Research Article

Darboux Transformations and Soliton-Like Solutions of a New System Associated with the Negative AKNS System

Shilong Huang and Hongmin Li

School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China

Correspondence should be addressed to Hongmin Li; lihongmin@hqu.edu.cn

Received 30 November 2021; Revised 24 March 2022; Accepted 28 March 2022; Published 15 April 2022

Copyright © 2022 Shilong Huang and Hongmin Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this letter, we consider a new two-component system. By its reciprocal relation with the first negative flow in the AKNS hierarchy, we construct three Darboux-Bäcklund transformations for the new system and obtain some soliton-like solutions.

1. Introduction

It is well known that Darboux transformation (DT) and Bäcklund transformation (BT) play important roles in establishing new exact solutions of integrable systems from old ones [1–4]. For example, it may be used to construct multi-soliton solutions from a trivial seed. For some classical integrable systems, one may construct DT/BT via the gauge transformation. But for others such as the Camassa-Holm equation and the complex short pulse equation, one can not construct their DTs/BTs directly, because they involve the independent variable $x$. Fortunately, the reciprocal transformation provides a useful tool [5–8]. There are many other methods to study exact solutions of integrable systems, see Refs. [9–13].

Recently, while discussing reciprocal transformations of negative flows of some classical integrable hierarchies, Li and Wu [14] proposed some new systems; one of them reads

$$u_t = 2bu^2 - 2au_x, \quad v_t = -2cv^2 - 2av_x,$$

$$b_x = -2av, \quad c_x = 2au, \quad a_x = cv - bu.$$  \hfill (1)

A Lax pair of (1) has been derived by using its reciprocal relation with the first negative flow in the AKNS hierarchy. Based on this Lax pair and using the standard approach, one may present in finitely many conserved quantities for it. However, its DT/BT and exact solutions are still unknown.

In this letter, we will construct three BTs and provide some soliton-like solutions for (1). The paper is arranged as follows. In Section 2, we will construct three BTs of (1) by using its reciprocal link with the first negative flow in the AKNS hierarchy. In Section 3, we will construct the multisoliton solutions of (1) by using the BTs discovered above.

2. Darboux Transformations for the Negative AKNS System

As pointed out in [14], the system (1) admits the following Lax pair

$$
\psi_x = \begin{bmatrix} \lambda uv & v \\ u & -\lambda uv \end{bmatrix} \psi, \\
\psi_t = \begin{bmatrix} a & -2\lambda uv \\ \frac{a}{\lambda} - 2au & b - 2av \\ c - 2au & -\frac{a}{\lambda} + 2\lambda uv \end{bmatrix} \psi,
$$  \hfill (2)

with $\psi = (\psi_1, \psi_2)^T$. The system (1) possesses a conservation law of the form $(uv)_t = -2(auv)_x$, which allows a reciprocal transformation

$$dy = uvdx - 2auvdt,$$

$$d\tau = dt.$$  \hfill (3)
Through this reciprocal transformation, it is easy to show that (1) is changed to the first negative flow in the AKNS hierarchy

\[
\begin{align*}
\dot{u} &= -2b, \quad \dot{v} = 2c, \\
\dot{b} &= -2a\ddot{u}, \quad \dot{c} = 2av, \quad \dot{a} = c\dot{u} - b\dot{v},
\end{align*}
\]

(4)

and the Lax pair (2) is converted to

\[
\begin{align*}
\Psi_\tau &= \begin{bmatrix} \lambda & \ddot{u} \\ \ddot{v} & -\lambda \end{bmatrix} \Psi, \\
\Psi_t &= \frac{1}{\lambda} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \Psi.
\end{align*}
\]

(5)

Now, we turn our attention to the negative AKNS system [15–17]. The famous DTs for the negative AKNS system may be summarized as the following proposition.

**Proposition 1.** The Lax pair (5) is covariant with respect to the following three DTs:

1. The first one is given by

\[
\begin{align*}
\Psi'[1] &= T_1 \Psi, \\
T_1 &= \begin{bmatrix} \lambda - \lambda_1 & \frac{\ddot{u}f_2}{2f_1} \\ -\frac{f_2}{f_1} & 1 \end{bmatrix},
\end{align*}
\]

(6)

where \((f_1, f_2)^T\) is a special solution of the Lax pair (5) at \(\lambda = \lambda_1\).

2. The second one reads

\[
\begin{align*}
\Psi'[1] &= T_2 \Psi, \\
T_2 &= \begin{bmatrix} 1 & \frac{g_1}{g_2} \\ -\frac{\ddot{v}}{2} & \lambda - \lambda_1 + \frac{\ddot{v}g_1}{2g_2} \end{bmatrix},
\end{align*}
\]

(7)

where \((g_1, g_2)^T\) is a special solution of the Lax pair (5) at \(\lambda = \lambda_1\).

3. The third one is provided by

\[
\begin{align*}
\Psi'[1] &= T_3 \Psi, \\
T_3 &= \lambda I - H \Delta H^{-1},
\end{align*}
\]

(8)

where \(\Lambda = \text{diag}(\lambda_1, \lambda_2), \ H = (h_{ij})_{2 \times 2}\), and \((h_{ij}, h_{ji})^T\) is a special solution of the Lax presentation (5) at \(\lambda = \lambda_1\).

### 3. DTs and Some Soliton-Like Solutions of the New System

In this section, we will use the three DTs for the negative AKNS system in Proposition 1 to construct three BTs, which involve the independent variable \(x\), for the new system (1) [8]. The key point of the procedure is to determine the coordinate transformation between \((x, t)\) and \((y, \tau)\) via the inverse transformation of (3).

Firstly, let us apply the first DT (6) to the Lax pair (5), which leads to the BT between \((\ddot{u}[1], \ddot{v}[1], a[1], b[1], c[1])\) and \((\ddot{u}, \ddot{v}, a, b, c)\). In particular, we have

\[
a[1] = a - \frac{c\ddot{u}}{2\lambda_1} + \frac{a\ddot{u}f_2}{\lambda_1 f_1} + \frac{bf_2}{f_1} + \frac{b\ddot{u}f_2}{2\lambda_1 f_1^2},
\]

\[
\ddot{u}[1] = -\lambda_1 \ddot{u} + \frac{\ddot{u}f_2}{2f_1},
\]

(9)

\[
\ddot{v}[1] = -\frac{2f_2}{f_1},
\]

which leads to

\[
\ddot{u}[1]\ddot{v}[1] = \ddot{u}\ddot{v} - \left(\frac{\ddot{u}f_2}{f_1}\right)_y,
\]

(10)

Considering the inverse of (3), we may obtain a BT of system (2) as

\[
\begin{align*}
\tau[1] &= x - \frac{\ddot{v}f_2}{f_1} + l_1, \\
u[1] &= -\frac{2f_1}{2\lambda_1 \ddot{u}f_1 - \ddot{u}y f_1 + \ddot{u}^2 f_2},
\end{align*}
\]

(11)

where \(x = \int \ddot{u}d\tau + \int 2ad\tau\) and \(l_1\) is an arbitrary constant.

Hereafter, as an example of use of this BT, let us choose a seed of (4) as

\[
\ddot{u} = c_i e^{k_i \tau}, \quad \ddot{v} = c_x e^{-k_i \tau}, \quad a = 0, \quad b = -\frac{1}{2} c_i k_x e^{k_i \tau}, \quad c = -\frac{1}{2} c_i k_x e^{-k_i \tau},
\]

(12)

where \(k_i \neq 0\) is an arbitrary constant. It follows immediately that

\[
f_1 = \alpha_i e^{k_i(2\tau + \xi)} + \beta_i e^{k_i(2\tau - \xi)},
\]

\[
f_2 = \frac{1}{c_i} \left[ \alpha_i (\sqrt{\gamma} - \lambda_1) e^{-(k_i/2)\tau - \xi} - \beta_i (\sqrt{\gamma} + \lambda_1) e^{-(k_i/2)\tau + \xi} \right],
\]

(13)
with \( \eta = \lambda_1^2 + c_1 c_2, \xi = \sqrt{\eta} (y - (k_1/2 \lambda_1) \tau). \) Now, substituting them into (11) yields

\[
x[1] = c_1 c_2 y - c_1 e^{k_1 \tau} f_1 / f_2 + l_2,
\]

\[
u[1] = -f_1 / 2 f_2,
\]

\[
u[1] = -f_1 / 2 f_2,
\]

which leads to

\[
\hat{u}[1] \hat{v}[1] = \hat{u} v - \left( \frac{\hat{v} g_1}{g_2} \right) y,
\]

\[
2 a[1] = 2 a - \left( \frac{\hat{v} g_1}{g_2} \right) y.
\]

Using the inverse of (3), we may obtain the second BT of system (2) as

\[
x[1] = x - \frac{\hat{v} g_1}{g_2} + l_2,
\]

\[
u[1] = \frac{g_2 f_1}{2 g_1},
\]

with \( x = \int u v d y + \int 2 a d \tau \) and \( l_2 \) is an arbitrary constant. Choosing the same seed as (12), one may obtain

\[
x[1] = c_1 c_2 y - c_2 e^{-k_2 \tau} f_1 / f_2 + l_2,
\]

\[
u[1] = f_1 / 2 f_2,
\]

The profile is similar to the first case, and hence, we omit it here.

A profile of 2-kink like solution is plotted in Figures 1 and 2 at \( c_1 = 1, c_2 = -1, k_1 = 1, \lambda_1 = 2, a_1 = 1, \beta_1 = 1, l_1 = -4. \)

Secondly, applying the DT (7) to the Lax presentation (5) yields

\[
a[1] = a - b \hat{v} g_1 - \frac{c g_1}{\lambda_1 g_2} + \frac{c \hat{v} g_1^2}{2 \lambda_1 g_2^2},
\]

\[
\hat{u}[1] = \frac{2 g_1}{g_2},
\]

\[
\hat{v}[1] = -\lambda_1 \hat{v} - \frac{\hat{v}^2 g_1}{2 g_2},
\]

Finally, imposing the DT (8) to the Lax pair (5) and setting \( S = H A H^{-1} \) leads to

\[
a[1] = a - (s_{11}) \lambda_1, \quad \hat{u}[1] = \hat{u} + 2 s_{12}, \quad \hat{v}[1] = \hat{v} - 2 s_{21},
\]

where

\[
s_{11} = \frac{\lambda_1 h_{11} h_{22} - \lambda_2 h_{12} h_{21}}{h_{11} h_{22} - h_{12} h_{21}},
\]

\[
s_{12} = \frac{(\lambda_2 - \lambda_1) h_{11} h_{12}}{h_{11} h_{22} - h_{12} h_{21}},
\]

\[
s_{21} = \frac{(\lambda_1 - \lambda_2) h_{21} h_{22}}{h_{11} h_{22} - h_{12} h_{21}}.
\]

Furthermore, a straightforward calculation shows that

\[
(s_{11}) \hat{v} + \hat{v} s_{12} - \hat{u} s_{21} - 2 s_{12} s_{21} = 0.
\]

Combining (19) and (21), we arrive at

\[
\hat{u}[1] \hat{v}[1] = \hat{u} v - 2 (s_{11}) \lambda_1, \quad 2 a[1] = 2 a - 2 (s_{11}) \lambda_1.
\]
Hence, we acquire the third BT of system (2), which is

\[
\begin{align*}
x[1] &= x - 2s_{11} + l_3, \\
u[1] &= \frac{1}{u + 2s_{12}}, \\
v[1] &= \frac{1}{v - 2s_{21}},
\end{align*}
\]

where \( x = \int \dddot{u} dy + \int 2\dot{a} dx \) and \( l_3 \) is an arbitrary constant. Choosing the same initial solution as (12) gives rise to

\[
\begin{align*}
h_{1i} &= \alpha_i \varepsilon^{(k_i/2)\tau} + \beta_i \varepsilon^{(-k_i/2)\tau + \xi_i}, \\
h_{2i} &= \frac{1}{c_1} \left[ \alpha_i (\sqrt{\eta_i} - \lambda_i) \varepsilon^{-(k_i/2)\tau + \xi_i} - \beta_i (\sqrt{\eta_i} + \lambda_i) \varepsilon^{-(k_i/2)\tau - \xi_i} \right],
\end{align*}
\]

where \( \eta_i = \lambda_i^2 + c_1 c_2 \xi_i, \eta_i = \sqrt{\eta_i}(y - (k_i/2\lambda_i)\tau), i = 1, 2. \)

Substituting them into (23), we get

\[
\begin{align*}
x[1] &= c_1 c_2 y - 2s_{11} + l_3, \\
u[1] &= \frac{1}{c_1 e^{k_1\tau} + 2s_{12}}, \\
v[1] &= \frac{1}{c_2 e^{-k_1\tau} - 2s_{21}},
\end{align*}
\]

A profile of 2-kink like solution is plotted in Figures 3 and 4 at \( c_1 = 5, c_2 = -2, k_1 = 1/10, \lambda_1 = -4, \lambda_2 = 5, \alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 1, l_3 = 0. \)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This work is partially supported by the National Natural Science Foundation of China (11805071, 11747010, and...
11871232), the Fujian Province Science Foundation for Youths (2019J05092), and the Fundamental Research Funds for the Central Universities (ZQN-803).

References


