FGS of a GS $G^* = (V, R_1, R_2, \dots, R_k)$. Then, $F = (\nu, \tau_1, \tau_2, \dots, \tau_k)$ is a PFSSGS of G if G , for $i = 1, 2, \dots, k$.

Definition 4 (see [11]). The strength of a λ_i -path $a_0 a_1 \dots a_n$ of a FGS G is $\bigwedge_{j=1}^n \lambda_i(a_{j-1} a_j)$, for $i = 1, 2, \dots, k$.

Definition 5 (see [11]). In a FGS G , $\lambda_i^2(ab) = (\lambda_i \circ \lambda_i)(ab) = \bigvee_c \{\lambda_i(ac) \wedge \lambda_i(bc)\}$, $\lambda_i^j(ab) = (\lambda_i^{j-1} \circ \lambda_i)(ab) = \bigvee_c \{\lambda_i^{j-1}(ac) \wedge \lambda_i(bc)\}$, $j = 1, 2, \dots, n$, for any $n \geq 2$. Also, $\lambda_i^\infty(ab) = \bigvee \{\lambda_i^j(ab), j = 1, 2, \dots\}$.

Definition 6 Reference [11] $G = (\nu, \lambda_1, \lambda_2, \dots, \lambda_k)$ is a λ_i -cycle if and only if $(\text{Supp}(\nu), \text{Supp}(\lambda_1), \text{Supp}(\lambda_2), \dots, \text{Supp}(\lambda_k))$ is a R_i -cycle.

Definition 7. Reference [11] $G = (\nu, \lambda_1, \lambda_2, \dots, \lambda_k)$ is a fuzzy λ_i -cycle if and only if $(\text{Supp}(\nu), \text{Supp}(\lambda_1), \text{Supp}(\lambda_2), \dots, \text{Supp}(\lambda_k))$ is a E_i -cycle and $\exists$ no unique ab in $\text{Supp}(\lambda_i)$ so that

$$\lambda_i(ab) = \bigwedge \{\lambda_i(cd) \mid cd \in \text{Supp}(\lambda_i)\}. \quad (1)$$

Definition 8 (see [3]). A VS A is a pair (t_A, f_A) on set X where t_A and f_A are taken as real valued functions which can be defined on $V \rightarrow [0, 1]$ so that $t_A(a) + f_A(b) \leq 1$, $\forall a \in X$.

Definition 9 (see [14]). A VG is a pair $G = (A, B)$, where $A = (t_A, f_A)$ is a VS on V and $B = (t_B, f_B)$ is a VS on $E \subseteq V \times V$ so that $t_B(ab) \leq \min(t_A(a), t_A(b))$ and $f_B(ab) \geq \max(f_A(a), f_A(b))$, for $ab \in E$.

All the basic notations are shown in Table 1.

3. New Concepts in Vague Graph Structure

Definition 10. $\widehat{G}_v = (A, B_1, B_2, \dots, B_n)$ is said to be a VGS of a GS $G^* = (V, R_1, R_2, \dots, R_n)$, if $A = (t_A, f_A)$ is a VS on V and for every $i = 1, 2, \dots, n$; $B_i = (t_{B_i}, f_{B_i})$ is a VS on R_i so that

$$\begin{aligned} t_{B_i}(ab) &\leq t_A(a) \wedge t_A(b), \\ f_{B_i}(ab) &\geq f_A(a) \vee f_A(b). \end{aligned} \quad (2)$$

$\forall ab \in R_i \subseteq V \times V$. Note that $t_{B_i}(ab) = 0 = f_{B_i}(ab)$, $\forall ab \in V \times V - R_i$ and $0 \leq t_{B_i}(ab) \leq 1$, $0 \leq f_{B_i}(ab) \leq 1$, $\forall ab \in R_i$, where V and R_i , ($i = 1, 2, \dots, n$) are named UVS and underlying i -edge set of $\widehat{G}_v$, respectively.

Example 1. Let $G^* = (V, R_1, R_2)$ be a GS so that $V = \{a_1, a_2, a_3, a_4\}$, $R_1 = \{a_1 a_2, a_2 a_3\}$, and $R_2 = \{a_3 a_4, a_1 a_4\}$. Let A, B_1 , and B_2 be vague subsets of V, R_1 , and R_2 , respectively, so that

$$\begin{aligned} A &= \{(a_1, 0.4, 0.5), (a_2, 0.4, 0.6), (a_3, 0.3, 0.4), (a_4, 0.4, 0.4)\}, \\ B_1 &= \{(a_1 a_2, 0.4, 0.6), (a_2 a_3, 0.3, 0.6)\}, \\ B_2 &= \{(a_3 a_4, 0.3, 0.4), (a_1 a_4, 0.3, 0.5)\}. \end{aligned} \quad (3)$$

Then, $\widehat{G}_v = (A, B_1, B_2)$ is a VGS of G^* drawn in Figure 1.

Definition 11

- (i) AVGS $\widetilde{H}_v = (C, D_1, D_2, \dots, D_n)$ is said to be a VSGS of a VGS $\widehat{G}_v = (A, B_1, B_2, \dots, B_n)$ with UVS V , if $C \subseteq A$ and $D_i \subseteq C_i$, $\forall i$, that is,

$$t_C(a) \leq t_A(a), f_C(a) \geq f_A(a); \quad \text{for all } a \in V, \quad (4)$$

and for $i = 1, 2, \dots, n$,

$$t_{D_i}(ab) \leq t_{B_i}(ab), f_{D_i}(ab) \geq f_{B_i}(ab); \quad \text{for all } ab \in V \times V. \quad (5)$$

- (ii) $\widetilde{H}_v$ is named a VSSGS of a VGS $\widehat{G}_v$, if $C = A$.
- (iii) $\widetilde{H}_v$ is named a VPSSGS of a VGS $\widehat{G}_v$, if it excludes some edges of $\widehat{G}_v$.

TABLE 1: Some basic notations.

Notation	Meaning
FG	Fuzzy graph
FS	Fuzzy set
VS	Vague set
GS	Graph structure
CGS	Complete graph structure
VGS	Vague graph structure
VSGS	Vague subgraph structure
SGS	Subgraph structure
FGS	Fuzzy graph structure
SVGS	Strong vague graph structure
UG	Underlying graph
EN	End node
SC	Self-complementary
SSC	Strong self-complementary
TSC	Totally self-complementary
TSSC	Totally strong self-complement
PFSSGS	Partial fuzzy spanning subgraph structure
UVS	Underlying vertex set
TSSC	Totally strong self-complementary

Example 2. Consider a VGS $\tilde{G}_v = (A, B_1, B_2)$ as shown in Figure 1. Let

$$\begin{aligned}
 C &= \{(a_1, 0.3, 0.6), (a_2, 0.0, 0.7), (a_3, 0.3, 0.4), (a_4, 0.3, 0.5)\}, \\
 D_1 &= \{(a_1a_2, 0, 0.7), (a_2a_3, 0, 0.7)\}, \\
 D_2 &= \{(a_3a_4, 0.2, 0.5), (a_1a_4, 0.3, 0.6)\}, \\
 C_1 &= \{(a_1a_2, 0.2, 0.6), (a_2a_3, 0.3, 0.6)\}, \\
 C_2 &= \{(a_3a_4, 0.2, 0.4), (a_1a_4, 0.3, 0.7)\}, \\
 F_1 &= \{(a_1a_2, 0.4, 0.6), (a_2a_3, 0.3, 0.6)\}, \\
 F_2 &= \{(a_1a_4, 0.3, 0.6)\}.
 \end{aligned} \tag{6}$$

It is easy to see that $\tilde{H}_v = (C, D_1, D_2)$, $\tilde{J}_v = (A, C_1, C_2)$, and $\tilde{K}_v = (A, F_1, F_2)$ are the VSGS, VSSGS, and VPSSGS of $\tilde{G}_v$, respectively.

Their corresponding graphs are drawn in Figures 2–4.

Definition 12. Let $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ be a VGS with UVS V . Then, there exists a B_i -edge among two nodes a and b of V so that we have

- (i) $t_{B_i}(ab) > 0$, $f_{B_i}(ab) > 0$, or
- (ii) $t_{B_i}(ab) > 0$, $f_{B_i}(ab) = 0$, or
- (iii) $t_{B_i}(ab) = 0$, $f_{B_i}(ab) > 0$, for some i .

Definition 13. For a VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$, the support of B_i is described as

$$\text{Supp}(B_i) = \{ab \in V \times V : t_{B_i}(ab) \neq 0 \text{ or } f_{B_i}(ab) \neq 0\}, \quad i = 1, 2, \dots, n. \tag{7}$$

Definition 14. B_i -path of a VGS $\tilde{G}_v$ with UVS V is a sequence of distinct nodes $v_1, v_2, \dots, v_m \in U$ (except the choice $v_m = v_1$) so that $v_{j-1}v_j$ is a B_i -edge, $\forall j = 2, 3, \dots, m$.

Definition 15. In a VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ with UVS V , two nodes a and b called B_i -connected, if they are connected by a B_i -path, for some $i \in \{1, 2, 3, \dots, n\}$.

FIGURE 1: VGS $\widehat{G}_v = (A, B_1, B_2)$. FIGURE 2: Vague subgraph structure $\check{H}_v = (C, D_1, D_2)$. FIGURE 3: Vague spanning subgraph structure $\check{J}_v = (A, C_1, C_2)$. FIGURE 4: Vague partial spanning subgraph structure $\check{K}_v = (A, F_1, F_2)$.

Definition 16. A VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ with UVS V is called to be B_i -strong, if $\forall B_i$ -edges ab ,

$$t_{B_i}(ab) = t_A(a) \wedge t_A(b), f_{B_i}(ab) = f_A(a) \vee f_A(b), \quad (8)$$

for some $i \in \{1, 2, 3, \dots, n\}$.

Example 3. Consider the VGS $\tilde{G}_v = (A, B_1, B_2)$, drawn in Figure 1. Then,

- (i) a_1a_2, a_2a_3 are B_1 -edges and a_3a_4, a_1a_4 are B_2 -edges
- (ii) $a_1a_2a_3$ and $a_3a_4a_1$ are B_1 and B_2 paths, respectively
- (iii) a_1 and a_3 are B_1 -connected nodes of U
- (iv) $\tilde{G}_v$ is a B_1 -strong since $\text{Supp}(B_1) = \{a_1a_2, a_2a_3\}$ and we have

$$\begin{aligned} t_{B_1}(a_1a_2) &= 0.4 = (t_A(a_1) \wedge t_A(a_2)), \\ f_{B_1}(a_1a_2) &= 0.6 = (f_A(a_1) \vee f_A(a_2)), \\ t_{B_1}(a_2a_3) &= 0.3 = (t_A(a_2) \wedge t_A(a_3)), \\ f_{B_1}(a_2a_3) &= 0.6 = (f_A(a_2) \vee f_A(a_3)). \end{aligned} \quad (9)$$

Definition 17. A VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ is named to be strong, if it is B_i -strong, $\forall i \in \{1, 2, 3, \dots, n\}$.

Definition 18. A VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ with UVS V , is named complete or $B_1, B_2, \dots, B_n$ -complete if

- (i) $\tilde{G}_v$ is a SVGS
- (ii) $\text{Supp}(B_i) \neq \emptyset, \forall i = 1, 2, 3, \dots, n$
- (iii) For each nodes $a, b \in V$, ab should be a B_i -edge

Example 4. Suppose $\tilde{G}_v = (A, B_1, B_2)$ drawn in Figure 5, be VGS of the GS $G^* = (V, R_1, R_2)$ where $V = \{a_1, a_2, a_3, a_4\}$, $R_1 = \{a_1a_2, a_2a_3, a_1a_3\}$, and $R_2 = \{a_1a_4, a_3a_4, a_2a_4\}$. Then, G_v is a SVGS, since it is both B_1 -strong and B_2 -strong. Moreover, $\text{Supp}(B_1) \neq \emptyset, \text{Supp}(B_2) \neq \emptyset$ and each nodes in V , is either a B_1 -edge or a B_2 -edge, so G_v is a complete or B_1B_2 -complete-VGS as well.

Definition 19. In a VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ with UVS V , t_{B_i} and f_{B_i} -strength of a B_i -path " $P_{B_i} = v_1v_2, \dots, v_m$ " are denoted by $\delta \cdot P_{B_i}$ and $\Delta \cdot P_{B_i}$, respectively, so that $\delta \cdot P_{B_i} = \bigwedge_{j=2}^m [t_{B_i}(a_{j-1}a_j)]$ and $\Delta \cdot P_{B_i} = \bigvee_{j=2}^m [f_{B_i}(a_{j-1}a_j)]$.

Then, we write strength of the path $P_{B_i} = (\delta \cdot P_{B_i}, \Delta \cdot P_{B_i})$.

Example 5. In $\tilde{G}_v = (A, B_1, B_2)$ shown in Figure 5, $P_1 = a_1a_2a_3a_1$ is a B_1 -path, and $a_1a_4a_3$ is a B_2 -path, we have

$$\begin{aligned} \delta \cdot P_1 &= t_{B_1}(a_1a_2) \wedge t_{B_1}(a_2a_3) \wedge t_{B_1}(a_3a_1) = 0.2 \wedge 0.3 \wedge 0.2 = 0.2, \\ \Delta \cdot P_1 &= f_{B_1}(a_1a_2) \vee f_{B_1}(a_2a_3) \vee f_{B_1}(a_3a_1) = 0.3 \vee 0.5 \vee 0.5 = 0.5, \\ \delta \cdot P_2 &= t_{B_2}(a_1a_4) \wedge t_{B_2}(a_4a_3) = 0.2 \wedge 0.4 = 0.2, \\ \Delta \cdot P_2 &= f_{B_2}(a_1a_4) \vee f_{B_2}(a_4a_3) = 0.6 \vee 0.6 = 0.6. \end{aligned} \quad (10)$$

Thus, strength of B_1 -path is $P_1 = (0.2, 0.5)$, and strength of P_2 -path is $P_2 = (0.2, 0.6)$.

Definition 20. In a VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ with UVS V ,

- (i) t_{B_i} -strength of connectedness between a and b is defined by $t_{B_i}^\infty(ab) = \bigvee_{j \geq 1} \{t_{B_i}^j(ab)\}$, where $t_{B_i}^j(ab) = (t_{B_i}^{j-1} \circ t_{B_i})(ab)$, for $j \geq 2$ and $t_{B_i}^2(ab) = (t_{B_i} \circ t_{B_i})(ab) = \bigvee_z \{t_{B_i}(ac) \wedge t_{B_i}(bc)\}$;
- (ii) f_{B_i} -strength of connectedness among a and b is described by $f_{B_i}^\infty(ab) = \bigwedge_{j \geq 1} \{f_{B_i}^j(ab)\}$, where $f_{B_i}^j(ab) = (f_{B_i}^{j-1} \circ f_{B_i})(ab)$, for $j \geq 2$ and $f_{B_i}^2(ab) = (f_{B_i} \circ f_{B_i})(ab) = \bigvee_z \{f_{B_i}(ac) \vee f_{B_i}(bc)\}$.

Example 6. Let $\tilde{G}_v = (A, B_1, B_2)$ be a VGS of GS $G^* = (V, R_1, R_2)$ as shown in Figure 6 so that $V = \{a_1, a_2, a_3\}$, $R_1 = \{a_1a_2, a_1a_3\}$, and $R_2 = \{a_2a_3\}$. Since $t_{B_1}(a_1a_2) = 0.2$, $t_{B_1}(a_1a_3) = 0.2$, and $t_{B_1}(a_2a_3) = 0$, therefore,

$$\begin{aligned} t_{B_1}^2(a_1a_2) &= (t_{B_1} \circ t_{B_1})(a_1a_2) = t_{B_1}(a_1a_3) \wedge t_{B_1}(a_3a_2) = 0.2 \wedge 0.3 = 0.2, \\ t_{B_1}^2(a_2a_3) &= (t_{B_1} \circ t_{B_1})(a_2a_3) = t_{B_1}(a_2a_1) \wedge t_{B_1}(a_1a_3) = 0.2 \wedge 0.2 = 0.2, \\ t_{B_1}^2(a_1a_3) &= (t_{B_1} \circ t_{B_1})(a_1a_3) = t_{B_1}(a_1a_2) \wedge t_{B_1}(a_2a_3) = 0.2 \wedge 0.3 = 0.2, \\ t_{B_1}^3(a_1a_2) &= (t_{B_1}^2 \circ t_{B_1})(a_1a_2) = t_{B_1}^2(a_1a_3) \wedge t_{B_1}(a_3a_2) = 0.2 \wedge 0.3 = 0.2, \\ t_{B_1}^3(a_2a_3) &= (t_{B_1}^2 \circ t_{B_1})(a_2a_3) = t_{B_1}^2(a_2a_1) \wedge t_{B_1}(a_1a_3) = 0.2 \wedge 0.2 = 0.2, \\ t_{B_1}^3(a_1a_3) &= (t_{B_1}^2 \circ t_{B_1})(a_1a_3) = t_{B_1}^2(a_1a_2) \wedge t_{B_1}(a_2a_3) = 0.2 \wedge 0.3 = 0.2. \end{aligned} \quad (11)$$

Thus, we have,

FIGURE 5: VGS $\tilde{G}_v = (A, B_1, B_2)$.

$$\begin{aligned}
 t_{B_1}^\infty(a_1 a_2) &= \vee\{0.2, 0.2, 0.2\} = 0.2, & \text{Since } f_{B_1}(a_1 a_2) &= 0.6, & f_{B_1}(a_1 a_3) &= 0.7, & \text{and} \\
 t_{B_1}^\infty(a_2 a_3) &= \vee\{0.2, 0.2, 0.2\} = 0.2, & f_{B_1}(a_2 a_3) &= 0, \text{ therefore,} \\
 t_{B_1}^\infty(a_1 a_3) &= \vee\{0.2, 0.2, 0.2\} = 0.2.
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 f_{B_1}^2(a_1 a_2) &= (f_{B_1} \circ f_{B_1})(a_1 a_2) = f_{B_1}(a_1 a_3) \vee f_{B_1}(a_3 a_2) = 0.7 \vee 0.7 = 0.7, \\
 f_{B_1}^2(a_2 a_3) &= (f_{B_1} \circ f_{B_1})(a_2 a_3) = f_{B_1}(a_2 a_1) \vee f_{B_1}(a_1 a_3) = 0.6 \vee 0.7 = 0.7, \\
 f_{B_1}^2(a_1 a_3) &= (f_{B_1} \circ f_{B_1})(a_1 a_3) = f_{B_1}(a_1 a_2) \vee f_{B_1}(a_2 a_3) = 0.6 \vee 0 = 0.6, \\
 f_{B_1}^3(a_1 a_2) &= (f_{B_1}^2 \circ f_{B_1})(a_1 a_2) = f_{B_1}^2(a_1 a_3) \vee f_{B_1}(a_3 a_2) = 0.6 \vee 0.7 = 0.7, \\
 f_{B_1}^3(a_2 a_3) &= (f_{B_1}^2 \circ f_{B_1})(a_2 a_3) = f_{B_1}^2(a_2 a_1) \vee f_{B_1}(a_1 a_3) = 0.7 \vee 0.7 = 0.7, \\
 f_{B_1}^3(a_1 a_3) &= (f_{B_1}^2 \circ f_{B_1})(a_1 a_3) = f_{B_1}^2(a_1 a_2) \vee f_{B_1}(a_2 a_3) = 0.7 \vee 0.7 = 0.7,
 \end{aligned} \tag{13}$$

and

$$\begin{aligned}
 f_{B_1}^4(a_1 a_2) &= (f_{B_1}^3 \circ f_{B_1})(a_1 a_2) = f_{B_1}^3(a_1 a_3) \vee f_{B_1}(a_3 a_2) = 0.7 \vee 0.7 = 0.7, \\
 f_{B_1}^4(a_2 a_3) &= (f_{B_1}^3 \circ f_{B_1})(a_2 a_3) = f_{B_1}^3(a_2 a_1) \vee f_{B_1}(a_1 a_3) = 0.7 \vee 0.7 = 0.7, \\
 f_{B_1}^4(a_1 a_3) &= (f_{B_1}^3 \circ f_{B_1})(a_1 a_3) = f_{B_1}^3(a_1 a_2) \vee f_{B_1}(a_2 a_3) = 0.7 \vee 0.7 = 0.7.
 \end{aligned} \tag{14}$$

Thus, we have

$$\begin{aligned}
 f_{B_1}^\infty(a_1 a_2) &= \vee\{0.6, 0.7, 0.7, 0.7\} = 0.7, \\
 f_{B_1}^\infty(a_2 a_3) &= \vee\{0.0, 0.7, 0.7, 0.7\} = 0.7, \\
 f_{B_1}^\infty(a_1 a_3) &= \vee\{0.7, 0.6, 0.7, 0.7\} = 0.7.
 \end{aligned} \tag{15}$$

By similarity way, we can calculate $t_{B_2}^\infty(a_1 a_2)$, $t_{B_2}^\infty(a_2 a_3)$, and $t_{B_2}^\infty(a_1 a_3)$.

Definition 21. A VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ of a GS $G^* = (V, R_1, R_2, \dots, R_n)$ is a B_i -cycle, if G^* is an R_i -cycle.

Definition 22. A VGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ of a GS $G^* = (V, R_1, R_2, \dots, R_n)$ is a vague B_i -cycle, for some i , if we have

(i) G is a B_i -cycle;

(ii) There is no unique B_i -edge ab in $\tilde{G}_v$ so that $t_{B_i}(ab) = \min\{t_{B_i}(zw): zw \in R_i = \text{Supp}(B_i)\}$ or $f_{B_i}(ab) = \max\{f_{B_i}(zw): zw \in R_i = \text{Supp}(B_i)\}$.

Example 7. VGS $\tilde{G}_v = (A, B_1, B_2)$ in Figure 7 is a B_1 -cycle as well as vague B_1 -cycle, since $(\text{Supp}(A), \text{Supp}(B_1), \text{Supp}(B_2))$ is an R_1 -cycle and there are two B_1 -edges with minimum degree of membership and two B_1 -edges with maximum degree of membership of all B_1 -edges.

Definition 23. A VGS $\tilde{G}_{v1} = (A_1, B_{11}, B_{12}, \dots, B_{1n})$ of GS $G_1^* = (V_1, R_{11}, R_{12}, \dots, R_{1n})$ is isomorphic to a VGS $\tilde{G}_{v2} = (A_2, B_{21}, B_{22}, \dots, B_{2n})$ of $G_2^* = (V_2, R_{21}, R_{22}, \dots, R_{2n})$ ($G_{v1} \cong G_{v2}$) if $\exists$ a bijective $g: V_1 \rightarrow V_2$ and a permutation ψ on the set $\{1, 2, \dots, n\}$ so that

FIGURE 6: VGS $\tilde{G}_v = (A, B_1, B_2)$. FIGURE 7: VGS $\tilde{G}_v = (A, B_1, B_2)$.

$$\begin{aligned} t_{A_1}(u_1) &= t_{A_2}(g(u_1)), \\ f_{A_1}(u_1) &= f_{A_2}(g(u_1)), \end{aligned} \quad (16)$$

$\forall u_1 \in V_1$, and for $\psi(i) = j$,

$$\begin{aligned} t_{B_{1i}}(u_1 u_2) &= t_{B_{2j}}(g(u_1)g(u_2)), & f_{B_{1i}}(u_1 u_2) &= \\ f_{B_{2j}}(g(u_1)g(u_2)), & \text{ for all } u_1 u_2 \in R_{1i}, i = 1, 2, \dots, n. \end{aligned}$$

Definition 24. A VGS $\tilde{G}_{v1} = (A_1, B_{11}, B_{12}, \dots, B_{1n})$ of GS $G_1^* = (V, R_{11}, R_{12}, \dots, R_{1n})$ is identical to a VGS $\tilde{G}_{v2} = (A_2, B_{21}, B_{22}, \dots, B_{2n})$ of $G_2^* = (V, R_{21}, R_{22}, \dots, R_{2n})$ if $\exists$ a bijection $g: V \longrightarrow V$ so that

$$\begin{aligned} t_{A_1}(u) &= t_{A_2}(g(u)), \\ f_{A_1}(u) &= f_{A_2}(g(u)), \text{ for all } u \in V, \end{aligned} \quad (17)$$

and

$$t_{B_{1i}}(u_1 u_2) = t_{B_{2i}}(g(u_1)g(u_2)), f_{B_{1i}}(u_1 u_2) = f_{B_{2i}}(g(u_1)g(u_2)), \text{ for all } u_1 u_2 \in R_{1i}, i = 1, 2, \dots, n. \quad (18)$$

Example 8. Let $\tilde{G}_{v1}$ and $\tilde{G}_{v2}$ be two VGSs of GS $G_1^* = (V_1, R_1, R_2)$ and $G_2^* = (V_2, R'_1, R'_2)$, respectively, drawn in Figure 8. Here, $\tilde{G}_{v1}$ is isomorphic (not identical) to $\tilde{G}_{v2}$

under the mapping $g: V_1 \longrightarrow V_2$, defined by $g(a_1) = b_1$, $g(a_2) = b_2$, and $g(a_3) = b_3$ and a permutation ψ given by $\psi(1) = 2$, $\psi(2) = 1$ so that

$$\begin{aligned} t_{A_1}(a_i) &= t_{A_2}(g(a_i)), f_{A_1}(a_i) = f_{A_2}(g(a_i)), \forall a_i \in V, \\ t_{B_k}(a_i a_j) &= t_{B_{\psi(k)}}(g(a_i)g(a_j)), f_{B_k}(a_i a_j) = f_{B_{\psi(k)}}(g(a_i)g(a_j)), \quad \forall a_i a_j \in R_k, k = 1, 2. \end{aligned} \quad (19)$$

Example 9. Let $\tilde{G}_{v1} = (A, B_1, B_2)$ and $\tilde{G}_{v2} = (A', B'_1, B'_2)$ be two VGSs of GSs $G_1^* = (V, R_1, R_2)$ and $G_2^* = (V', R'_1, R'_2)$, respectively, as shown in Figure 9. Here, $\tilde{G}_{v1}$ is identical with

$\tilde{G}_{v2}$ under the mapping $g: V \longrightarrow V'$, defined by $g(a_1) = b_6$, $g(a_2) = b_2$, $g(a_3) = b_4$, $g(a_4) = b_5$, $g(a_5) = b_1$, and $g(a_6) = b_3$ so that

$$\begin{aligned} t_A(a_i) &= t_{A'}(g(a_i)), f_A(a_i) = f_{A'}(g(a_i)), \quad \forall a_i \in V, \\ t_{B_k}(a_i a_j) &= t_{B'_k}(g(a_i)g(a_j)), f_{B_k}(a_i a_j) = f_{B'_k}(g(a_i)g(a_j)), \quad \forall a_i a_j \in R_k, k = 1, 2. \end{aligned} \quad (20)$$

Definition 25. Let $\tilde{G}_v$ be a VGS of a GS G^* . Let ψ denote a permutation on the set $\{R_1, R_2, \dots, R_n\}$ and the corresponding permutation on $\{B_1, B_2, \dots, B_n\}$, i.e., $\psi(B_i) = B_j$ if

$\psi(R_i) = E_j$, $\forall i$. If $ab \in B_r$ for some r and $t_{B_i}^\psi(ab) = t_A(a) \wedge t_A(b) - \bigvee_{j \neq i} t_{\psi(B_j)}(ab)$, $f_{B_i}^\psi(ab) = f_A(a) \vee f_A(b) - \bigvee_{j \neq i} f_{\psi(B_j)}(ab)$, $i = 1, 2, \dots, n$,

FIGURE 8: Isomorphic vague graph structures.

FIGURE 9: Identical vague graph structures.

then, $ab \in B_m^\psi$ while m is chosen so that $t_{B_m^\psi}(ab) \geq t_{B_i^\psi}(ab)$ and $f_{B_m^\psi}(ab) \geq f_{B_i^\psi}(ab)$, $\forall i$. Then, VGS $(A, B_1^\psi, B_2^\psi, \dots, B_n^\psi)$ denoted by G_v^ψ , is named the ψ -complement of VGS.

Example 10. Consider VGS $\tilde{G}_v = (A, B_1, B_2)$ shown in Figure 10 and let ψ be a permutation on the set $\{B_1, B_2\}$ so that $\psi(B_1) = B_2$ and $\psi(B_2) = B_1$. Now, for $a_1 a_2 \in B_1$,

$$\begin{aligned}
 t_{B_1}^\psi(a_1 a_2) &= t_A(a_1) \wedge t_A(a_2) - \bigvee_{j \neq 1} \left[t_{\psi B_j}(a_1 a_2) \right] = 0.2 - 0.2 = 0, \\
 f_{B_1}^\psi(a_1 a_2) &= f_A(a_1) \vee f_A(a_2) - \bigvee_{j \neq 1} \left[f_{\psi B_j}(a_1 a_2) \right] = 0.5 - 0.5 = 0, \\
 t_{B_2}^\psi(a_1 a_2) &= t_A(a_1) \wedge t_A(a_2) - \bigvee_{j \neq 2} \left[t_{\psi B_j}(a_1 a_2) \right] = 0.2 - 0 = 0.2, \\
 f_{B_2}^\psi(a_1 a_2) &= f_A(a_1) \vee f_A(a_2) - \bigvee_{j \neq 2} \left[f_{\psi B_j}(a_1 a_2) \right] = 0.5 - 0 = 0.5.
 \end{aligned} \tag{21}$$

Clearly, $t_{B_2^\psi}(a_1a_2) = 0.2 > 0 = t_{B_1^\psi}(a_1a_2)$ and $f_{B_2^\psi}(a_1a_2) = 0.5 > 0 = f_{B_1^\psi}(a_1a_2)$, so $a_1a_2 \in B_2^\psi$.

$$\begin{aligned} t_{B_1^\psi}(a_1a_3) &= t_A(a_1) \wedge t_A(a_3) - \vee [t_{\psi B_2}(a_1a_3)] = 0.2 - 0.2 = 0, \\ f_{B_1^\psi}(a_1a_3) &= f_A(a_1) \vee f_A(a_3) - \vee [f_{\psi B_2}(a_1a_3)] = 0.4 - 0.4 = 0, \\ t_{B_2^\psi}(a_1a_3) &= t_A(a_1) \wedge t_A(a_3) - \vee [t_{\psi B_1}(a_1a_3)] = 0.2 - 0 = 0.2, \\ f_{B_2^\psi}(a_1a_3) &= f_A(a_1) \vee f_A(a_3) - \vee [f_{\psi B_1}(a_1a_3)] = 0.4 - 0 = 0.4. \end{aligned} \quad (22)$$

So, $a_1a_3 \in B_2^\psi$. Also, for a_2a_3 , we have

$$\begin{aligned} t_{B_1^\psi}(a_2a_3) &= t_A(a_2) \wedge t_A(a_3) - \vee [t_{\psi B_2}(a_2a_3)] = 0.2 - 0 = 0.2, \\ f_{B_1^\psi}(a_2a_3) &= f_A(a_2) \vee f_A(a_3) - \vee [f_{\psi B_2}(a_2a_3)] = 0.5 - 0 = 0.5, \\ t_{B_2^\psi}(a_2a_3) &= t_A(a_2) \wedge t_A(a_3) - \vee [t_{\psi B_1}(a_2a_3)] = 0.2 - 0.2 = 0, \\ f_{B_2^\psi}(a_2a_3) &= f_A(a_2) \vee f_A(a_3) - \vee [f_{\psi B_1}(a_2a_3)] = 0.5 - 0.5 = 0. \end{aligned} \quad (23)$$

So, $a_2a_3 \in B_1^\psi$.

Theorem 1. A ψ -complement of a VGS is a SVGS. In addition, if $\psi(i) = r$, for $r, i \in \{1, 2, \dots, n\}$, then all B_r -edges in $\widetilde{G}_v = (A, B_1, B_2, \dots, B_n)$ become B_i^ψ -edges in $G_v = (A, B_1^\psi, B_2^\psi, \dots, B_n^\psi)$.

Proof. The proof of the first part is clear according to the definition of the ψ -complement $\widetilde{G}_v$ of VGS $\widetilde{G}_v$, since for any B_i^ψ -edge ab , $t_{B_i^\psi}(ab)$ and $f_{B_i^\psi}(ab)$, respectively, have the maximum values of

$$\begin{aligned} [t_A(a) \wedge t_A(b)] - \vee_{j \neq i} t_{\psi B_j}(ab) \\ [f_A(a) \vee f_A(b)] - \vee_{j \neq i} f_{\psi B_j}(ab). \end{aligned} \quad (24)$$

That is,

$$\begin{aligned} t_{B_i^\psi}(ab) &= t_A(a) \wedge t_A(b), \\ f_{B_i^\psi}(ab) &= f_A(a) \vee f_A(b), \end{aligned} \quad (25)$$

for all edges ab in $\widetilde{G}_v$; hence, $\widetilde{G}_v$ is a SVGS. Now assume on contrary that $\psi(i) = r$, but ab is B_s -edge in $\widetilde{G}_v$ with $s \neq r$ that shows $\psi B_i \neq B_s$.

By comparing equations (24) and (25), we have

$$\begin{aligned} \vee_{j \neq i} t_{\psi B_j}(xy) &= 0, \\ \vee_{j \neq i} f_{\psi B_j}(xy) &= 0, \end{aligned} \quad (26)$$

that it is not true because $B_s = \psi B_j$, for some $j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$. Therefore, our assumption is wrong and ab should be a B_r -edge. Thus, we get if $\psi(i) = r$, then, all B_r -edges in VGS $\widetilde{G}_v = (A, B_1, B_2, \dots, B_n)$ become

B_i^ψ -edges in $\widetilde{G}_v^{\psi c} = (A, B_1^\psi, B_2^\psi, \dots, B_n^\psi)$, for $r, i \in \{1, 2, \dots, n\}$. $\square$

Definition 26. Let $\widetilde{G}_v = (A, B_1, B_2, \dots, B_n)$ be a VGS and ψ be a permutation on the set $\{1, 2, \dots, n\}$. Then,

- (i) $\widetilde{G}_v$ is SC, if $\widetilde{G}_v \cong \widetilde{G}_v^{\psi c}$,
- (ii) $\widetilde{G}_v$ is SSC, if it is identical to $\widetilde{G}_v^{\psi c}$.
- (iii) $\widetilde{G}_v$ is TSC if $\widetilde{G}_v \cong \widetilde{G}_v^{\psi c}$, $\forall$ permutations ψ on the set $\{1, 2, \dots, n\}$.
- (iv) $\widetilde{G}_v$ is TSSC, if it is identical to $\widetilde{G}_v^{\psi c}$, $\forall$ permutations ψ on the set $\{1, 2, \dots, n\}$.

Theorem 2. A VGS $\widetilde{G}_v$ is strong if $\widetilde{G}_v$ is TSC.

Proof. Assume $\widetilde{G}_v$ be a SVGS and ψ be a permutation on the set $\{1, 2, \dots, n\}$. If $\psi^{-1}(i) = j$, then, by Theorem 1, all $B_{\psi(i)}$ -edges in $\widetilde{G}_v = (A, B_1, B_2, \dots, B_n)$ become B_j^ψ -edges in $G_v = (A, B_1^\psi, B_2^\psi, \dots, B_n^\psi)$. Also, G_v is strong, so,

$$\begin{aligned} t_{B_i}(a_1a_2) &= t_A(a_1) \wedge t_A(a_2) = t_{B_j^\psi}(a_1a_2), \\ f_{B_i}(a_1a_2) &= f_A(a_1) \vee f_A(a_2) = f_{B_j^\psi}(a_1a_2). \end{aligned} \quad (27)$$

Then, $\widetilde{G}_v$ is isomorphic to $\widetilde{G}_v^{\psi c}$, under the identity mapping $g: V \longrightarrow V$ and a permutation ψ , so that

$$t_A(a) = t_A(g(a)), f_A(a) = f_A(g(a)), \text{ for all } a \in V, \quad (28)$$

and

$$\begin{aligned} t_{B_i}(a_1a_2) &= t_{B_j^\psi}(a_1a_2) = t_{B_j^\psi}(g(a_1)g(a_2)), \\ f_{B_i}(a_1a_2) &= f_{B_j^\psi}(a_1a_2) = f_{B_j^\psi}(g(a_1)g(a_2)), \end{aligned} \quad (29)$$

FIGURE 10: Vague graph structures $\tilde{G}_v$ and $\tilde{G}_v^{\psi c}$.

for all $a_1 a_2 \in R_i$. This holds for all the permutation on the set $\{1, 2, \dots, n\}$. Thus, G_v is TSSC. Conversely, let ψ be any permutation on the set $\{1, 2, \dots, n\}$ and G_v and $G_v^{\psi c}$ be

isomorphic. According to the definitions of ψ -complement and isomorphism, we get

$$\begin{aligned} t_{B_i}(a_1 a_2) &= t_{B_j^{\psi}}(g(a_1)g(a_2)) = t_A(g(a_1)) \wedge t_A(g(a_2)) = t_A(a_1) \wedge t_A(a_2), \\ f_{B_i}(a_1 a_2) &= f_{B_j^{\psi}}(g(a_1)g(a_2)) = f_A(g(a_1)) \vee f_A(g(a_2)) = f_A(a_1) \vee f_A(a_2), \end{aligned} \quad (30)$$

for all $a_1 a_2 \in E_i$, $i = 1, 2, \dots, n$. Hence, $\tilde{G}_v$ is a SVGS. $\square$

Theorem 3. If $GS G^* = (V, R_1, R_2, \dots, R_n)$ is TSSC and A is a VS of V with constant fuzzy mapping t_A and f_A , then, a SVGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ of G^* is TSSC.

Proof. Consider a SVGS $\tilde{G}_v = (A, B_1, B_2, \dots, B_n)$ of a GS $G^* = (V, R_1, R_2, \dots, R_n)$. Assume that G^* is TSSC and that

for some constants $s, t \in [0, 1]$, $A = (t_A, f_A)$ is a VS of u so that $t_A(u) = s$, $f_A(u) = t$, $\forall u \in V$. Then, we have to prove that $\tilde{G}_v$ is TSSC. Let ψ be an arbitrary permutation on the set $\{1, 2, \dots, n\}$ and $\psi^{-1}(j) = i$. Since G^* is TSSC, so $\exists$ a bijection $g: V \rightarrow V$ so that for each R_i -edge $a_1 a_2$ in G^* , $g(a_1)g(a_2)$ is an R_i -edge in $(G^*)^{\psi^{-1}c}$. Consequently, for every B_i -edge $a_1 a_2$ in $\tilde{G}_v$, $g(a_1)g(a_2)$ is a B_i^{ψ} -edge in $\tilde{G}_v^{\psi c}$. From the definition of A and the definition of SVGS $\tilde{G}_v$,

$$t_A(a) = s = t_A(g(a)), f_A(a) = t = f_A(g(a)), \text{ for all } a, g(a) \in V, \quad (31)$$

and

$$\begin{aligned} t_{B_i}(a_1 a_2) &= t_A(a_1) \wedge t_A(a_2) = t_A(g(a_1)) \wedge t_A(g(a_2)) = t_{B_i^{\psi}}(g(a_1)g(a_2)), \\ f_{B_i}(a_1 a_2) &= f_A(a_1) \vee f_A(a_2) = f_A(g(a_1)) \vee f_A(g(a_2)) = f_{B_i^{\psi}}(g(a_1)g(a_2)), \end{aligned} \quad (32)$$

for all $a_1 a_2 \in B_i$, $i = 1, 2, \dots, n$, which shows $\tilde{G}_v$ is SSC. Hence, $\tilde{G}_v$ is TSSC, since ψ is arbitrary. $\square$

Example 11. A VGS $\tilde{G}_v = (A, B_1, B_2, B_3)$ of GS $G^* = (V, R_1, R_2, R_3)$ as shown in Figure 11 is TSSC.

4. Application

Providing services to the people in the shortest possible time is one of the most important issues that governments strive to do in the best possible way. Unfortunately, in the past, due to the lack of medical and welfare services as well as the lack of educational centers, people had to travel to distant cities to

provide these necessary facilities and equipment, and in this way, unfortunately, they incurred a lot of costs. But today, with the help of governments as well as the equipping of medical and welfare centers, people can move to their nearest neighboring city to benefit from these facilities. A vague graph structure helps us to understand which route is more appropriate for a particular service. It also shows us which service is increasing so that governments can strengthen it in the same way and also try to increase the rest of the services so that people can benefit from them. Most importantly, a VGS can also be useful to the mayor of the area, as it helps the mayor make appropriate policies for the city to develop appropriate services. For example, if one of

FIGURE 11: Totally strong self-complementary.

the welfare services in that area is more than other services (the number of people referring to that welfare center is higher), then, the VGS shows this issues and as a result, the mayor takes the necessary measures to maintain and increase this opportunity. Suppose A is a set consisting of 7 cities as $A = \{\text{Neka, Behshahr, Sari, Ghaemshahr, Babol, Bahnamir, Babolsar}\}$, and H is a VS on A as Table 2.

In Table 2, true membership value indicates the amount of welfare, educational, medical, and peace facilities in that area, and the false-membership value reveals the lack of necessary facilities in each of the medical services. In Tables 3–9, we show the membership rate of various welfare

services on the route between each pair of cities. The relations that considered on set A are as follows:

$R_1 = \text{Grocery stores}$, $R_2 = \text{Banking services}$,
 $R_3 = \text{Car daelership}$, $R_4 = \text{Hospitals}$, and
 $R_5 = \text{University and Schools}$ so that $(A, R_1, R_2, R_3, R_4, R_5)$ is a graph structure. Each object in a particular relation indicates the amount of services provided between the two cities. In these relations, each element is determined by its corresponding membership rate, and the corresponding sets are FSs on R_1, R_2, R_3, R_4 , and R_5 , respectively, which are shown by B_1, B_2, B_3, B_4 , and B_5 , respectively.

Suppose that

$$\begin{aligned} R_1 &= \{(\text{Neka, Sari}), (\text{Sari, Behshahr})\}, \\ R_2 &= \{(\text{Behshahr, Babol}), (\text{Babolsar, Babol})\}, \\ R_3 &= \{(\text{Babol, Ghaemshahr}), (\text{Sari, Ghaemshahr}), (\text{Bahnamir, Babolsar})\}, \\ R_4 &= \{(\text{Neka, Bahnamir}), (\text{Sari, Bahnamir})\}, \\ R_5 &= \{(\text{Ghaemshahr, Bahnamir}), (\text{Babolsar, Sari})\}. \end{aligned} \quad (33)$$

Now, corresponding vague sets are

$$\begin{aligned} B_1 &= \{((\text{Neka, Sari}), (0.6, 0.2)), ((\text{Sari, Behshahr}), (0.7, 0.2))\}, \\ B_2 &= \{((\text{Behshahr, Babol}), (0.7, 0.2)), ((\text{Babolsar, Babol}), (0.8, 0.2))\}, \\ B_3 &= \left\{ ((\text{Babol, Ghaemshahr}), (0.5, 0.1)), ((\text{Sari, Ghaemshahr}), (0.7, 0.3)), \right. \\ &\quad \left. ((\text{Bahnamir, Babolsar}), (0.7, 0.2)) \right\}, \\ B_4 &= \{((\text{Neka, Bahnamir}), (0.7, 0.1)), ((\text{Sari, Bahnamir}), (0.6, 0.2))\}, \\ B_5 &= \{((\text{Ghaemshahr, Bahnamir}), (0.7, 0.2)), ((\text{Babolsar, Sari}), (0.6, 0.3))\}. \end{aligned} \quad (34)$$

TABLE 2: Vague set H on A .

City	Membership value
Neka	(0.3, 0.4)
Behshahr	(0.6, 0.3)
Sari	(0.8, 0.2)
Ghaemshahr	(0.5, 0.3)
Babol	(0.5, 0.2)
Bahnamir	(0.3, 0.2)
Babolsar	(0.6, 0.3)

TABLE 3: Vague set of services on roads connecting Neka and other cities.

Services	(Neka, Behshahr)	(Neka, Sari)	(Neka, Bahnamir)	(Neka, Babolsar)
Grocery store	(0.6, 0.2)	(0.5, 0.2)	(0.5, 0.2)	(0.7, 0.2)
Banking services	(0.6, 0.2)	(0.6, 0.2)	(0.4, 0.3)	(0.6, 0.3)
Car dealership	(0.5, 0.3)	(0.5, 0.2)	(0.3, 0.2)	(0.5, 0.4)
Hospitals	(0.6, 0.2)	(0.5, 0.3)	(0.4, 0.2)	(0.6, 0.3)
University and schools	(0.5, 0.4)	(0.5, 0.3)	(0.6, 0.2)	(0.5, 0.2)

TABLE 4: Vague set of services on roads connecting Behshahr and other cities.

Services	(Behshahr, Neka)	(Behshahr, Babol)	(Behshahr, Sari)
Grocery store	(0.7, 0.2)	(0.6, 0.2)	(0.7, 0.2)
Banking services	(0.8, 0.1)	(0.7, 0.2)	(0.4, 0.3)
Car dealership	(0.6, 0.2)	(0.6, 0.3)	(0.8, 0.2)
Hospitals	(0.8, 0.2)	(0.7, 0.2)	(0.6, 0.2)
University and schools	(0.7, 0.2)	(0.5, 0.3)	(0.6, 0.3)

TABLE 5: Vague set of services on roads connecting Sari and other cities.

Services	(Sari, Behshahr)	(Sari, Neka)	(Sari, Ghaemshahr)
Grocery store	(0.7, 0.2)	(0.5, 0.2)	(0.6, 0.2)
Banking services	(0.5, 0.3)	(0.4, 0.2)	(0.4, 0.3)
Car dealership	(0.6, 0.2)	(0.5, 0.3)	(0.6, 0.2)
Hospitals	(0.6, 0.4)	(0.4, 0.3)	(0.7, 0.3)
University and schools	(0.7, 0.2)	(0.5, 0.2)	(0.6, 0.2)

TABLE 6: Vague set of services on roads connecting Babol and other cities.

Services	(Babol, Behshahr)	(Babol, Babolsar)	(Babol, Ghaemshahr)
Grocery store	(0.5, 0.2)	(0.6, 0.2)	(0.4, 0.3)
Banking services	(0.6, 0.3)	(0.6, 0.1)	(0.4, 0.2)
Car dealership	(0.7, 0.2)	(0.5, 0.3)	(0.4, 0.3)
Hospitals	(0.6, 0.4)	(0.6, 0.2)	(0.4, 0.2)
University and schools	(0.6, 0.2)	(0.7, 0.1)	(0.5, 0.1)

TABLE 7: Vague set of services on roads connecting Ghaemshahr and other cities.

Services	(Ghaemshahr, Babol)	(Ghaemshahr, Sari)	(Ghaemshahr, Bahnamir)
Grocery store	(0.7, 0.1)	(0.7, 0.2)	(0.6, 0.2)
Banking services	(0.6, 0.3)	(0.8, 0.1)	(0.4, 0.3)
Car dealership	(0.6, 0.2)	(0.7, 0.3)	(0.7, 0.2)
Hospitals	(0.5, 0.4)	(0.6, 0.2)	(0.4, 0.2)
University and schools	(0.6, 0.2)	(0.5, 0.4)	(0.6, 0.3)

TABLE 8: Vague set of services on roads connecting Babolsar and other cities.

Services	(Babolsar, Babol)	(Babolsar, Sari)	(Babolsar, Bahnamir)
Grocery store	(0.7, 0.1)	(0.5, 0.2)	(0.6, 0.2)
Banking services	(0.5, 0.3)	(0.4, 0.3)	(0.4, 0.3)
Car dealership	(0.8, 0.2)	(0.6, 0.3)	(0.5, 0.2)
Hospitals	(0.8, 0.2)	(0.5, 0.3)	(0.4, 0.3)
University and schools	(0.7, 0.2)	(0.5, 0.3)	(0.5, 0.3)

TABLE 9: Vague set of services on roads connecting Bahnamir and other cities.

Services	(Bahnamir, Neka)	(Bahnamir, Sari)	(Bahnamir, Babolsar)
Grocery store	(0.5, 0.2)	(0.5, 0.3)	(0.5, 0.3)
Banking services	(0.6, 0.3)	(0.6, 0.2)	(0.6, 0.2)
Car dealership	(0.6, 0.3)	(0.5, 0.2)	(0.7, 0.2)
Hospitals	(0.7, 0.1)	(0.6, 0.2)	(0.5, 0.3)
University and schools	(0.6, 0.2)	(0.5, 0.3)	(0.4, 0.3)

FIGURE 12: VGS showing the most important services along the roads connecting the two cities.

It is obvious that $(H, B_1, B_2, B_3, B_4, B_5)$ is a VGS and is denoted in Figure 12. In the VGS shown in Figure 12, each edge represents the most services that occur on the connecting route between the two cities. For example, most

services on the way from Babolsar to Bahnamir are grocery services, and their membership value are $(0.7, 0.2)$. It is clear that the city of Sari has the highest membership in terms of banking services, which means that this city is the most

sensitive city for providing banking services. According to the VGS in Figure 12, the most common services is car dealership. Therefore, we conclude that government should identify places that are suitable for providing services between cities so that people can benefit from these services at the lowest cost and in the fastest possible time.

5. Conclusion

Fuzzy graph has various uses in modern science and technology, especially in the fields of neural networks, computer science, operation research, and decision making. VGS have more precision, flexibility, and compatibility, as compared to the fuzzy graphs. Today, VGSs play an important role in social networks and allow users to find the most effective person in a group or organization. So, in this paper, we presented certain notions, including VGS, SVGS, and vague B_i -cycle and illustrated these notions by several examples. We investigated ψ -complement, SC, SSC, and TSSC in VGS and studied some of their properties. Finally, an application of VGS has been introduced. In our future work, we will introduce neighborly irregular, highly irregular, strongly irregular, and edge irregularity in VGSs and investigate some of their properties.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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