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Research Article

Superconvergence of Semidiscrete Splitting Positive Definite Mixed Finite Elements for Hyperbolic Optimal Control Problems

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In this paper, we consider semidiscrete splitting positive definite mixed finite element methods for optimal control problems governed by hyperbolic equations with integral constraints. The state and costate are approximated by the lowest order Raviart-Thomas mixed rectangular finite element, and the control is approximated by piecewise constant functions. We derive some convergence and superconvergence results for the control, the state and the adjoint state. A numerical example is provided to demonstrate our theoretical results.

1. Introduction

There have been extensive studies in error estimates of standard finite element methods (FEMs) for optimal control problems (OCPs). The convergence or superconvergence results of standard FEMs for elliptic and parabolic OCPs can be found in [1–13], respectively.

Because large temperature gradients during cooling or heating may lead to its destruction in temperature control problems, the gradient stands for Darcy velocity in flow control problems, stiffness optimization in nonlinear pantographic structures [14], and topology optimization of a cycloidal metamaterial [15]; their objective functionals contain not only the primal state variable but also its gradient. At this time, mixed finite element methods (MFEMs) will be a very good choice for solving this kind of OCPs. The convergence or superconvergence results of MFEMs for elliptic and parabolic OCPs can be found in [16–20], respectively. However, mixed finite element spaces have to satisfy the Ladyženskaja-Babuška-Brezzi (LBB) condition, which brings very little available approximation spaces and expensive computing costs.

In order to avoid the limitation of LBB condition, a splitting positive definite MFEM was first proposed for solving miscible displacement of compressible flow in a porous medium [21]. Compared with the classic MFEMs, the main advantages of this method are that the original problems can be split into two independent symmetric positive definite subschemes and that the LBB condition is not necessary. Recently, splitting positive definite MFEMs have been used to solve hyperbolic equations [22, 23], elliptic OCPs [24], and parabolic OCPs [25]. To our best knowledge, most of the published papers on different FEMs or MFEMs for OCPs are focused on elliptic or parabolic cases. Although Xu in [26] established a priori error estimates and superconvergence results of splitting positive definite MFEM for pseudo-hyperbolic integrodifferential OCPs and Lu et al. in [27] derived the convergence of finite volume element method for nonlinear hyperbolic OCPs, there are very little studies on hyperbolic OCPs, .

The goal of this paper is to investigate splitting positive definite MFEMs for hyperbolic OCPs and derive the convergence and superconvergence.

We are interested in the following hyperbolic OCPs:

$$\min_{u\in K}\left\{\frac{1}{2}\int_0^T\left(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\}, \tag{1}$$

$$c(x)y_{tt}(x,t) + \operatorname{div} \boldsymbol{p}(x,t) = f(x,t) + u(x,t), x \in \Omega, t \in J, \tag{2}$$

$$\boldsymbol{p}(x,t) = -A(x)\nabla y(x,t), x \in \Omega, t \in J, \tag{3}$$

$$y(x,t) = 0, x \in \partial\Omega, t \in J, \tag{4}$$

$$y(x,0) = y_0(x), y_t(x,0) = y_1(x), x \in \Omega, \tag{5}$$

where $\Omega \subset \mathbf{R}^2$ is a rectangle domain, $J = (0,T]$, $\boldsymbol{p}_d \in (L^2(J;L^2(\Omega)))^2$, $y_d, f \in L^2(J;L^2(\Omega))$, $c \in W^{1,\infty}(\Omega)$ with $0 < c_{\min} \le c(x) \le c_{\max}$ and $y_0, y_1 \in H^2(\Omega)$. The coefficient matrix $A(x) = (a_{ij}(x))_{2\times 2} \in W^{1,\infty}(\bar{\Omega};\mathbf{R}^{2\times 2})$ is a symmetric matrix, and there are constants $c_1, c_2 > 0$ satisfying any vector $\mathbf{X} \in \mathbf{R}^2$, $c_1\|\mathbf{X}\|_{\mathbf{R}^2}^2 \le \mathbf{X}^t A\mathbf{X} \le c_2\|\mathbf{X}\|_{\mathbf{R}^2}^2$. K is a set defined by

$$K = \left\{u \in L^2\left(J;L^2(\Omega)\right): \int_0^T\int_\Omega u\, dxdt \ge 0\right\}. \tag{6}$$

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m}\|D^\alpha v\|_{L^p(\Omega)}^p$, a seminorm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. For $p=2$, we set $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^1(\Omega) = \{v \in H^1(\Omega): v|_{\partial\Omega} = 0\}$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(J;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \left(\int_0^T\|v\|_{W^{m,p}(\Omega)}^s dt\right)^{1/s}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. For simplicity of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^1(J;W^{m,p}(\Omega))$ and $C^k(J;W^{m,p}(\Omega))$. In addition, C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

The plan of this paper is as follows. In Section 2, we give an equivalent optimality conditions for the OCP (1)–(5) and construct its splitting positive definite mixed finite element approximation scheme. In Section 3, we derive the convergence for the control variable, the state variables, and the adjoint state variables. In Section 4, we derive the superconvergence properties between the RT projections and the approximation solutions of the control and the state variables. In the last section, we present a numerical example to illustrate our theoretical results.

2. Splitting Positive Definite MFEMs for OCPs

In this section, we shall construct a splitting positive definite mixed finite element approximation of the control problems (1)–(5). To fix the idea, we shall take the state spaces $\mathbf{L} = H^2(J;\boldsymbol{V})$ and $Q = H^2(J;W)$, where $\boldsymbol{V}$ and W are defined as follows.

Let

$$\boldsymbol{V} = H(\operatorname{div};\Omega) = \left\{\boldsymbol{v} \in \left(L^2(\Omega)\right)^2, \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\right\}, W = L^2(\Omega), \tag{7}$$

and the inner products

$$\begin{aligned}(f_1, f_2) &= \int_\Omega f_1 f_2, \forall f_1, f_2 \in L^2(\Omega),\\ (\varphi, \psi) &= \sum_{i=1}^2 (\varphi_i, \psi_i), \forall \varphi, \psi \in \left(L^2(\Omega)\right)^2.\end{aligned} \tag{8}$$

Let $b = 1/c(x)$, a mixed weak form of (2), and (3) can be given by

$$\left(A^{-1}\boldsymbol{p}, \boldsymbol{v}\right) = (y, \operatorname{div} \boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{9}$$

$$(y_{tt}, w) + (b \operatorname{div} \boldsymbol{p}, w) = (bf, w) + (bu, w), \forall w \in W, t \in J. \tag{10}$$

As in [24], taking $w = \operatorname{div}\ \mathbf{v}, \forall \mathbf{v} \in \mathbf{V}$ in (10), (9) differentiating twice with respect to t, and then substituting the two resulting equations, we derive

$$\left(A^{-1}\boldsymbol{p}_{tt}, \boldsymbol{v}\right) + (b\ \operatorname{div} \boldsymbol{p}, \operatorname{div} \boldsymbol{v}) = (bf, \operatorname{div} \boldsymbol{v}) + (bu, \operatorname{div} \boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J. \tag{11}$$

By using (10) and (11), we get the following new mixed variational form:

$$\left(A^{-1}\boldsymbol{p}_{tt}, \boldsymbol{v}\right) + (b \operatorname{div} \boldsymbol{p}, \operatorname{div} \boldsymbol{v}) = (bf, \operatorname{div} \boldsymbol{v}) + (bu, \operatorname{div} \boldsymbol{v}), \forall \boldsymbol{v} \in \mathbf{V}, t \in J, \tag{12}$$

$$(y_{tt}, w) + (b \operatorname{div} \boldsymbol{p}, w) = (bf, w) + (bu, w), \forall w \in W, t \in J. \tag{13}$$

It is easily seen that (12) is separated from (13) so that $\boldsymbol{p}$ can be solved independently from (12).

We recast (1)–(5) as the following weak form: find $(\boldsymbol{p}, y, u) \in \mathbf{L} \times Q \times K$ such that

$$\min_{u\in K}\left\{\frac{1}{2}\int_0^T\left(\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2\right)dt\right\}, \tag{14}$$

$$\left(A^{-1}\boldsymbol{p}_{tt}, \boldsymbol{v}\right) + (b \operatorname{div} \boldsymbol{p}, \operatorname{div} \boldsymbol{v}) = (bf, \operatorname{div} \boldsymbol{v}) + (bu, \operatorname{div} \boldsymbol{v}), \forall \boldsymbol{v} \in \mathbf{V}, t \in J, \tag{15}$$

$$\boldsymbol{p}(x,0) = -A\nabla y_0(x), \boldsymbol{p}_t(x,0) = -A\nabla y_1(x), \forall x \in \Omega, \tag{16}$$

$$(y_{tt}, w) + (b \operatorname{div} \boldsymbol{p}, w) = (bf, w) + (bu, w), \forall w \in W, t \in J, \tag{17}$$

$$y(x,0) = y_0(x), y_t(x,0) = y_1(x), \forall x \in \Omega. \tag{18}$$

It then follows from [28] that the optimal control problems (14)–(18) have a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (14)–(18) if and only if there is a costate $(\boldsymbol{q}, z) \in \mathbf{L} \times Q$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$\left(A^{-1}\boldsymbol{p}_{tt}, \boldsymbol{v}\right) + (b \operatorname{div} \boldsymbol{p}, \operatorname{div} \boldsymbol{v}) = (bf, \operatorname{div} \boldsymbol{v}) + (bu, \operatorname{div} \boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{19}$$

$$\boldsymbol{p}(x, 0) = -A\nabla y_0(x), \boldsymbol{p}_t(x, 0) = -A\nabla y_1(x), \forall x \in \Omega, \tag{20}$$

$$(y_{tt}, w) + (b \operatorname{div} \boldsymbol{p}, w) = (bf, w) + (bu, w), \forall w \in W, t \in J, \tag{21}$$

$$y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \forall x \in \Omega, \tag{22}$$

$$\left(A^{-1}\boldsymbol{q}_{tt}, \boldsymbol{v}\right) + (b \operatorname{div} \boldsymbol{p}, \operatorname{div} \boldsymbol{v}) + (bz, \operatorname{div} \boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{23}$$

$$\boldsymbol{q}(x, T) = 0, \boldsymbol{q}_t(x, T) = 0, \forall x \in \Omega, \tag{24}$$

$$(z_{tt}, w) = -(y - y_d, w), \forall w \in W, t \in J, \tag{25}$$

$$z(x, T) = 0, z_t(x, T) = 0, \forall x \in \Omega, \tag{26}$$

$$\int_0^T (u - bz - b \operatorname{div} \boldsymbol{q}, \tilde{u} - u)dt \geq 0, \forall \tilde{u} \in K. \tag{27}$$

The inequality (27) can be expressed as

$$u = \max\left\{0, -\bar{G}\right\} + G, \tag{28}$$

where $G = (bz + b \operatorname{div} \boldsymbol{q})$ and $\bar{G} = \int_0^T \int_\Omega G dx dt / (|\Omega| \times T)$.

Let $\boldsymbol{T}_h$ be a uniform rectangulation of the domain Ω, and h_e denotes the diameter of element e and $h = \max_{e \in \boldsymbol{T}_h}\{h_e\}$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denote the lowest order Raviart-Thomas mixed finite element space [29, 30], namely

$$\begin{aligned} \boldsymbol{V}_h &:= \left\{\boldsymbol{v}_h \in \boldsymbol{V} : \forall e \in \boldsymbol{T}_h, \boldsymbol{v}_h|_e \in Q_{1,0}(e) \times Q_{0,1}(e)\right\}, \\ W_h &:= \left\{w_h \in W : \forall e \in \boldsymbol{T}_h, w_h|_e \in Q_{0,0}(e)\right\}, \end{aligned} \tag{29}$$

where $Q_{m,n}(e)$ indicates the space of polynomials of degree no more than m and n in x_1 and x_2 on e, respectively. And the approximated space of control is given by

$$K_h := L^2(J; W_h) \cap K. \tag{30}$$

We introduce two projection operators. First, we define the standard $L^2(\Omega)$-projection [29] $P_h : W \longrightarrow W_h$, which satisfies the following: for any $\phi \in W$

$$(P_h\phi - \phi, w_h) = 0, \forall w_h \in W_h, \tag{31}$$

$$\|\phi - P_h\phi\|_{-s,\rho} \leq Ch^{1+s}\|\phi\|_{1,\rho}, s = 0, 1, 2 \leq \rho \leq \infty, \forall \phi \in W^{1,\rho}(\Omega). \tag{32}$$

Second, we recall the Fortin projection (see [29, 31]) $\Pi_h : \boldsymbol{V} \longrightarrow \boldsymbol{V}_h$, which satisfies the following: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\operatorname{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \forall w_h \in W_h, \tag{33}$$

$$\|\mathbf{q} - \Pi_h\mathbf{q}\|_{0,\rho} \leq Ch\|\mathbf{q}\|_{1,\rho}, 2 \leq \rho \leq \infty, \forall \boldsymbol{q} \in \left(W^{1,\rho}(\Omega)\right)^2, \tag{34}$$

$$\|\operatorname{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\| \leq Ch\|\operatorname{div} \boldsymbol{q}\|_1, \forall \operatorname{div} \boldsymbol{q} \in H^1(\Omega). \tag{35}$$

Then the splitting positive definite mixed finite element discretization of (14)–(18) is as follows: find $(\boldsymbol{p}_h, y_h, u_h) \in H^2(J; \boldsymbol{V}_h) \times H^2(J; W_h) \times K_h$ such that

$$\min_{u_h \in K_h}\left\{\frac{1}{2}\int_0^T \left(\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2\right)dt\right\}, \tag{36}$$

$$\begin{aligned} \left(A^{-1}\boldsymbol{p}_{h,tt}, \boldsymbol{v}_h\right) + (b \operatorname{div} \boldsymbol{p}_h, \operatorname{div} \boldsymbol{v}_h) &= (bf, \operatorname{div} \boldsymbol{v}_h) \\ &+ (bu_h, \operatorname{div} \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \end{aligned} \tag{37}$$

$$\boldsymbol{p}_h(x, 0) = \Pi_h\boldsymbol{p}(x, 0), \boldsymbol{p}_{h,t}(x, 0) = \Pi_h\boldsymbol{p}_t(x, 0), \forall x \in \Omega, \tag{38}$$

$$(y_{h,tt}, w_h) + (b \operatorname{div} \boldsymbol{p}_h, w_h) = (bf, w_h) + (bu_h, w_h), \forall w_h \in W_h, t \in J, \tag{39}$$

$$y_h(x, 0) = P_h y_0(x), y_{h,t}(x, 0) = P_h y_1(x), \forall x \in \Omega. \tag{40}$$

The optimal control problems (36)–(40) again have a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (36)–(40) if and only if there is a costate $(\mathbf{q}_h, z_h)$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$\begin{aligned} \left(A^{-1}\boldsymbol{p}_{h,tt}, \boldsymbol{v}_h\right) + (b \operatorname{div} \boldsymbol{p}_h, \operatorname{div} \boldsymbol{v}_h) &= (bf, \operatorname{div} \boldsymbol{v}_h) \\ &+ (bu_h, \operatorname{div} \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \end{aligned} \tag{41}$$

$$\boldsymbol{p}_h(x, 0) = \Pi_h\boldsymbol{p}(x, 0), \boldsymbol{p}_{h,t}(x, 0) = \Pi_h\boldsymbol{p}_t(x, 0), \forall x \in \Omega, \tag{42}$$

$$(y_{h,tt}, w_h) + (b \operatorname{div} \boldsymbol{p}_h, w_h) = (bf, w_h) + (bu_h, w_h), \forall w_h \in W_h, t \in J, \tag{43}$$

$$y_h(x, 0) = P_h y_0(x), y_{h,t}(x, 0) = P_h y_0(x), \forall x \in \Omega, \tag{44}$$

$$\begin{aligned} &\left(A^{-1}\boldsymbol{q}_{h,tt}, \boldsymbol{v}_h\right) + (b \operatorname{div} \boldsymbol{q}_h, \operatorname{div} \boldsymbol{v}_h) + (bz_h, \operatorname{div} \boldsymbol{v}_h) \\ &= -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \end{aligned} \tag{45}$$

$$\boldsymbol{q}_h(x, T) = 0, \boldsymbol{q}_{h,t}(x, T) = 0, \forall x \in \Omega, \tag{46}$$

$$(z_{h,tt}, w_h) = -(y_h - y_d, w_h), \forall w_h \in W_h, t \in J, \tag{47}$$

$$z_h(x, T) = 0, z_{h,t}(x, T) = 0, \forall x \in \Omega, \tag{48}$$

$$\int_0^T (u_h - bz_h - b \operatorname{div} \mathbf{q}_h, \tilde{u}_h - u_h)dt \geq 0, \forall \tilde{u}_h \in K_h. \tag{49}$$

Similar to (28), we have

$$u_h = \max\left\{0, -\bar{G}_h\right\} + G_h, \tag{50}$$

where $G_h = P_h b(z_h + \operatorname{div} \mathbf{q}_h)$ and $\bar{G}_h = \int_0^T \int_\Omega G_h dx dt / (|\Omega| \times T)$.

3. Convergence Analysis

In this section, we will derive the convergence of splitting positive definite MFEMs for hyperbolic OCPs. For $\forall \tilde{u} \in K$, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u}$ which satisfies

$$\begin{aligned}&\left(A^{-1}\boldsymbol{p}_{h,tt}(\tilde{u}), \boldsymbol{v}_h\right) + \left(b \operatorname{div} \boldsymbol{p}_h(\tilde{u}), \operatorname{div} \boldsymbol{v}_h\right)\\ &= \left(bf, \operatorname{div} \boldsymbol{v}_h\right) + \left(b\tilde{u}, \operatorname{div} \boldsymbol{v}_h\right), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J,\end{aligned} \tag{51}$$

$$\boldsymbol{p}_h(\tilde{u})(x,0) = \Pi_h \boldsymbol{p}(x,0), \boldsymbol{p}_{h,t}(\tilde{u})(x,0) = \Pi_h \boldsymbol{p}_t(x,0), \forall x \in \Omega, \tag{52}$$

$$\left(y_{h,tt}(\tilde{u}), w_h\right) + \left(b \operatorname{div} \boldsymbol{p}_h(\tilde{u}), w_h\right) = \left(bf, w_h\right) + \left(b\tilde{u}, w_h\right), \forall w_h \in W_h, t \in J, \tag{53}$$

$$y_h(\tilde{u})(x,0) = P_h y_0(x), y_{h,t}(\tilde{u})(x,0) = P_h y_1(x), \forall x \in \Omega, \tag{54}$$

$$\begin{aligned}&\left(A^{-1}\boldsymbol{q}_{h,tt}(\tilde{u}), \boldsymbol{v}_h\right) + \left(b \operatorname{div} \boldsymbol{q}_h(\tilde{u}), \operatorname{div} \boldsymbol{v}_h\right) + \left(b z_h(\tilde{u}), \operatorname{div} \boldsymbol{v}_h\right)\\ &= -\left(\mathrm{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h\right), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J,\end{aligned} \tag{55}$$

$$\boldsymbol{q}_h(\tilde{u})(x,T) = 0, \boldsymbol{q}_{h,t}(\tilde{u})(x,T) = 0, \forall x \in \Omega, \tag{56}$$

$$\left(z_{h,tt}(\tilde{u}), w_h\right) = -\left(y_h(\tilde{u}) - y_d, w_h\right), \forall w_h \in W_h, t \in J, \tag{57}$$

$$z_h(\tilde{u})(x,T) = 0, z_{h,t}(\tilde{u})(x,T) = 0, \forall x \in \Omega. \tag{58}$$

It is clear that the exact solution and its approximation can be written in the following way:

$$\begin{aligned}(\boldsymbol{p}, y, \boldsymbol{q}, z) &= (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),\\ (\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) &= (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).\end{aligned} \tag{59}$$

Lemma 1. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z)$ be the solution of (19)–(26) and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the solution of (51)–(58) with $\tilde{u} = u$, respectively. If the solution satisfies*

$$\boldsymbol{p}, \boldsymbol{q} \in \left[L^\infty(H^1)\right]^2 \cap \left[L^2(H^2)\right]^2 \cap \left[H^2(H^1)\right]^2 \text{ and } y, z \in L^\infty(H^1), \tag{60}$$

then we have

$$\begin{aligned}&\|y - y_h(u)\|_{L^\infty(L^2)} + \|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_{L^\infty(L^2)} \le Ch,\\ &\|z - z_h(u)\|_{L^\infty(L^2)} + \|\boldsymbol{q} - \boldsymbol{q}_h(u)\|_{L^\infty(L^2)} \le Ch,\\ &\|\operatorname{div}(\boldsymbol{p} - \boldsymbol{p}_h(u))\|_{L^2(L^2)} + \|\operatorname{div}(\boldsymbol{q} - \boldsymbol{q}_h(u))\|_{L^2(L^2)} \le Ch.\end{aligned} \tag{61}$$

Proof. For ease of presentation, we set $\rho_{\mathbf{p}} = \Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)$, $\rho_y = P_h y - y_h(u)$, $\rho_{\boldsymbol{q}} = \Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)$, $\rho_z = P_h z - z_h(u)$. From Equations (19)–(26) and (51)–(58) and the fact that div $\boldsymbol{V}_h \subset W_h$, with the definition of P_h and Π_h, we can easily obtain the following error equations:

$$\begin{aligned}&\left(A^{-1}\rho_{\boldsymbol{p},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \rho_{\boldsymbol{p}}, \operatorname{div} \boldsymbol{v}_h\right) = -\left(b \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \operatorname{div} \boldsymbol{v}_h\right)\\ &\quad - \left(A^{-1}(\boldsymbol{p}_{tt} - \Pi_h \boldsymbol{p}_{tt}), \boldsymbol{v}_h\right), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{62}$$

$$\begin{aligned}&\left(\rho_{y,tt}, w_h\right) + \left(b \operatorname{div} \rho_{\boldsymbol{p}}, w_h\right) = -\left(b \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), w_h\right)\\ &\quad - \left(y_{tt} - P_h y_{tt}, w_h\right), \forall w_h \in W_h,\end{aligned} \tag{63}$$

$$\begin{aligned}&\left(A^{-1}\rho_{\boldsymbol{q},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \rho_{\boldsymbol{p}}, \operatorname{div} \boldsymbol{v}_h\right) + \left(b \operatorname{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \operatorname{div} \boldsymbol{v}_h\right)\\ &\quad + \left(b(z - P_h z), \operatorname{div} \boldsymbol{v}_h\right) = -\left(A^{-1}(\boldsymbol{q}_{tt} - \Pi_h \boldsymbol{q}_{tt}), \boldsymbol{v}_h\right)\\ &\quad - \left(\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h\right) - \left(\rho_{\boldsymbol{p}}, \boldsymbol{v}_h\right) - \left(b\rho_z, \operatorname{div} \boldsymbol{v}_h\right), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{64}$$

$$\left(\rho_{z,tt}, w_h\right) = -\left(\rho_y, w_h\right) - \left(z_{tt} - P_h z_{tt}, w_h\right), \forall w_h \in W_h. \tag{65}$$

Setting $\boldsymbol{v}_h = \rho_{\boldsymbol{p},t}$ in (62), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\left\|A^{-\frac{1}{2}}\rho_{\boldsymbol{p},t}\right\|^2 + \left\|b^{\frac{1}{2}} \operatorname{div} \rho_{\boldsymbol{p}}\right\|^2\right) &= -\left(b \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \operatorname{div} \rho_{\boldsymbol{p},t}\right)\\ &\quad - \left(A^{-1}(\boldsymbol{p}_{tt} - \Pi_h \boldsymbol{p}_{tt}), \rho_{\boldsymbol{p},t}\right).\end{aligned} \tag{66}$$

Notice that

$$\left(b \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \operatorname{div} \rho_{\boldsymbol{p},t}\right) = -\left(b \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p})_t, \operatorname{div} \rho_{\boldsymbol{p}}\right). \tag{67}$$

Substitute (67) into (66), integrating the resulting equation from 0 to t, and using Hölder's inequality, Young's inequality, Gronwall's inequality, and the assumption on A and (34) and (35), we have

$$\left\|\rho_{\boldsymbol{p},t}\right\|_{L^\infty(L^2)} + \left\|\operatorname{div} \rho_{\boldsymbol{p}}\right\|_{L^\infty(L^2)} \le Ch. \tag{68}$$

Letting $w_h = \rho_{y,t}$ in (63) and $w_h = -\rho_{z,t}$ in (65), respectively. Then integrating the resulting equations from 0 to t and t to T, respectively, we get

$$\begin{aligned}\left\|\rho_{y,t}\right\|_{L^\infty(L^2)} &\le C\left\|\operatorname{div} \rho_{\boldsymbol{p}}\right\|_{L^2(L^2)} + Ch,\\ \left\|\rho_{z,t}\right\|_{L^\infty(L^2)} &\le C\left\|\rho_y\right\|_{L^2(L^2)} + Ch,\end{aligned} \tag{69}$$

where we also used Hölder's inequality, Young's inequality, and Gronwall's inequality.

At last, setting $\mathbf{v}_h = -\rho_{\mathbf{q},t}$ as the test function in (64) and integrating the resulting equation from t to T, similar to

(68), we arrive at

$$\left\|\rho_{\mathbf{q},t}\right\|_{L^\infty(L^2)} + \left\|\operatorname{div}\rho_{\boldsymbol{p}}\right\|_{L^\infty(L^2)} \le Ch + C\left(\left\|\rho_{\boldsymbol{p}}\right\|_{L^2(L^2)} + \|\rho_z\|_{L^2(L^2)}\right). \tag{70}$$

Note that $\rho_{\mathbf{p}}(0)=\rho_y(0)=\rho_{\boldsymbol{q}}(T)=\rho_z(T)=0$, and we have

$$\left\|\rho_{\boldsymbol{p}}\right\|_{L^\infty(L^2)} \le C\left\|\rho_{\mathbf{p},t}\right\|_{L^2(L^2)}, \tag{71}$$

$$\left\|\rho_y\right\|_{L^\infty(L^2)} \le C\left\|\rho_{y,t}\right\|_{L^2(L^2)}, \tag{72}$$

$$\left\|\rho_{\mathbf{q}}\right\|_{L^\infty(L^2)} \le C\left\|\rho_{\boldsymbol{q},t}\right\|_{L^2(L^2)}, \tag{73}$$

$$\|\rho_z\|_{L^\infty(L^2)} \le C\|\rho_{z,t}\|_{L^2(L^2)}. \tag{74}$$

Combining (68)–(74), (32), (34), and (35) and the triangle inequality, we complete the proof. □

Using the same estimates as in Lemma 1, we get the following.

Lemma 2. *Let $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h)$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the solutions of (51)–(58) with $\tilde{u}=u_h$ and $\tilde{u}=u$, respectively. Then we have*

$$\begin{aligned}
\|y_h - y_h(u)\|_{L^\infty(L^2)} + \|\boldsymbol{p}_h - \boldsymbol{p}_h(u)\|_{L^\infty(L^2)} &\le C\|u-u_h\|_{L^2(L^2)},\\
\|z_h - z_h(u)\|_{L^\infty(L^2)} + \|\boldsymbol{q}_h - \boldsymbol{q}_h(u)\|_{L^\infty(L^2)} &\le C\|u-u_h\|_{L^2(L^2)},\\
\|\operatorname{div}(\boldsymbol{p}_h - \boldsymbol{p}_h(u))\|_{L^2(L^2)} + \|\operatorname{div}(\boldsymbol{q}_h - \boldsymbol{q}_h(u))\|_{L^2(L^2)} &\le C\|u-u_h\|_{L^2(L^2)}.
\end{aligned} \tag{75}$$

Now, from the above Lemmas 1 and 2, we can derive the following convergence results.

Theorem 3. *Let u be the solution of (19)–(27) and u_h be the solution of (41)–(49), respectively. Assume that all the assumptions in Lemma 1 are valid. Then we have*

$$\|u-u_h\|_{L^2(L^2)} \le Ch, \tag{76}$$

$$\|y-y_h\|_{L^\infty(L^2)} + \|\boldsymbol{p}-\boldsymbol{p}_h\|_{L^\infty(L^2)} \le Ch, \tag{77}$$

$$\|z-z_h\|_{L^\infty(L^2)} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^\infty(L^2)} \le Ch, \tag{78}$$

$$\|\operatorname{div}(\boldsymbol{p}-\boldsymbol{p}_h)\|_{L^2(L^2)} + \|\operatorname{div}(\boldsymbol{q}-\boldsymbol{q}_h)\|_{L^2(L^2)} \le Ch. \tag{79}$$

Proof. Let $w_h=1$ in (31), and we have

$$\int_\Omega P_h u dx = \int_\Omega u dx. \tag{80}$$

By (80), we have

$$\int_0^T\int_\Omega P_h u dx dt = \int_0^T\int_\Omega u dx dt \ge 0. \tag{81}$$

Thus, we know that $P_h u \in K_h$.
It follows from (27) and (49) that

$$\begin{aligned}
\|u-u_h\|^2_{L^2(L^2)} &= \int_0^T (u-u_h, u-u_h)dt = \int_0^T (u - bz - b\operatorname{div}\boldsymbol{q}, u-u_h)\\
&\cdot dt + \int_0^T (bz - bz_h(u) + b\operatorname{div}(\boldsymbol{q}-\boldsymbol{q}_h(u)), u-u_h)\\
&\cdot dt + \int_0^T (bz_h(u) - bz_h + b\operatorname{div}(\boldsymbol{q}_h(u)-\boldsymbol{q}_h), u-u_h)\\
&\cdot dt + \int_0^T (bz_h + b\operatorname{div}\boldsymbol{q}_h - u_h, u-u_h)\\
&\cdot dt \le \int_0^T (bz - bz_h(u) + b\operatorname{div}(\boldsymbol{q}-\boldsymbol{q}_h(u)), u-u_h)\\
&\cdot dt - \int_0^T (u - bz - b\operatorname{div}\boldsymbol{q}, u - P_h u)\\
&\cdot dt + \int_0^T (bz_h(u) - bz_h + b\operatorname{div}(\boldsymbol{q}_h(u)-\boldsymbol{q}_h), u-u_h)\\
&\cdot dt + \int_0^T (u-u_h, u-P_h u)dt + \int_0^T (bz_h - bz_h(u)\\
&+ b\operatorname{div}(\boldsymbol{q}_h - \boldsymbol{q}_h(u)), u - P_h u)\\
&\cdot dt + \int_0^T (bz_h(u) - bz + b\operatorname{div}(\boldsymbol{q}_h(u) - \boldsymbol{q}), u - P_h u)\\
&\cdot dt := \sum_{i=1}^6 I_i.
\end{aligned} \tag{82}$$

Next, we estimate (82) term by term. For I_1, using Hölder's inequality, Young's inequality, and Lemma 1, we have

$$I_1 \le Ch^2 + \frac{1}{4}\|u-u_h\|^2_{L^2(L^2)}. \tag{83}$$

From (28), we find that

$$u - bz - b\operatorname{div}\boldsymbol{q} = const. \tag{84}$$

Thus

$$I_2 = -const\int_0^T\int_\Omega (u - P_h u)dx dt = 0. \tag{85}$$

Set $\eta_{\boldsymbol{p}} = \boldsymbol{p}_h(u) - \boldsymbol{p}_h$, $\eta_y = y_h(u) - y_h$, $\eta_{\boldsymbol{q}} = \boldsymbol{q}_h(u) - \boldsymbol{q}_h$, and $\eta_z = z_h(u) - z_h$; then from (51)–(58), we have the following error equations:

$$\left(A^{-1}\eta_{\boldsymbol{p},tt}, \boldsymbol{v}_h\right) + \left(b\operatorname{div}\eta_{\boldsymbol{p}}, \operatorname{div}\boldsymbol{v}_h\right) = (bu - bu_h, \operatorname{div}\boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \tag{86}$$

$$\left(\eta_{y,tt}, w_h\right) + \left(b\ \operatorname{div}\ \eta_{\mathrm{p}}, w_h\right) = (bu - bu_h, w_h), \forall w_h \in W_h, t \in J, \tag{87}$$

$$\left(A^{-1}\eta_{\boldsymbol{q},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \eta_{\boldsymbol{q}}, \operatorname{div} \boldsymbol{v}_h\right) = -\left(\eta_{\boldsymbol{p}}, \boldsymbol{v}_h\right) - (b\eta_z, \operatorname{div} \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \quad (88)$$

$$\left(\eta_{z,tt}, w_h\right) = -\left(\eta_y, w_h\right), \forall w_h \in W_h, t \in J. \quad (89)$$

choosing $\mathbf{v}_h = \eta_{\mathbf{q}}$ in (86), $w_h = \eta_z$ in (87), $\mathbf{v}_h = \eta_{\mathbf{p}}$ in (88), and $w_h = \eta_y$ in (89), respectively. Since $\eta_{\mathbf{p}}(0) = \eta_y(0) = \eta_{\mathbf{p},t}(0) = \eta_{y,t}(0) = \eta_{\mathbf{q}}(T) = \eta_z(T) = \eta_{\mathbf{q},t}(T) = \eta_{z,t}(T) = 0$, integrating the resulting equations from 0 to T, we can see that

$$I_3 = \int_0^T \left(\eta_z + \operatorname{div} \eta_{\boldsymbol{q}}, bu - bu_h\right) dt = -\left\|\eta_{\mathbf{p}}\right\|^2 - \left\|\eta_y\right\|^2 \leq 0. \quad (90)$$

For I_4 and I_5, using Hölder's inequality, Young's inequality, Lemma 2, and (32), we get

$$I_4 \leq Ch^2 + \frac{1}{4}\|u - u_h\|^2_{L^2(L^2)}, \quad (91)$$

$$\begin{aligned} I_5 &\leq \left(\|\eta_z\|_{L^2(L^2)} + \left\|\operatorname{div} \eta_{\boldsymbol{q}}\right\|_{L^2(L^2)}\right)\|u - P_h u\|_{L^2(L^2)} \\ &\leq \|u - u_h\|_{L^2(L^2)}\|u - P_h u\|_{L^2(L^2)} \leq Ch^2 + \frac{1}{4}\|u - u_h\|^2_{L^2(L^2)}. \end{aligned} \quad (92)$$

For I_6, by Hölder's inequality, the triangle inequality, Lemma 1, and (32), we arrive at

$$I_6 \leq \|b\|_{0,\infty}\left(\|z_h(u) - z\|_{L^2(L^2)} + \left\|\operatorname{div} \eta_{\boldsymbol{q}}\right\|_{L^2(L^2)}\right)\|u - P_h u\|_{L^2(L^2)} \leq Ch^2. \quad (93)$$

Combining (82), (83), (85), and (90)–(92), we derive (76). Using (76), Lemmas 1 and 2, and the triangle inequality, we complete the proof. □

4. Superconvergence Properties

In this section, we will derive some superconvergence properties for the optimal control problems. In order to derive the main results, we need the following lemmas.

Lemma 4. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z)$ *be the solution of (19)–(27) and* $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ *be the solution of (51)–(58) with* $\tilde{u} = u$*, respectively. If the solution satisfies*

$$\boldsymbol{p}, \boldsymbol{q} \in \left[L^\infty(H^1)\right]^2 \cap \left[L^2(H^2)\right]^2 \cap \left[H^1(H^2)\right]^2 \text{and } y, z \in L^\infty(H^1), \quad (94)$$

then we have

$$\|P_h y - y_h(u)\|_{L^\infty(L^2)} + \|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)\|_{L^\infty(L^2)} \leq Ch^{\frac{3}{2}}, \quad (95)$$

$$\|P_h z - z_h(u)\|_{L^\infty(L^2)} + \|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)\|_{L^\infty(L^2)} \leq Ch^{\frac{3}{2}}, \quad (96)$$

$$\|\operatorname{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))\|_{L^2(L^2)} + \|\operatorname{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u))\|_{L^2(L^2)} \leq Ch^{\frac{3}{2}}. \quad (97)$$

Proof. At first, for any $\boldsymbol{p} \in \boldsymbol{V}$ and $\boldsymbol{v} \in \boldsymbol{V}_h$, by applying the proof of Theorems 4.1, 5.1, and Example 6.2 in [32], we can prove

$$\left(A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h\right) \leq Ch^{\frac{3}{2}}\|\mathbf{p}\|_2(\|\mathbf{v}_h\| + \|\operatorname{div} \boldsymbol{v}_h\|). \quad (98)$$

Moreover, using (32) and (35), we have

$$\begin{aligned} (b \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \operatorname{div} \boldsymbol{v}_h) &= ((b - P_h b) \operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \operatorname{div} \boldsymbol{v}_h) \\ &\leq Ch^2\|b\|_{1,\infty}\|\mathbf{p}\|_2\|\operatorname{div} \boldsymbol{v}_h\|, \\ (b(y - P_h y), \operatorname{div} \boldsymbol{v}_h) &= ((b - P_h b)(y - P_h y), \operatorname{div} \boldsymbol{v}_h) \\ &\leq Ch^2\|b\|_{1,\infty}\|y\|_1\|\operatorname{div} \boldsymbol{v}_h\|. \end{aligned} \quad (99)$$

Similar to Lemma 1, using (98), we can prove (95)–(97). We omit the proof here. □

Lemma 5. *Let* $(\boldsymbol{p}_h(P_h u), y_h(P_h u), \boldsymbol{q}_h(P_h u), z_h(P_h u))$ *and* $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ *be the solutions of (51)–(58) with* $\tilde{u} = P_h u$ *and* $\tilde{u} = u$*, respectively. Then we have*

$$\begin{aligned} &\|y_h(u) - y_h(P_h u)\|_{L^\infty(L^2)} + \|\boldsymbol{p}_h(u) - \boldsymbol{p}_h(P_h u)\|_{L^\infty(L^2)} = 0, \\ &\|z_h(u) - z_h(P_h u)\|_{L^\infty(L^2)} + \|\boldsymbol{q}_h(u) - \boldsymbol{q}_h(P_h u)\|_{L^\infty(L^2)} = 0, \\ &\|\operatorname{div}(\boldsymbol{p}_h(u) - \boldsymbol{p}_h(P_h u))\|_{L^2(L^2)} + \|\operatorname{div}(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(P_h u))\|_{L^2(L^2)} = 0. \end{aligned} \quad (100)$$

Proof. First, set $\zeta_{\boldsymbol{p}} = \boldsymbol{p}_h(u) - \boldsymbol{p}_h(P_h u)$, $\zeta_y = y_h(u) - y_h(P_h u)$, $\zeta_{\boldsymbol{q}} = \boldsymbol{q}_h(u) - \boldsymbol{q}_h(P_h u)$, and $\zeta_z = z_h(u) - z_h(P_h u)$, and we choose $\tilde{u} = P_h u$ and $\tilde{u} = u$ in (51)–(58), respectively; then we obtain the following error equations:

$$\left(A^{-1}\zeta_{\boldsymbol{p},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \zeta_{\boldsymbol{p}}, \operatorname{div} \boldsymbol{v}_h\right) = (bu - bP_h u, \operatorname{div} \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \quad (101)$$

$$\left(\zeta_{y,tt}, w_h\right) + \left(b \operatorname{div} \zeta_{\boldsymbol{p}}, w_h\right) = (bu - bP_h u, w_h), \forall w_h \in W_h, t \in J, \quad (102)$$

$$\left(A^{-1}\left(\zeta_{\boldsymbol{q},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \zeta_{\boldsymbol{q}}, \operatorname{div} \boldsymbol{v}_h\right) = -\left(\zeta_{\boldsymbol{p}}, \boldsymbol{v}_h\right) - (b\zeta_z, \operatorname{div} \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t \in J, \quad (103)$$

$$\left(\zeta_{z,tt}, w_h\right) = -\left(\zeta_y, w_h\right), \forall w_h \in W_h, t \in J. \quad (104)$$

Noting from the fact that $\operatorname{div} \mathbf{V}_h \subset W_h$ and (31)

$$\begin{aligned} &(bu - bP_h u, \operatorname{div} \boldsymbol{v}_h) \leq Ch^2\|b\|_{1,\infty}\|u\|_1\|\operatorname{div} \boldsymbol{v}_h\|, \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ &(bu - bP_h u, w_h) \leq Ch^2h^2\|b\|_{1,\infty}\|u\|_1\|w_h\|, \forall w_h \in W_h, \end{aligned} \quad (105)$$

then, as in Lemma 1, using the stability estimates, we complete the proof. □

Lemma 6. *Let $(\boldsymbol{p}_h(P_hu), y_h(P_hu), \boldsymbol{q}_h(P_hu), z_h(P_hu))$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h)$ be the solutions of (51)–(58) with $\tilde{u} = P_hu$ and $\tilde{u} = u_h$, respectively. Then we have*

$$\int_0^T [(bz_h(P_hu) - bz_h, P_hu - u_h) + (b \operatorname{div} (\boldsymbol{q}_h(P_hu) - \boldsymbol{q}_h), P_hu - u_h)]dt \le 0. \tag{106}$$

Proof. Set $\theta_{\boldsymbol{p}} = \boldsymbol{p}_h - \boldsymbol{p}_h(P_hu)$, $\theta_y = y_h - y_h(P_hu)$, $\theta_{\boldsymbol{q}} = \boldsymbol{q}_h - \boldsymbol{q}_h(P_hu)$, and $\theta_z = z_h - z_h(P_hu)$, similar to (101)–(104), and we obtain the following error equations:

$$\left(A^{-1}\theta_{\boldsymbol{p},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \left(\theta_{\boldsymbol{p}}, \operatorname{div} \boldsymbol{v}_h\right) = (bu_h - bP_hu, \operatorname{div} \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\right. \tag{107}$$

$$\left(\theta_{y,tt}, w_h\right) + \left(b \operatorname{div} \theta_{\boldsymbol{p}}, w_h\right) = (bu_h - bP_hu, w_h), \forall w_h \in W_h, \tag{108}$$

$$\left(A^{-1}\left(\theta_{\boldsymbol{q},tt}, \boldsymbol{v}_h\right) + \left(b \operatorname{div} \theta_{\boldsymbol{q}}, \operatorname{div} \boldsymbol{v}_h\right) = -\left(\theta_{\boldsymbol{p}}, \boldsymbol{v}_h\right) + \left(b\theta_z, \operatorname{div} \boldsymbol{v}_h\right), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\right. \tag{109}$$

$$\left(\theta_{z,tt}, w_h\right) = -\left(\theta_y, w_h\right), \forall w_h \in W_h, \tag{110}$$

choosing $\boldsymbol{v}_h = \theta_{\boldsymbol{q}}$ in (107), $w_h = \theta_z$ in (108), $\boldsymbol{v}_h = -\theta_{\boldsymbol{p}}$ in (109), and $w_h = -\theta_y$ in (110), respectively. Integrating the resulting equations from 0 to T, we derive

$$\begin{aligned} &\int_0^T [(z_h(P_hu) - z_h, bP_hu - bu_h) \\ &\quad + (\operatorname{div} (\boldsymbol{q}_h(P_hu) - \boldsymbol{q}_h), bP_hu - bu_h)] \\ &\quad \cdot dt = -\left\|\theta_y\right\|^2_{L^2\left(L^2\right)} - \left\|\theta_{\boldsymbol{p}}\right\|^2_{L^2\left(L^2\right)}, \end{aligned} \tag{111}$$

which yields to (106). □

Theorem 7. *Let u be the solution of (19)–(27) and u_h be the solution of (41)–(49), respectively. Assume that all the assumptions in Lemma 1 are valid. Then, we have*

$$\|P_hu - u_h\|_{L^2\left(L^2\right)} \le Ch^{\frac{3}{2}}. \tag{112}$$

Proof. We choose $\tilde{u} = u_h$ in (23) and $\tilde{u}_h = P_hu$ in (44) to get the following two inequalities:

$$\begin{aligned} &\int_0^T (u - bz - b \operatorname{div} \boldsymbol{q}, u_h - u)dt \ge 0, \\ &\int_0^T (u_h - bz_h - b \operatorname{div} \boldsymbol{q}_h, P_hu - u_h)dt \ge 0. \end{aligned} \tag{113}$$

Note that $u_h - u = u_h - P_hu + P_hu - u$. Adding the above two inequalities, we get

$$\begin{aligned} &\int_0^T (u_h - u + bz - bz_h + b \operatorname{div} (\boldsymbol{q} - \boldsymbol{q}_h), P_hu - u_h) \\ &\quad \cdot dt + \int_0^T (u - bz - b \operatorname{div} \boldsymbol{q}, P_hu - u)dt \ge 0. \end{aligned} \tag{114}$$

Thus, by (114) and (26), we find that

$$\begin{aligned} \|P_hu - u_h\|^2_{L^2\left(L^2\right)} &= \int_0^T (P_hu - u, P_hu - u_h) \\ &\cdot \mathrm{d}t + \int_0^T (u - u_h, P_hu - u_h) \\ &\cdot dt \le \int_0^T (bz - bz_h + b \operatorname{div} \boldsymbol{q} - b \operatorname{div} \boldsymbol{q}_h, P_hu - u_h) \\ &\cdot dt + \int_0^T (u - bz - b \operatorname{div} \boldsymbol{q}, P_hu - u) \\ &\cdot dt = \int_0^T (bz - bP_hz + b \operatorname{div} \boldsymbol{q} - b \operatorname{div} \Pi_h\boldsymbol{q}, P_hu - u_h) \\ &\cdot dt + \int_0^T (bP_hz - bz_h(\mathrm{u}) + b \operatorname{div} \Pi_h\boldsymbol{q} \\ &- \operatorname{div} \boldsymbol{q}_h(u), P_hu - u_h)dt + \int_0^T (bz_h(u) - bz_h(P_hu) \\ &+ b \operatorname{div} \boldsymbol{q}_h(u) - b \operatorname{div} \boldsymbol{q}_h(P_hu), P_hu - u_h) \\ &\cdot dt + \int_0^T (bz_h(P_hu) - bz_h + b \operatorname{div} \boldsymbol{q}(P_hu) \\ &- b \operatorname{div} \boldsymbol{q}_h, P_hu - u_h)dt + \int_0^T (u - bz \\ &- b \operatorname{div} \boldsymbol{q}, P_hu - u)dt \coloneqq \sum_{i=1}^5 J_i. \end{aligned} \tag{115}$$

At first, from Cauchy inequality, (26), (28), and (24), it is easy to get

$$\begin{aligned} J_1 &= \int_0^T (bz - bP_hz + b \operatorname{div} \boldsymbol{q} - b \operatorname{div} \Pi_h\boldsymbol{q}, P_hu - u_h) \\ &\quad \cdot dt \le Ch^4 + \frac{1}{4}\|P_hu - u_h\|^2_{L^2\left(L^2\right)}. \end{aligned} \tag{116}$$

Using Hölder's inequality, Young's inequality, and Lemmas 4 and 5, we arrive at

$$J_2 \le Ch^3 + \frac{1}{4}\|P_hu - u_h\|^2_{L^2\left(L^2\right)}, \tag{117}$$

$$J_3 \le Ch^3 + \frac{1}{4}\|P_hu - u_h\|^2_{L^2\left(L^2\right)}. \tag{118}$$

Combining (84), (85), and (115)–(118) and Lemma 6, we derive (112). □

Using Theorem 7 and the stability estimates as in Lemma 1, we can arrive at the following.

Table 1: Numerical results of convergence.

$h=\tau$	$\|e_u\|_{2,2}$	Rate	$\|e_y\|_{\infty,2}$	Rate	$\|e_z\|_{\infty,2}$	Rate
1/10	4.9561e − 2	—	3.5081e − 2	—	4.0448e − 2	—
1/20	2.5280e − 2	0.97	1.8105e − 2	0.95	2.0416e − 2	0.99
1/40	1.2642e − 2	1.00	9.1468e − 3	0.99	1.0208e − 2	1.00
1/80	6.2663e − 3	1.01	4.5615e − 3	1.00	5.1032e − 3	1.00

Lemma 8. *Let $(\boldsymbol{p}_h(P_hu), y_h(P_hu), \boldsymbol{q}_h(P_hu), z_h(P_hu))$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h)$ be the solutions of (51)–(58) with $\tilde{u}=P_hu$ and $\tilde{u}=u_h$, respectively. Assume that all the assumptions in Theorem 7 are valid. Then we have*

$$\begin{aligned}
&\|y_h - y_h(P_hu)\|_{L^\infty(L^2)} + \|\boldsymbol{p}_h - \boldsymbol{p}_h(P_hu)\|_{L^\infty(L^2)} \le Ch^{\frac{3}{2}},\\
&\|z_h - z_h(P_hu)\|_{L^\infty(L^2)} + \|\boldsymbol{q}_h - \boldsymbol{q}_h(P_hu)\|_{L^\infty(L^2)} \le Ch^{\frac{3}{2}},\\
&\|\text{div}\,(\boldsymbol{p}_h - \boldsymbol{p}_h(P_hu))\|_{L^2(L^2)} + \|\text{div}\,(\boldsymbol{q}_h - \boldsymbol{q}_h(P_hu))\|_{L^2(L^2)} \le Ch^{\frac{3}{2}}.
\end{aligned} \tag{119}$$

Combining Lemma 4–8 and using the triangle inequality, we derive the following superconvergence results.

Theorem 9. *Let $(y, \boldsymbol{p}, z, \boldsymbol{q})$ and $(y_h, \boldsymbol{p}_h, z_h, \boldsymbol{q}_h)$ be the solutions of (19)–(27) and (41)–(49), respectively. Assume that all the assumptions in Theorem 7 are valid. Then we have*

$$\begin{aligned}
&\|P_hy - y_h\|_{L^\infty(L^2)} + \|\Pi_h\boldsymbol{p} - \boldsymbol{p}_h\|_{L^\infty(L^2)} \le Ch^{\frac{3}{2}},\\
&\|P_hz - z_h\|_{L^\infty(L^2)} + \|\Pi_h\boldsymbol{q} - \boldsymbol{q}_h\|_{L^\infty(L^2)} \le Ch^{\frac{3}{2}},\\
&\|\text{div}\,(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h)\|_{L^2(L^2)} + \|\text{div}\,(\Pi_h\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(L^2)} \le Ch^{\frac{3}{2}}.
\end{aligned} \tag{120}$$

5. Numerical Experiments

In this section, we present a numerical example to validate our convergence and superconvergence results. The hyperbolic OCP was dealt numerically with codes developed based on AFEPack. The package is freely available and the details can be found in [33].

Let $\tau>0, N=T/\tau\in\mathbb{Z}$, and $t_n=n\tau, n=0,1,\cdots,N$. Set

$$\phi^n=\phi(x,t_n), \phi_{tt}^n=\left(\phi^{n+1}-2\phi^n+\phi^{n-1}\right)/\tau^2. \tag{121}$$

Then, a fully discrete splitting positive definite mixed finite element solution $(\mathbf{p}_h^n, y_h^n, \mathbf{q}_h^n, z_h^n, u_h^n)$ of (1)–(6) satisfies the following system:

$$\begin{aligned}
&\left(A^{-1}\boldsymbol{p}_{h,tt}^n, \boldsymbol{v}_h\right) + (\text{div}\,\boldsymbol{p}_h^n, \text{div}\,\boldsymbol{v}_h) = (f^n, \text{div}\,\boldsymbol{v}_h)\\
&\quad + (u_h^n, \text{div}\,\boldsymbol{v}_h), \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, n=1,2,\cdots,N.\\
&\left(\frac{\boldsymbol{p}_h^1-\boldsymbol{p}_h^{-1}}{2\tau}, \boldsymbol{v}_h\right) = (\boldsymbol{p}_t(x,0), \boldsymbol{v}_h), \forall \boldsymbol{v}_h\in\boldsymbol{V}_h,\\
&\boldsymbol{p}_h^0 = \Pi_h\boldsymbol{p}(x,0), \forall x\in\Omega,\\
&\left(y_{h,tt}^n, w_h\right) + (\text{div}\,\boldsymbol{p}_h^n, w_h) = (f^n, w_h)\\
&\quad + (u_h^n, w_h), \forall w_h\in W_h, n=1,2,\cdots,N,\\
&\left(\frac{y_h^1-y_h^{-1}}{2\tau}, w_h\right) = (y_1(x), w_h), \forall w_h\in W_h,\\
&y_h^0 = P_hy_0(x), \quad \forall x\in\Omega,\\
&\left(A^{-1}\boldsymbol{q}_{h,tt}^n, \boldsymbol{v}_h\right) - (\text{div}\,\boldsymbol{q}_h^n, \text{div}\,\boldsymbol{v}_h) = (\boldsymbol{p}_h^n - \boldsymbol{p}_d^n, \boldsymbol{v}_h)\\
&\quad + (z_h^n, \text{div}\,\boldsymbol{v}_h), \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, n=N, N-1,\cdots,1,\\
&\left(\frac{\boldsymbol{q}_h^{N-1}-\boldsymbol{q}_h^{N+1}}{2\tau}, \boldsymbol{v}_h\right) = (\boldsymbol{q}_t(x,T), \boldsymbol{v}_h), \forall \boldsymbol{v}_h\in\boldsymbol{V}_h,\\
&\boldsymbol{q}_h^N = 0, \forall x\in\Omega,\\
&\left(z_{h,tt}^n, w_h\right) = (y_h^n - y_d^n, w_h), \forall w_h\in W_h, n=N, N-1,\cdots,1,\\
&\left(\frac{z_h^{N-1}-z_h^{N+1}}{2\tau}, w_h\right) = (z_t(x,T), w_h), \forall w_h\in W_h,\\
&z_h^N = 0, \forall x\in\Omega,\\
&u_h^n = \max\left\{0, -(z_h + \bar{\text{d}}\text{iv}\,\boldsymbol{q}_h)\right\} + z_h^n + \text{div}\,\boldsymbol{q}_h^n, \forall n=0,1,\cdots,N.
\end{aligned} \tag{122}$$

Example 10. Let $\Omega=(0,1)\times(0,1), T=1, c(x)=1, A=E$, where E denotes identity matrix. The data under testing are as follows:

$$\begin{aligned}
&y(x,t) = t^2\sin(2\pi x_1)\sin(2\pi x_2),\\
&\boldsymbol{p}(x,t) = -\begin{pmatrix} 2\pi t^2\cos(2\pi x_1)\sin(2\pi x_2)\\ 2\pi t^2\sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
&\boldsymbol{q}(x,t) = \begin{pmatrix} (1-t)^2\cos(2\pi x_1)\sin(2\pi x_2)\\ (1-t)^2\sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
&z(x,t) = -\,\text{div}\,\boldsymbol{q}(x,t) + 2(1-t)\sin(2\pi x_1)\sin(2\pi x_2),\\
&y_d(x,t) = y(x,t) - z_{tt}(x,t),\\
&\boldsymbol{p}_d(x,t) = \boldsymbol{p}(x,t) - \boldsymbol{q}_{tt}(x,t) + \begin{pmatrix} 2\pi(1-t)\cos(2\pi x_1)\sin(2\pi x_2)\\ 2\pi(1-t)\sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
&u(x,t) = \max\left\{0, -z(x,t) + \bar{\text{d}}\text{iv}\,\boldsymbol{q}(x,t)\right\} + z(x,t) + \text{div}\,\boldsymbol{q}(x,t),\\
&f(x,t) = y_{tt}(x,t) + \text{div}\,\boldsymbol{p}(x,t) - u(x,t).
\end{aligned} \tag{123}$$

Let $e_u=u-u_h, e_y=y-y_h, e_{\mathbf{p}}=\boldsymbol{p}-\boldsymbol{p}_h, e_z=z-z_h, e_{\boldsymbol{q}}=\boldsymbol{q}-\boldsymbol{q}_h$ and $E_u=P_hu-u_h, E_y=P_hy-y_h, E_{\mathbf{p}}=\Pi_h\boldsymbol{p}-\boldsymbol{p}_h, E_z=$

Table 2: Numerical results of convergence.

$h=\tau$	$\|e_p\|_{\infty,2}$	Rate	$\|e_q\|_{\infty,2}$	Rate	$\|\text{div } e_p\|_{2,2}$	Rate	$\|\text{div } e_q\|_{2,2}$	Rate
1/10	$4.7647e-2$	—	$4.9557e-2$	—	$5.5145e-2$	—	$6.7192e-2$	—
1/20	$2.4324e-2$	00.97	$2.5128e-2$	0.98	$2.8071e-2$	0.97	$3.3987e-2$	0.98
1/40	$1.2153e-2$	1.00	$1.28077e-2$	0.97	$1.4045e-2$	1.00	$1.6943e-2$	1.00
1/80	$6.0748e-3$	1.00	$6.4102e-3$	1.00	$7.0231e-3$	1.00	$8.4158e-3$	1.01

Table 3: Numerical results of superconvergence.

h	τ	$\|E_u\|_{2,2}$	Rate	$\|E_y\|_{\infty,2}$	Rate	$\|E_z\|_{\infty,2}$	Rate
1/10	1/10	$4.8542e-2$	—	$3.4063e-2$	—	$4.0284e-2$	—
1/20	1/30	$1.7282e-2$	1.49	$1.2154e-2$	1.49	$1.4404e-2$	1.48
1/40	1/90	$6.0473e-3$	1.51	$4.3104e-3$	1.50	$5.0486e-3$	1.51
1/80	1/270	$2.1158e-3$	1.52	$1.4835e-3$	1.54	$1.7635e-3$	1.52

Table 4: Numerical results of superconvergence.

h	τ	$\|E_p\|_{\infty,2}$	Rate	$\|E_q\|_{\infty,2}$	Rate	$\|\text{div } E_p\|_{2,2}$	Rate	$\|\text{div } E_q\|_{2,2}$	Rate
1/10	$\frac{1}{10}$	$4.6684e-2$	—	$4.7694e-2$	—	$5.4134e-2$	—	$6.5048e-2$	—
1/20	1/30	$1.6902e-2$	1.47	$1.7120e-2$	1.48	$1.9402e-2$	1.48	$2.3406e-2$	1.47
1/40	1/90	$5.9615e-3$	1.50	$6.0452e-3$	1.50	$6.8342e-3$	1.51	$8.1906e-3$	1.51
1/80	1/270	$2.0512e-3$	1.54	$2.1124e-3$	1.52	$2.3448e-3$	1.54	$2.8302e-3$	1.53

$P_h z - z_h, E_{\boldsymbol{q}} = \Pi_h \boldsymbol{q} - \boldsymbol{q}_h$. We set $\|\cdot\|_{L^\infty(L^2)} = \|\cdot\|_{\infty,2}$ and $\|\cdot\|_{L^2(L^2)} = \|\cdot\|_{2,2}$. The numerical results of convergence and superconvergence based on a sequence of uniformly refined meshes are reported in Tables 1–4, respectively. They are consistent with our theoretical results.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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