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Research Article

Single-Soliton Solution of KdV Equation via Hirota's Direct Method under the Time Scale Framework

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Hirota's direct method is one significant way to obtain solutions of soliton equations, but it is rarely studied under the time scale framework. In this paper, the generalized KdV equation on time-space scale is deduced from one newly constructed Lax equation and zero curvature equation by using the AKNS method, which can be reduced to the classical and discrete KdV equation by considering different time scales. What is more, it is the first time that the single-soliton solution of the KdV equation under the time scale framework is obtained by using the idea of Hirota's direct method.

1. Introduction

In 1990, measure chain was proposed by Hilger as a bridge connecting continuous and discrete cases [1]. The time scale is a special case of the measure chain, which is an arbitrary nonempty closed subset of the real number $\mathbb{R}$ [2], and it has been widely used in biology, medicine, and physics [3–8]. For example, in biology, responses of both land sensitivity and Earth's terrestrial biomes to drought were assessed by using drought time scale [3]. In medicine, hyperpolarized 3He diffusion MRI was used to compare the lung microstructure of healthy subjects and asthmatic patients on two different time scales [5]. In physics, the interannual time scale was used to assess the coupling and interaction between monsoon system and Pacific trade wind field [7]. In addition, it has also been widely used in differential equations. However, due to the complexity of nonlinear partial differential equations (PDEs), most of the researches on time-space scale are still aimed at ordinary differential equations [9–12]. Therefore, we will focus on the soliton equations on time-space scale, which are one of the significant branches of nonlinear partial differential equations [13–15].

There are many methods to solve the soliton equations, mainly including Riemann-Hilbert's method [16, 17], Darboux's transformation [18], Bäcklunds's transformation [19], and Hirota's direct method [20]. Hirota's direct method was proposed by Hirota in 1977 to simplify PDEs by linearizing them with various transformations [21]. Compared with the first three methods or other methods [22–24], the direct method is more universal because of targeting the equation itself rather the Lax pair of the soliton equations.

Soliton equations have been widely studied in continuous or discrete case [25–33], such as KdV equation and Toda lattice equation. The N-soliton solutions for a class of more generalized KdV equations have been carefully studied via the Hirota bilinear method [31]. The Wronskian technique was used to obtain more general soliton solutions of the KdV equation in the continuous case [32]. In addition, Casorati's determinant was used to obtain more general soliton solutions of the Toda lattice in the discrete case [33]. However, soliton equations have rarely been studied in the simultaneous existence of continuous and discrete case in a system via direct method. Therefore, in this paper, we focus on the research of KdV equation under the time scale framework via Hirota's direct method.

The structure of this paper is as follows. In the second section, some basic knowledge on time-space scale are introduced. The third section is important that new AKNS system is constructed, and specific parameters are selected to obtain the KdV equation on time-space scale, which can be simplified into classical and discrete KdV equation. In Section 4, the single-soliton solution of KdV equation under the time scale framework is constructed by using the idea of direct method, and the nonlinear dispersion relationship of the equation is obtained. In particular, solutions of KdV equation on two different time scales are obtained. The last part is our conclusion.

2. Time-Space Scale Calculus

Some significant definitions and lemmas for the time-space scale calculus are recalled, which is used throughout the rest of this paper [34–37].

Definition 1. For $(t, x) \in \mathbb{T} \times \mathbb{X}$ (time-space scale), backward jump operators are defined as

$$\begin{gathered} \varsigma : \mathbb{T} \longrightarrow \mathbb{T}, \\ \rho : \mathbb{X} \longrightarrow \mathbb{X}, \\ \varsigma(t) = \sup \{m \in \mathbb{T} : m < t\}, \\ \rho(x) = \sup \{n \in \mathbb{X} : n < x\}. \end{gathered} \tag{1}$$

Definition 2. If $\mathbb{T}$ has a right-scattered minimum n, define $\mathbb{T}_\kappa := \mathbb{T} - \{n\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. The backward graininess $\mu : \mathbb{T}_\kappa \longrightarrow \mathbb{R}_0^+$ is defined by $\mu(t) = t - \varsigma(t)$. Assume that $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$.

(i) If f is nabla differentiable at t, then f is continuous at t

(ii) If f is continuous at a left-scattered t, then f is nabla differentiable at t with

$$\nabla_t f(t) = \frac{f(t) - f^\varsigma(t)}{\mu(t)} \tag{2}$$

(iii) If t is left-dense, then f is nabla differentiable at t iff the limit

$$\lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s} \tag{3}$$

exists as a finite number. In this case,

$$\nabla_t f(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s} \tag{4}$$

(iv) If f is nabla differentiable at t, then

$$f^\varsigma(t) = f(t) - \mu(t)\nabla_t f(t) \tag{5}$$

Definition 3. The function b is μ-regressive if

$$1 - \mu(t)b(t) \neq 0 \text{ for all } t \in \mathbb{T}_\kappa. \tag{6}$$

Define the μ-regressive class of functions on $\mathbb{T}_\kappa$ to be

$$\mathscr{R}_\mu = \{b : \mathbb{T} \longrightarrow \mathbb{R} : b \text{ is ld-continuous and } \mu\text{-regressive}\}. \tag{7}$$

Definition 4. If $b \in \mathscr{R}_\mu$, then we define the nabla exponential function by

$$\widehat{e}_b(t, s) := \exp\left(\int_s^t \widehat{\xi}_{\mu(\tau)}(b(\tau))\nabla\tau\right) \text{ for } s, t \in \mathbb{T}, \tag{8}$$

where the μ-cylinder transformation

$$\widehat{\xi}_h(z) := -\frac{1}{h} \log(1 - zh), \tag{9}$$

and for $h = 0$, we define $\widehat{\xi}_0(z) = z$ for all $z \in \mathbb{C}_0 := \mathbb{C}$.

Definition 5. If $b \in \mathscr{R}_\mu$ and $y = \widehat{e}_b(t, s)$, then the first-order linear dynamic equation

$$\nabla_y = b(t)y \tag{10}$$

is called μ-regressive.

Lemma 6. *Take $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{R}$. The backward jump operators are as follows:*

$$\begin{gathered} \varsigma(t) = \sup(-\infty, t) = t, \\ \rho(x) = \sup(-\infty, x) = x, \end{gathered} \tag{11}$$

and the graininess functions are as follows:

$$\begin{gathered} \mu(t) = t - \varsigma(t) = 0, \\ \nu(x) = x - \rho(x) = 0. \end{gathered} \tag{12}$$

Lemma 7. *Take $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{Z}$. The backward jump operators are as follows:*

$$\begin{gathered} \varsigma(t) = \sup(-\infty, t) = t, \\ \rho(x) = \sup\{x - 1, x - 2, \cdots\} = x - 1, \end{gathered} \tag{13}$$

and the graininess functions are as follows:

$$\begin{gathered} \mu(t) = t - \varsigma(t) = 0, \\ \nu(x) = x - \rho(x) = 1. \end{gathered} \tag{14}$$

Lemma 8. *If the $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable on the $t \in \mathbb{T}_\kappa$, the following formula is easily proved by the definition of the ∇:*

$$
\begin{gathered}
\nabla_t(af + bg) = a\nabla_t f + b\nabla_t g, a, b \in \mathbb{R}, \\
\nabla_t(fg) = (\nabla_t f)g + f^\varsigma(\nabla_t g) = f(\nabla_t g) + (\nabla_t f)g^\varsigma, \\
gg^\varsigma \neq 0, \nabla_t\left(\frac{1}{g}\right) = -\frac{\nabla_t g}{gg^\varsigma}, \\
gg^\varsigma \neq 0, \nabla_t\left(\frac{f}{g}\right) = \frac{(\nabla_t f)g - f(\nabla_t g)}{gg^\varsigma}.
\end{gathered}
\tag{15}
$$

3. KdV Equation on Time-Space Scale

In this section, in order to obtain the KdV equation on time-space scale, we postulate the Lax pair of equation on time-space scale:

$$
\begin{aligned}
\nabla_x \kappa &= C\kappa, \\
\nabla_t \kappa &= D\kappa,
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
C &= \begin{pmatrix} -i\vartheta & u \\ l & i\vartheta \end{pmatrix}, \\
D &= \begin{pmatrix} I & J \\ K & -I \end{pmatrix}.
\end{aligned}
\tag{17}
$$

Here, i is a imaginary number. κ is a 2-dimensional vector function of x, t. C and D are 2×2 matrices, the elements of which contain spectral parameter ϑ and potential functions $u(x, t)$ and $l(x, t)$ with x, t as independent variables.

According to the compatibility condition $\nabla_{xt}\kappa = \nabla_{tx}\kappa$ and ∇-derivative product rules, the zero curvature equation on time-space scale is obtained:

$$
\nabla_t C - \nabla_x D + C^\varsigma D - D^\rho C = 0. \tag{18}
$$

Then substituting Equation (17) into Equation (18), these relations are obtained:

$$
\begin{gathered}
-\nabla_x I - i\vartheta I + u^\varsigma K + i\vartheta I^\rho - lJ^\rho = 0, \\
\nabla_t u - \nabla_x J - i\vartheta J - u^\varsigma I - uI^\rho - i\vartheta J^\rho = 0, \\
\nabla_t l - \nabla_x K + l^\varsigma I + i\vartheta k + i\vartheta K^\rho + lI^\rho = 0, \\
\nabla_x I + l^\varsigma J - i\vartheta I - uK^\rho + i\vartheta I^\rho = 0.
\end{gathered}
\tag{19}
$$

Taking I, J, and K as the cubic polynomials of ϑ,

$$
\begin{aligned}
I &= \sum_{j=0}^{3} a_j \vartheta^j, \\
J &= \sum_{j=0}^{3} b_j \vartheta^j, \\
K &= \sum_{j=0}^{3} c_j \vartheta^j,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
u &= \frac{U + U^\rho}{2}, \\
l &= \frac{L + L^\rho}{2}.
\end{aligned}
\tag{21}
$$

Then substituting Equations (20) and (21) into Equation (19) and comparing coefficients of $\vartheta^j (j = 0, \cdots, 3)$, these relations are obtained:

$$
\begin{gathered}
a_3 = a_3^0, \\
b_3 = 0, \\
c_3 = 0, \\
a_2 = a_2^0, \\
b_2 = i\left(\frac{U + U^\varsigma}{2}\right)a_3^0, \\
c_2 = i\left(\frac{L + L^\varsigma}{2}\right)a_3^0, \\
b_1 = (2 - \nu\nabla_x)^{-1}\left[-i\frac{U^\varsigma + U^{\rho\varsigma} + U + U^\rho}{2}a_2^0 - \nabla_x\frac{U + U^\varsigma}{2}a_3^0\right], \\
c_1 = (2 - \nu\nabla_x)^{-1}\left[i\frac{L^\varsigma + L^{\rho\varsigma} + L + L^\rho}{2}a_2^0 + \nabla_x\frac{L + L^\varsigma}{2}a_3^0\right], \\
a_1 = \nabla_x^{-1}\frac{u^\varsigma c_1 + uc_1^\rho - lb_1^\rho - l^\varsigma b_1}{2} + a_1^0, \\
b_0 = i(2 - \nu\nabla_x)^{-1}\left(\nabla_x b_1 + u^\varsigma a_1 + ua_1^\rho\right), \\
c_0 = -i(2 - \nu\nabla_x)^{-1}\left(\nabla_x c_1 - l^\varsigma a_1 - la_1^\rho\right), \\
a_0 = \frac{1}{2}\nabla_x^{-1}\left(u^\varsigma c_0 - lb_0^\rho + uc_0^\rho - l^\varsigma b_0\right) + a_0^0.
\end{gathered}
\tag{22}
$$

The evolution equations on time-space scale are obtained:

$$
\begin{aligned}
\nabla_t u &= \nabla_x b_0 + u^\varsigma a_0 + u{a_0}^\rho, \\
\nabla_t l &= \nabla_x c_0 - l^\varsigma a_0 - l{a_0}^\rho.
\end{aligned}
\tag{23}
$$

Taking specific values

$$
\begin{aligned}
a_0^0 = a_1^0 = a_2^0 &= 0, \\
a_3^0 &= -4i, \\
l &= -1,
\end{aligned}
\tag{24}
$$

then the KdV equation on time-space scale is obtained:

$$
\begin{aligned}
\nabla_t\left(\frac{U+U^{\rho}}{2}\right) = {} & i(2-\nu\nabla_x)^{-1}\nabla_x\left[\nabla_x(2-\nu\nabla_x)^{-1}(2i\nabla_x(U+U^{\varsigma}))+\frac{U^{\varsigma}+U^{\rho\varsigma}}{2}i(U+U^{\varsigma})+\frac{U+U^{\rho}}{2}i(U^{\rho}+U^{\varsigma\rho})\right] \\
& +\frac{U^{\sigma}+U^{\rho\varsigma}}{2}\frac{1}{2}\nabla_x^{-1}\left\{\frac{U^{\varsigma}+U^{\rho\varsigma}}{2}(2-\nu\nabla_x)^{-1}(U+U^{\varsigma}+U^{\rho}+U^{\varsigma\rho})\right. \\
& +i\left[\nabla_x(2-\nu\nabla_x)^{-1}(2i\nabla_x(U+U^{\varsigma}))+\frac{U^{\varsigma}+U^{\rho\varsigma}}{2}i(U+U^{\varsigma})+\frac{U+U^{\rho}}{2}i(U^{\rho}+U^{\varsigma\rho})\right] \\
& \left.+\frac{U+U^{\rho}}{2}(1-\nu\nabla_x)(2-\nu\nabla_x)^{-1}(U+U^{\varsigma}+U^{\rho}+U^{\varsigma\rho})\right\} \\
& +\frac{U+U^{\rho}}{2}(1-\nu\nabla_x)\frac{1}{2}\nabla_x^{-1}\left\{\frac{U^{\varsigma}+U^{\rho\varsigma}}{2}(2-\nu\nabla_x)^{-1}(U+U^{\varsigma}+U^{\rho}+U^{\varsigma\rho})\right. \\
& +i\left[\nabla_x(2-\nu\nabla_x)^{-1}(2i\nabla_x(U+U^{\varsigma}))+\frac{U^{\varsigma}+U^{\rho\varsigma}}{2}i(U+U^{\varsigma})+\frac{U+U^{\rho}}{2}i(U^{\rho}+U^{\varsigma\rho})\right] \\
& \left.+\frac{U+U^{\rho}}{2}(1-\nu\nabla_x)(2-\nu\nabla_x)^{-1}(U+U^{\varsigma}+U^{\rho}+U^{\varsigma\rho})\right\}.
\end{aligned}
\tag{25}
$$

In the case $\mu(t)=0, \nu(x)\neq 0$, Equation (25) becomes

$$
\begin{aligned}
-4\nabla_x^3(2-\nu\nabla_x)^{-2}U-(2-\nu\nabla_x)^{-1}\nabla_x[(2-\nu\nabla_x)U]^2 \\
-[(2-\nu\nabla_x)U](\nabla_x U)=\left[\frac{(2-\nu\nabla_x)U}{2}\right]_t.
\end{aligned}
\tag{26}
$$

In the following, two special cases of Equation (26) are given, respectively.

Case 1. Taking $\mathbb{T}\times\mathbb{X}=\mathbb{R}\times\mathbb{R}$, we find $\mu(t)=0, \nu(x)=0$.
Equation (22) is reduced to

$$
\begin{aligned}
u &= U, \\
b_0 &= -u_{xx}-2u^2, \\
a_0 &= -u_{xx}, \\
l &= -1, \\
c_0 &= 2u.
\end{aligned}
\tag{27}
$$

The classical KdV equation is obtained:

$$
u_t+6uu_x+u_{xxx}=0. \tag{28}
$$

Case 2. Taking $\mathbb{T}\times\mathbb{X}=\mathbb{R}\times\mathbb{Z}$, we find $\mu(t)=0, \nu(x)=1$.

$$
\begin{aligned}
f^{\varsigma}(x,t) &= f(x,t), \\
f^{\rho}(x,t) &= Ef(x,t)=f(x,t)-(1-E)f(x,t),
\end{aligned}
\tag{29}
$$

where E is the shift operator.

Equation (26) becomes

$$
\begin{aligned}
-4(1-E)^3(1+E)^{-2}U-(1+E)^{-1}(1-E)[(1+E)U]^2 \\
-[(1+E)U][(1-E)U]=\left[\frac{(1+E)U}{2}\right]_t.
\end{aligned}
\tag{30}
$$

4. Direct Method of KdV Equation on Time-Space Scale

In this section, we utilize the idea of direct method to transform the KdV equation and obtain its single-soliton solution on time-space scale.

To eliminate the negative power term, assuming

$$
U=\frac{1}{4}(2-\nu\nabla_x)^2 F, \tag{31}
$$

Equation (26) becomes

$$\left[\frac{1}{8}(2-\nu\nabla_x)^3F\right]_t+\nabla_x^3F+(2-\nu\nabla_x)^{-1}\nabla_x\left[\frac{1}{4}(2-\nu\nabla_x)^3F\right]^2+\left[\frac{1}{4}(2-\nu\nabla_x)^3F\right]\cdot\left[\frac{1}{4}\nabla_x(2-\nu\nabla_x)^2F\right]=0. \tag{32}$$

Then assuming

$$F=\nabla_x w, \tag{33}$$

it is easy to obtain

$$\frac{1}{8}\left[(2-\nu\nabla_x)^4w\right]_t+(2-\nu\nabla_x)\nabla_x^3w+\left[\frac{1}{4}(2-\nu\nabla_x)^3\nabla_x w\right]^2+(2-\nu\nabla_x)\left[\frac{1}{4}(2-\nu\nabla_x)^2\nabla_x w\right]^2=0. \tag{34}$$

Finally, assuming

$$w=2\frac{\nabla_x G}{G}, \tag{35}$$

then the formula for G is obtained:

$$\frac{1}{4}\left[(2-\nu\nabla_x)^4\left(\frac{\nabla_x G}{G}\right)\right]_t+2(2-\nu\nabla_x)\nabla_x^3\left(\frac{\nabla_x G}{G}\right)+\left[\frac{1}{2}(2-\nu\nabla_x)^3\nabla_x\left(\frac{\nabla_x G}{G}\right)\right]^2+(2-\nu\nabla_x)\left[\frac{1}{2}(2-\nu\nabla_x)^2\nabla_x\left(\frac{\nabla_x G}{G}\right)\right]^2=0. \tag{36}$$

In order to get the single-solution of KdV equation on time-space scale, the perturbation method is used to expand G into a power series with a small parameter ε:

$$G=1+\varepsilon g_1+\varepsilon^2g_2+\cdots. \tag{37}$$

Substituting the expansion of G into Equation (36), rearrange it according to the power of ε:

$$\begin{aligned}
&\varepsilon^0:0=0,\\
&\varepsilon^1:\left[(2-\nu\nabla_x)^4\nabla_x g_{1t}\right]+8(2-\nu\nabla_x)\nabla_x^4g_1=0,\\
&\varepsilon^2:\left[(2-\nu\nabla_x)^4\nabla_x g_{2t}\right]+(2-\nu\nabla_x)^4[(\nabla_x g_{1t})g_1-(\nabla_x g_1)g_{1t}]\\
&\quad+8(2-\nu\nabla_x)\left[\nabla_x^4g_2+(\nabla_x^4g_1)g_1-(\nabla_x^3g_1)(\nabla_x g_1^{\rho\rho})-2(\nabla_x^3g_1)(\nabla_x g_1^{\rho})-2(\nabla_x^2g_1)^{\rho}(\nabla_x^2g_1^{\rho})-(\nabla_x^3g_1)(\nabla_x g_1)-(\nabla_x^2g_1)^{\rho}(\nabla_x^2g_1)\right]\\
&\quad+\left[(2-\nu\nabla_x)^3\nabla_x^2g_1\right]^2+(2-\nu\nabla_x)\left[(2-\nu\nabla_x)^2\nabla_x^2g_1\right]^2=0,\\
&\quad\vdots
\end{aligned} \tag{38}$$

When

$$g_1=e^{at}\hat{e}_b(x),\quad g_i=0,\ i\geq 2, \tag{39}$$

the relation between ν and b is obtained:

$$(2-\nu b)\nu b^5\left(-\nu^4b^4+9\nu^3b^3-21\nu^2b^2+26\nu b-12\right)=0. \tag{40}$$

In the following, several situations are discussed.

When $\nu\neq 2/b$,

$$a=\frac{-8(2-\nu b)b^3}{16-32\nu b+24\nu^2b^2-8\nu^3b^3+\nu^4b^4},\quad u=\frac{1}{4}(2-\nu\nabla_x)^3\cdot\left[\frac{b^2}{(1/(e^{at}\hat{e}_b(x))e^{at}\hat{e}_b(x))+2+(1-\nu b)e^{at}\hat{e}_b(x)-\nu b}\right]. \tag{41}$$

The relation between a and b is known as the nonlinear dispersion relation, and the expression of single-soliton solution is given.

Then, we discuss two special cases.

Case 1. Taking $\mathbb{T}\times\mathbb{X}=\mathbb{R}\times\mathbb{R}$, we find $\mu(t)=0$, $\nu(x)=0$.

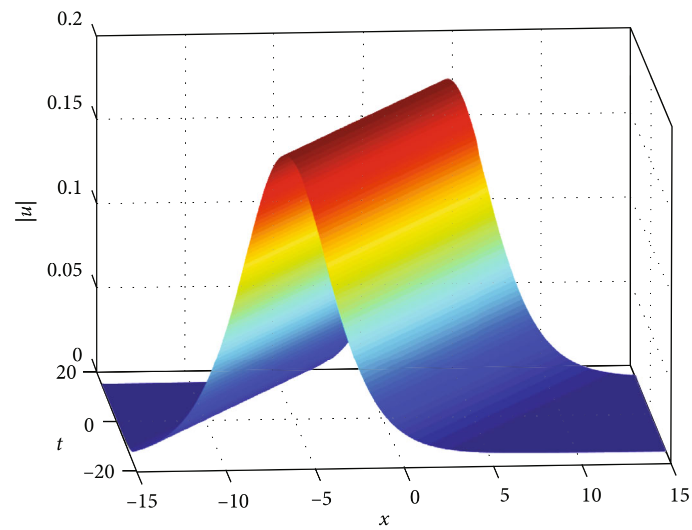

FIGURE 1: The single-soliton solution with $b = 0.6$ and $\eta_1^0 = 0.6$.

The classical KdV equation is obtained:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{42}$$

Setting

$$g_1 = e^{\eta_1}, \eta_1 = at + bx + \eta_1^0, \tag{43}$$

it is easy to obtain the classic nonlinear dispersion relation and the expression of single-soliton solution of KdV equation.

$$\begin{aligned} a &= -b^3, \\ u &= \frac{b^2}{2} \sec h^2 \frac{at + bx + \eta_1^0}{2}. \end{aligned} \tag{44}$$

The graph of the single-soliton solution is presented in Figure 1.

Case 2. Taking $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{Z}$, we find $\mu(t) = 0, \nu(x) = 1$.

$$\begin{aligned} a &= \frac{-8(2-b)b^3}{16 - 32b + 24b^2 - 8b^3 + b^4}, \\ u &= \frac{1}{4}(1+E)^3 \cdot \left[\frac{b^2}{(1/(e^{at}(1/(1-b))^{x-x_0})e^{at}(1/1-b)^{x-x_0}) + 2 + (1-b)e^{at}(1/(1-b))^{x-x_0} - b} \right]. \end{aligned} \tag{45}$$

At this time, the nonlinear dispersion relation and the expression of single-soliton solution are given, respectively.

When $\nu = 2/b, u = 0$. This is a trivial solution.

5. Conclusions

In this paper, a method of generating integrable system on time-space scale is introduced. Starting from the ∇-dynamical system, the coupled KdV equation on time-space scale is derived from the Lax pair and zero curvature equation. When different time scales are considered, different soliton equations can be obtained. In addition, the variable transformation of the KdV equation on time-space scale is constructed to obtain its single-soliton solution.

As we all know, the KdV equation appears in the study of many different physical systems, such as water waves, plasma physics, anharmonic lattices, and elastic rods, which serves as a model equation governing weakly nonlinear long waves whose phase speed attains a simple maximum for waves of infinite length [32] and describes the long-time evolution of dispersive waves of small but finite amplitude [38]. In the continuous case, the nonlinear dispersion relation in the single-soliton solution of the classical KdV equation explains the equilibrium phenomenon of shallow water wave motion. In the discrete case, the single-soliton solution of the KdV equation can explain some basic principles in quantum mechanics [39].

The results obtained in this paper effectively unify the continuous and discrete cases. Equations (44) and (45) are the nonlinear dispersion relations in the classical continuous and discrete cases, respectively. In addition, we expect this model to describe the phenomenon which contain both continuous and discrete case and provide a new idea for solving the complex model. Due to the limitations of computer algorithms, it is difficult to obtain its dynamic graph in this case, and the problem of defining an operator on time-space scale to make the equation billinearized and further solved remains to be studied. Therefore, we mainly focus on finding a more efficient way to simplify the structure of equation (25) or solution (45) in the future research and apply this model in more fields.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

H.H.D. and Y.F. were responsible for the conceptualization; Y.K. and X.C.L. were responsible for the methodology; M.S.L. was responsible for the software; C.M.W. and X.Q.H. were responsible for the validation; X.C.L. was responsible for the formal analysis; Y.K. and X.C.L. were responsible for the investigation; X.C.L. wrote the original draft; Y.F. wrote, reviewed, and edited the manuscript. All authors have read and agreed to the published version of the manuscript. Yuan Kong, Xing-Chen Li, and Huan-He Dong contributed equally to this work.

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