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Research Article

Darboux Transformations and Soliton-Like Solutions of a New System Associated with the Negative AKNS System

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In this letter, we consider a new two-component system. By its reciprocal relation with the first negative flow in the AKNS hierarchy, we construct three Darboux-Bäcklund transformations for the new system and obtain some soliton-like solutions.

1. Introduction

It is well known that Darboux transformation (DT) and Bäcklund transformation (BT) play important roles in establishing new exact solutions of integrable systems from old ones [1–4]. For example, it may be used to construct multisoliton solutions from a trivial seed. For some classical integrable systems, one may construct DT/BT via the gauge transformation. But for others such as the Camassa-Holm equation and the complex short pulse equation, one can not construct their DTs/BTs directly, because they involve the independent variable x. Fortunately, the reciprocal transformation provides a useful tool [5–8]. There are many other methods to study exact solutions of integrable systems, see Refs. [9–13].

Recently, while discussing reciprocal transformations of negative flows of some classical integrable hierarchies, Li and Wu [14] proposed some new systems; one of them reads

$$\begin{gathered} u_t = 2bu^2 - 2au_x, \, v_t = -2cv^2 - 2av_x, \\ b_x = -2av, \, c_x = 2au, \, a_x = cv - bu. \end{gathered} \tag{1}$$

A Lax pair of (1) has been derived by using its reciprocal relation with the first negative flow in the AKNS hierarchy. Based on this Lax pair and using the standard approach, one may present infinitely many conserved quantities for it. However, its DT/BT and exact solutions are still unknown.

In this letter, we will construct three BTs and provide some soliton-like solutions for (1). The paper is arranged as follows. In Section 2, we will construct three BTs of (1) by using its reciprocal link with the first negative flow in the AKNS hierarchy. In Section 3, we will construct the multisoliton solutions of (1) by using the BTs discovered above.

2. Darboux Transformations for the Negative AKNS System

As pointed out in [14], the system (1) admits the following Lax pair

$$\begin{gathered} \psi_x = \begin{bmatrix} \lambda uv & v \\ u & -\lambda uv \end{bmatrix} \psi, \\ \psi_t = \begin{bmatrix} \dfrac{a}{\lambda} - 2\lambda auv & \dfrac{b}{\lambda} - 2av \\ \dfrac{c}{\lambda} - 2au & -\dfrac{a}{\lambda} + 2\lambda auv \end{bmatrix} \psi, \end{gathered} \tag{2}$$

with $\psi = (\psi_1, \psi_2)^T$. The system (1) possesses a conservation law of the form $(uv)_t = -2(auv)_x$, which allows a reciprocal transformation

$$\begin{gathered} dy = uvdx - 2auvdt, \\ d\tau = dt. \end{gathered} \tag{3}$$

Through this reciprocal transformation, it is easy to show that (1) is changed to the first negative flow in the AKNS hierarchy

$$\begin{gathered}\tilde{u}_\tau = -2b, \tilde{v}_\tau = 2c,\\ b_y = -2a\tilde{u}, c_y = 2a\tilde{v}, a_y = c\tilde{u} - b\tilde{v},\end{gathered} \tag{4}$$

and the Lax pair (2) is converted to

$$\begin{gathered}\psi_y = \begin{bmatrix}\lambda & \tilde{u}\\ \tilde{v} & -\lambda\end{bmatrix}\psi,\\ \psi_\tau = \frac{1}{\lambda}\begin{bmatrix}a & b\\ c & -a\end{bmatrix}\psi.\end{gathered} \tag{5}$$

Now, we turn our attention to the negative AKNS system [15–17]. The famous DTs for the negative AKNS system may be summarized as the following proposition.

Proposition 1. *The Lax pair (5) is covariant with respect to the following three DTs:*

(1) The first one is given by

$$\begin{gathered}\psi[1] = T_1\psi,\\ T_1 = \begin{bmatrix}\lambda - \lambda_1 - \dfrac{\tilde{u}f_2}{2f_1} & \dfrac{\tilde{u}}{2}\\ -\dfrac{f_2}{f_1} & 1\end{bmatrix},\end{gathered} \tag{6}$$

where $(f_1, f_2)^T$ is a special solution of the Lax pair (5) at $\lambda = \lambda_1$.

(2) The second one reads

$$\begin{gathered}\psi[1] = T_2\psi,\\ T_2 = \begin{bmatrix}1 & -\dfrac{g_1}{g_2}\\ -\dfrac{\tilde{v}}{2} & \lambda - \lambda_1 + \dfrac{\tilde{v}g_1}{2g_2}\end{bmatrix},\end{gathered} \tag{7}$$

where $(g_1, g_2)^T$ is a special solution of the Lax pair (5) at $\lambda = \lambda_1$.

(3) The third one is provided by

$$\begin{gathered}\psi[1] = T_3\psi,\\ T_3 = \lambda I - H\Lambda H^{-1},\end{gathered} \tag{8}$$

where $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2)$, $H = (h_{i,j})_{2\times 2}$, and $(h_{1i}, h_{2i})^T$ is a special solution of the Lax presentation (5) at $\lambda = \lambda_i$.

3. DTs and Some Soliton-Like Solutions of the New System

In this section, we will use the three DTs for the negative AKNS system in Proposition 1 to construct three BTs, which involve the independent variable x, for the new system (1) [8]. The key point of the procedure is to determine the coordinate transformation between (x, t) and (y, τ) via the inverse transformation of (3).

Firstly, let us apply the first DT (6) to the Lax pair (5), which leads to the BT between $(\tilde{u}[1], \tilde{v}[1], a[1], b[1], c[1])$ and $(\tilde{u}, \tilde{v}, a, b, c)$. In particular, we have

$$\begin{gathered}a[1] = a - \frac{c\tilde{u}}{2\lambda_1} + \frac{a\tilde{u}f_2}{\lambda_1 f_1} + \frac{bf_2}{f_1} + \frac{b\tilde{u}f_2^2}{2\lambda_1 f_1^2},\\ \tilde{u}[1] = -\lambda_1\tilde{u} + \frac{\tilde{u}_y}{2} - \frac{\tilde{u}^2 f_2}{2f_1},\\ \tilde{v}[1] = -\frac{2f_2}{f_1},\end{gathered} \tag{9}$$

which leads to

$$\begin{gathered}\tilde{u}[1]\tilde{v}[1] = \tilde{u}\tilde{v} - \left(\frac{\tilde{u}f_2}{f_1}\right)_y,\\ 2a[1] = 2a - \left(\frac{\tilde{u}f_2}{f_1}\right)_\tau.\end{gathered} \tag{10}$$

Considering the inverse of (3), we may obtain a BT of system (2) as

$$\begin{gathered}x[1] = x - \frac{\tilde{u}f_2}{f_1} + l_1,\\ u[1] = -\frac{2f_1}{2\lambda_1\tilde{u}f_1 - \tilde{u}_y f_1 + \tilde{u}^2 f_2},\\ v[1] = -\frac{f_1}{2f_2},\end{gathered} \tag{11}$$

where $x = \int \tilde{u}\tilde{v}dy + \int 2ad\tau$ and l_1 is an arbitrary constant.

Hereafter, as an example of use of this BT, let us choose a seed of (4) as

$$\tilde{u} = c_1 e^{k_1\tau}, \tilde{v} = c_2 e^{-k_1\tau}, a = 0, b = -\frac{1}{2}c_1 k_1 e^{k_1\tau}, c = -\frac{1}{2}c_2 k_1 e^{-k_1\tau}, \tag{12}$$

where $k_1 \neq 0$ is an arbitrary constant. It follows immediately that

$$\begin{gathered}f_1 = \alpha_1 e^{(k_1/2)\tau + \xi} + \beta_1 e^{(k_1/2)\tau - \xi},\\ f_2 = \frac{1}{c_1}\left[\alpha_1(\sqrt{\eta} - \lambda_1)e^{-(k_1/2)\tau + \xi} - \beta_1(\sqrt{\eta} + \lambda_1)e^{-(k_1/2)\tau - \xi}\right],\end{gathered} \tag{13}$$

with $\eta=\lambda_1^2+c_1c_2, \xi=\sqrt{\eta}(y-(k_1/2\lambda_1)\tau)$. Now, substituting them into (11) yields

$$x[1]=c_1c_2y-c_1e^{k_1\tau}\frac{f_2}{f_1}+l_1,$$
$$u[1]=-\frac{2f_1}{2\lambda_1c_1f_1e^{k_1\tau}+c_1^2f_2e^{2k_1\tau}}, \qquad (14)$$
$$v[1]=-\frac{f_1}{2f_2}.$$

A profile of 2-kink like solution is plotted in Figures 1 and 2 at $c_1=1, c_2=-1, k_1=1, \lambda_1=2, \alpha_1=1, \beta_1=1, l_1=-4$.

Secondly, applying the DT (7) to the Lax presentation (5) yields

$$a[1]=a-\frac{b\tilde{v}}{2\lambda_1}-\frac{a\tilde{v}g_1}{\lambda_1g_2}-\frac{cg_1}{g_2}+\frac{c\tilde{v}g_1^2}{2\lambda_1g_2^2},$$
$$\tilde{u}[1]=\frac{2g_1}{g_2}, \qquad (15)$$
$$\tilde{v}[1]=-\lambda_1\tilde{v}-\frac{\tilde{v}_y}{2}+\frac{\tilde{v}^2g_1}{2g_2},$$

which leads to

$$\tilde{u}[1]\tilde{v}[1]=\tilde{u}\tilde{v}-\left(\frac{\tilde{v}g_1}{g_2}\right)_y,$$
$$2a[1]=2a-\left(\frac{\tilde{v}g_1}{g_2}\right)_\tau. \qquad (16)$$

Using the inverse of (3), we may obtain the second BT of system (2) as

$$x[1]=x-\frac{\tilde{v}g_1}{g_2}+l_2,$$
$$u[1]=\frac{g_2}{2g_1}, \qquad (17)$$
$$v[1]=\frac{2g_2}{-2\lambda_1\tilde{v}g_2-\tilde{v}_yg_2+\tilde{v}^2g_1},$$

with $x=\int\tilde{u}\tilde{v}dy+\int 2ad\tau$ and l_2 is an arbitrary constant. Choosing the same seed as (12), one may obtain

$$x[1]=c_1c_2y-c_2e^{-k_1\tau}\frac{f_1}{f_2}+l_2,$$
$$u[1]=\frac{f_2}{2f_1}, \qquad (18)$$
$$v[1]=\frac{2f_2}{-2\lambda_1c_2f_2e^{-k_1\tau}+c_2^2f_1e^{-2k_1\tau}}.$$

The profile is similar to the first case, and hence, we omit it here.

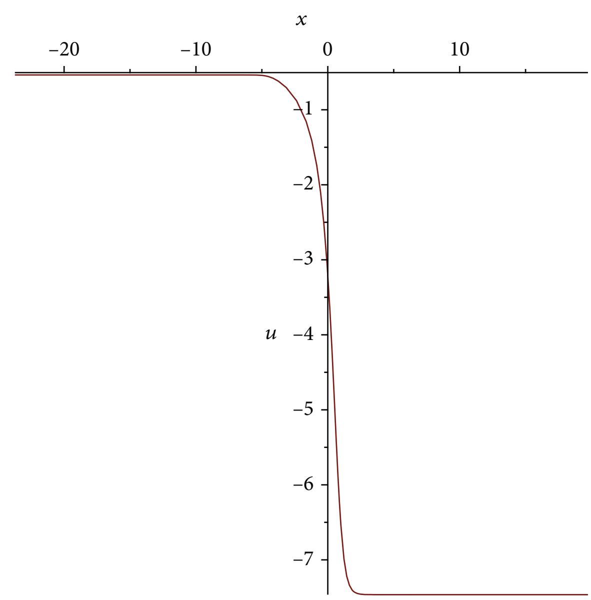

Figure 1: One antikink for u.

Finally, imposing the DT (8) to the Lax pair (5) and setting $S=H\Lambda H^{-1}$ leads to

$$a[1]=a-(s_{11})_\tau,$$
$$\tilde{u}[1]=\tilde{u}+2s_{12}, \qquad (19)$$
$$\tilde{v}[1]=\tilde{v}-2s_{21},$$

where

$$s_{11}=\frac{\lambda_1h_{11}h_{22}-\lambda_2h_{12}h_{21}}{h_{11}h_{22}-h_{12}h_{21}},$$
$$s_{12}=\frac{(\lambda_2-\lambda_1)h_{11}h_{12}}{h_{11}h_{22}-h_{12}h_{21}}, \qquad (20)$$
$$s_{21}=\frac{(\lambda_1-\lambda_2)h_{21}h_{22}}{h_{11}h_{22}-h_{12}h_{21}}.$$

Furthermore, a straightforward calculation shows that

$$(s_{11})_y+\tilde{v}s_{12}-\tilde{u}s_{21}-2s_{12}s_{21}=0. \qquad (21)$$

Combining (19) and (21), we arrive at

$$\tilde{u}[1]\tilde{v}[1]=\tilde{u}\tilde{v}-2(s_{11})_y,$$
$$2a[1]=2a-2(s_{11})_\tau. \qquad (22)$$

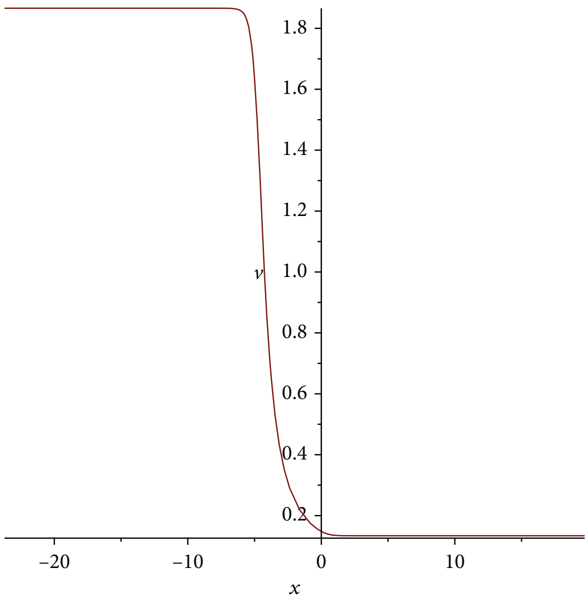

FIGURE 2: One antikink for v.

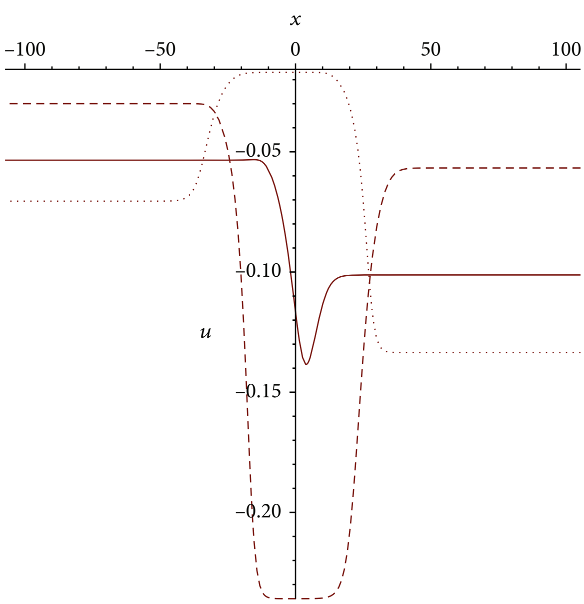

FIGURE 3: Two kinks for u ($t=-200$ for $u/10^9$, linestyle is dot; $t=10$ for u, linestyle is solid; $t=200$ for $10^8 u$, linestyle is dash).

Hence, we acquire the third BT of system (2), which is

$$\begin{aligned} x[1] &= x - 2s_{11} + l_3, \\ u[1] &= \frac{1}{\tilde{u} + 2s_{12}}, \\ v[1] &= \frac{1}{\tilde{v} - 2s_{21}}, \end{aligned} \tag{23}$$

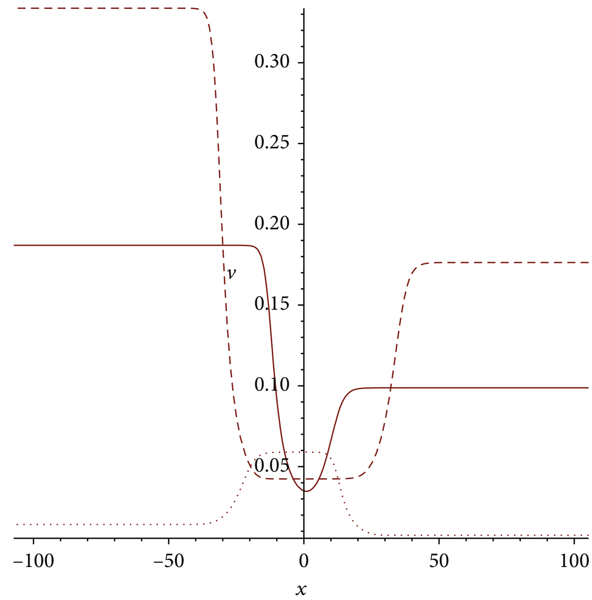

FIGURE 4: Two kinks for v ($t=-200$ for $10^7 v$, linestyle is dot; $t=10$ for $v/10$, linestyle is solid; $t=200$ for $v/10^9$, linestyle is dash.).

where $x = \int \tilde{u}\tilde{v}dy + \int 2ad\tau$ and l_3 is an arbitrary constant. Choosing the same initial solution as (12) gives rise to

$$\begin{aligned} h_{1i} &= \alpha_i e^{(k_1/2)\tau+\xi_i} + \beta_i e^{(k_1/2)\tau-\xi_i}, \\ h_{2i} &= \frac{1}{c_1}\left[\alpha_i\left(\sqrt{\eta_i} - \lambda_i\right)e^{-(k_1/2)\tau+\xi_i} - \beta_i\left(\sqrt{\eta_i} + \lambda_i\right)e^{-(k_1/2)\tau-\xi_i}\right], \end{aligned} \tag{24}$$

where $\eta_i = \lambda_i^2 + c_1 c_2$, $\xi_i = \sqrt{\eta_i}(y - (k_1/2\lambda_i)\tau)$, $i = 1, 2$. Substituting them into (23), we get

$$\begin{aligned} x[1] &= c_1 c_2 y - 2s_{11} + l_3, \\ u[1] &= \frac{1}{c_1 e^{k_1\tau} + 2s_{12}}, \\ v[1] &= \frac{1}{c_2 e^{-k_1\tau} - 2s_{21}}. \end{aligned} \tag{25}$$

A profile of 2-kink like solution is plotted in Figures 3 and 4 at $c_1 = 5$, $c_2 = -2$, $k_1 = 1/10$, $\lambda_1 = -4$, $\lambda_2 = 5$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 1$, $l_3 = 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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