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*Research Article*

# Fuzzy Fixed Point Results of Fuzzy Mappings on $b$-Metric Spaces via $(\alpha_*, F)$-Contractions

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In this manuscript, we establish some fixed point results for fuzzy mappings via $(\alpha_*, F)$ -contractions. For validation of the proved results, some nontrivial examples are presented. Few interesting consequences are also stated which authenticate that our results generalize many existing ones in the literature.

## 1. Introduction

Fixed point theory has attained an important and significant role in analysis. The literature of the last four decades ornamented with results which discover fixed points of self and non-self nonlinear operators in a metric space. This branch of mathematics provides a strong tool for finding out the solution of integral, differential, and eigenvalue equations. Since 1922, the Banach contraction principle (BCP) has become the center of the attention for researchers working in different areas. The eminent Banach contraction principle (BCP) [1] states that for a complete metric space $(\Omega, d)$, each self-mapping $\eta$ on $\Omega$ satisfying

$$d(\eta\xi_1, \eta\xi_2) \leq \lambda d(\xi_1, \xi_2), \forall \xi_1, \xi_2 \in \Omega, \text{ where } \lambda \in [0, 1) \quad (1)$$

has a unique fixed point.

A lot of literature can be found on generalizations and extensions of the famous BCP by changing either the space under consideration or the condition on the mapping. In recent years, various authors presented interesting generalizations of a metric space, for example, uniform space [2, 3], $b$-metric space [4], $C^*$-algebra valued metric space [5, 6], $G_{p_b}$-metric space [7], and $b_2$-metric space [8].

A very interesting generalization of metric space is cone metric space. Gupta and Chauhan [9] presented an analogous to the BCP on cone $b$-metric spaces in 2021. In contrast, Wardowski [10] extended the Banach contraction to a more generalized form, known as $F$-contractions, and established a fixed point theorem in complete metric spaces. Furthermore, many mathematicians used $F$-contractions for the existence of a fixed point, see [11–13]. Nadler's theorem [14] states that for a complete metric space $(\Omega, d)$, each non-self-mapping $\eta : \Omega \longrightarrow C(\Omega)$ satisfying

$$H(\eta\xi_1, \eta\xi_2) \leq \lambda d(\xi_1, \xi_2), \forall \xi_1, \xi_2 \in \Omega, \text{ where } \lambda \in [0, 1) \quad (2)$$

has a fixed point. Here, $H$ denotes the Hausdorff metric defined on $C(\Omega)$, the set of bounded and closed subsets of $\Omega$. Throughout the article, a $b-$metric space and a fuzzy fixed point are denoted by a bms and a ffp, respectively. Some important definitions are presented before constructing the main results. These definitions are inevitable for next discussion.

*Definition 1.* Let $\Omega$ be a nonempty set. A function $\mathscr{D}_b : \Omega \times \Omega \longrightarrow [0,\infty)$ is called a $b$-metric if, for all $\xi_1, \xi_2, \xi_3 \in \Omega$ and $b \geq 1$, the following assertions are satisfied:

(i) $\mathscr{D}_b(\xi_1, \xi_2) = 0$, if and only if $\xi_1 = \xi_2$

(ii) $\mathscr{D}_b(\xi_1, \xi_2) = \mathscr{D}_b(\xi_2, \xi_1)$

(iii) $\mathscr{D}_b(\xi_1, \xi_2) \leq b[\mathscr{D}_b(\xi_1, \xi_3) + \mathscr{D}_b(\xi_3, \xi_2)]$

The pair $(\Omega, \mathscr{D}_b)$ is called a bms. if we consider $b = 1$, then above definition coincides with the definition of a metric space.

*Example 2.* Let $\Omega = \mathbb{R}$. Define $\mathscr{D}_b : \Omega \times \Omega \longrightarrow \mathbb{R}$ by $\mathscr{D}_b(\xi_1, \xi_2) = (\xi_1 - \xi_2)^2$. Then, it is easy to show that $(\mathscr{D}_b, \Omega)$ is a bms with $b = 2$.

*Definition 3.* Let $(\Omega, \mathscr{D}_b)$ be a bms.

(i) A sequence $\{\xi_n\} \in (\Omega, \mathscr{D}_b)$ is said to be a Cauchy sequence if, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, we have $\mathscr{D}_b(\xi_m, \xi_n) < \varepsilon$ or $\lim_{n,m\longrightarrow\infty}\mathscr{D}_b(\xi_m, \xi_n) = 0$

(ii) A sequence $\{\xi_n\}$ converges to $\xi$ if $\lim_{n\longrightarrow\infty}\mathscr{D}_b(\xi_n, \xi) = 0$

Among numerous advances of fuzzy sets theory, a significant development is made to find the fuzzy analogues of fixed point results of the classical fixed point theorems. The concept of fuzzy sets along with its related notions is given by Zadeh in [15]. In 1975, the idea of fuzzy metric space is introduced. This interesting concepts is investigated by many researchers with different contractions, see for example [16, 17].

Furthermore, Weiss [18] and Butnariu [19] used the notions of fuzzy maps to established various results. Heilpern [20] proved a fixed point theorem for fuzzy maps which is the analogue of Nadler's multivalued result [14] in metric spaces. Notion of fuzzy maps is investigated by many mathematician in different directions. In this context, in [21], the existence of common fuzzy fixed points under a rational contractive condition has been established which is further generalized by Shoaib et al. [22] in dislocated complete metric spaces. Motivated by these research ideas, we formulate some results for fuzzy contractions using the Hausdorff metric. Namely, the present article provides certain fixed point results for fuzzy mappings via $(\alpha_*, F)$ contractions. A mapping $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is said to be an $F$-mapping [10], if the following conditions are satisfied:

(F1): $F$ is a strictly increasing function, that is, for $\xi_1, \xi_2 \in \mathbb{R}^+$, with $\xi_1 < \xi_2$ then $F(\xi_1) < F(\xi_2)$

(F2): for each sequence $\{\xi_n\}$ of the positive real numbers $\mathbb{R}^+$,

$$\lim_{n\longrightarrow\infty}\xi_n = 0 \text{ if and only if } \lim_{n\longrightarrow\infty} F(\xi_n) = -\infty. \tag{3}$$

(F3): There is a real number $c \in (0, 1)$ such as

$$\lim_{\xi\longrightarrow 0^+} \xi^c F(\xi) = 0. \tag{4}$$

Afterward, Altun et al. [13] altered the above definition by adding another condition given below:

(F4): $F(\inf (A)) = \inf (F(A)) \forall A \subset (0,\infty)$

The set of all functions satisfying $(F_1)$ to $(F_4)$ is denoted by $\mathscr{F}$. Let $(\Omega, d)$ be a metric space and $\eta : \Omega \longrightarrow \Omega$ be a self mapping on $\Omega$. $\eta$ is said to be an $F$-contraction [10] if there exists $\tau > 0$ and $F$ satisfying $(F_1)$-$(F_3)$ such that

$$d(\eta\xi_1, \eta\xi_2) > 0 \Rightarrow \tau + F(d(\eta\xi_1, \eta\xi_2)) \leq F(d(\xi_1, \xi_2)) \forall \xi_1, \xi_2 \in \Omega. \tag{5}$$

*Definition 4.* (see [22]). Let $M$ be a nonempty subset of a bms $(\Omega, \mathscr{D}_b)$, and let $\xi_1 \in \Omega$. An element $\xi_0 \in M$ is called the best approximation in $M$ if $\mathscr{D}_b(\xi_1, M) = \mathscr{D}_b(\xi_1, \xi_0)$, where

$$\mathscr{D}_b(\xi_1, M) = \inf_{\xi_2\in\Omega} \mathscr{D}_b(\xi_1, \xi_2). \tag{6}$$

The set $M$ is called a proximinal set if every $\xi \in \Omega$ has at least one best approximation in $M$. The set of all proximinal subsets of $\Omega$ is denoted by $P(\Omega)$.

*Definition 5.* (see [22]). The function, $H_{\mathscr{D}_b} : P(\Omega) \times P(\Omega) \longrightarrow \mathbb{R}$, defined by

$$H_{\mathscr{D}_b}(M, N) = \max \left\{ \sup_{\xi_1\in M} \mathscr{D}_b(\xi_1, N), \sup_{\xi_2\in N} \mathscr{D}_b(M, \xi_2) \right\} \tag{7}$$

is called a Hausdorff $b$-metric on $P(\Omega)$.

A function with domain $\Omega$ and values in $[0, 1]$ is called a fuzzy set. We will denote the collection of all fuzzy sets in $\Omega$ by $F(\Omega)$. If $\xi \in \Omega$, and $M$ is a fuzzy set, then $M(\xi)$ is called the grade of membership of $\xi$ in $M$. The set $[M]_\alpha$ is called an $\alpha$-level set of a fuzzy set $M$, and it is given by

$$\begin{gathered} [M]_\alpha = \{\xi : M(\xi) \geq \alpha\}, \text{where } \alpha \in (0, 1], \\ [M]_0 = \{\xi : M(\xi) > 0\}. \end{gathered} \tag{8}$$

If $\eta$ is a mapping from $\Omega$ into $F(\Omega,)$ then $\eta$ is called a fuzzy mapping. A fuzzy mapping $\eta$ is a fuzzy subset on $\Omega \times \Omega$ with membership function $\eta(\xi_1)(\xi_2)$. The function $\eta(\xi_1)(\xi_2)$ is the grade of membership of $\xi_2$ in $\eta(\xi_1)$.

*Definition 6.* (see [21]). Let $\eta : \Omega \longrightarrow F(\Omega)$ be a fuzzy mapping. A point $\xi_1 \in \Omega$ is called an $\alpha$ – fuzzy fixed point (ffp) if there exists $\alpha \in (0, 1]$ such that $\xi \in [\eta\xi_1]_\alpha$.

**Lemma 7.** (see [22]). *Let $M$ and $N$ be nonempty proximal subsets of a bms $(\Omega, \mathscr{D}_b)$.*

*If $\xi_1 \in M$, then $\mathscr{D}_b(\xi_1, N) \leq H_{\mathscr{D}_b}(M, N)$.*

**Lemma 8.** (see [22]). *Let $(\Omega, \mathscr{D}_b)$ be a bms and suppose that $(P(\Omega), H_{\mathscr{D}_b})$ is a Hausdorff bms. If, for all $M, N \in P(\Omega)$ and for each $\xi_1 \in M$, there exists $\xi_2 \in N$ satisfying $D_b(\xi_1, N) = \mathscr{D}_b(\xi_1, \xi_2)$, then $H_{\mathscr{D}_b}(M, N) \geq \mathscr{D}_b(\xi_1, \xi_2)$.*

## 2. Main Results

*Definition 9.* Consider a bms $(\Omega, \mathscr{D}_b)$. A mapping $\eta : \Omega \longrightarrow F(\Omega)$ is said to be a fuzzy $(\alpha_*, F)$-contractive mapping if for a function $\alpha_* : \Omega \times \Omega \longrightarrow [0,\infty)$, there exists $F : (0,+\infty) \longrightarrow \mathbb{R}$ such that condition $(F1)$ holds, and the following conditions are satisfied:

(Fb2): for each sequence $\{\xi_n\}$ of positive numbers, if $\lim_{n\longrightarrow\infty} F(\xi_n) = -\infty$, then $\lim_{n\longrightarrow\infty}\xi_n = 0$

(Fb3): $k \in (0, (1/1 + \log_2 b))$ such that $\xi^k F(\xi) = 0$

(Fb4): there exists $\tau > 0$ for which

$$\begin{aligned}
&\tau + F\left(\alpha_*(\xi_1, \xi_2) H\left([\eta(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_2)]_{\alpha(\xi_2)}\right)\right) \\
&\quad \leq F\left(a_1 \mathscr{D}_b\left(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)}\right) + a_2 \mathscr{D}_b\left(\xi_2, [\eta(\xi_2)]_{\alpha(\xi_2)}\right)\right. \\
&\qquad + a_3 \mathscr{D}_b\left(\xi_1, [\eta(\xi_2)]_{\alpha(\xi_2)}\right) + a_4 \mathscr{D}_b\left(\xi_2, [\eta(\xi_1)]_{\alpha(\xi_1)}\right) \\
&\qquad \left. + a_5 \mathscr{D}_b(\xi_1, \xi_2) + a_6 \mathscr{D}_b\left(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)}\right)\left(\frac{1 + \mathscr{D}_b(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)})}{1 + \mathscr{D}_b(\xi_1, \xi_2)}\right)\right),
\end{aligned} \tag{9}$$

for all $\xi_1, \xi_2 \in \Omega$, and $a_i \geq 0$ for $1 \leq i \leq 6$, also $a_1 + a_2 + 2ba_3 + a_4 + a_5 + a_6 < 1$.

**Theorem 10.** *Let $(\Omega, \mathscr{D}_b)$ be a complete bms with $b \geq 1$. Let $\eta : \Omega \longrightarrow F(\Omega)$ be a fuzzy $(\alpha_*, F)$-contractive mapping. Then, $\eta$ has an $\alpha - ffp$ if following conditions are satisfied:*

*(i) $\eta$ is an $\alpha_*$-admissible mapping*

*(ii) There exists $\xi_0 \in \Omega$ and $\xi_1 \in [\eta(\xi_0)]_{\alpha(\xi_0))}$ such that $\alpha_*(\xi_0, \xi_1) \geq 1$*

*(iii) For any sequence $\{\xi_n\} \subset \Omega$ which converges to $\xi \in \Omega$ with $\alpha_*(\xi_n, \xi_{n+1}) \geq 1 \forall n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\longrightarrow\infty} \alpha_*(\xi_n, \xi) \geq 1$*

*Proof.* Let $\xi_0 \in \Omega$ be an arbitrary point of $\Omega$. Choose $\xi_1 \in [\eta(\xi_0)]_{\alpha(\xi_0)}$. If $\xi_1 \in [\eta(\xi_1)]_{\alpha(\xi_1))}$, then $\xi_1$ is a fixed point, and there is nothing to prove; so, $\xi_1 \notin [\eta(\xi_1)]_{\alpha(\xi_1))}$. Therefore, $[\eta(\xi_0)]_{\alpha(\xi_0))} \neq [\eta(\xi_1)]_{\alpha(\xi_1))}$. Now, by Lemma 8, there exists $\xi_1 \in [\eta(\xi_2)]_{\alpha(\xi_2))}$ such that

$$\mathscr{D}_b(\xi_1, \eta\xi_1) < H\left([\eta(\xi_0)]_{\alpha(\xi_0)}, [\eta(\xi_1)]_{\alpha(\xi_1)}\right). \tag{10}$$

Then,

$$\mathscr{D}_b(\xi_1, \xi_2) < H\left([\eta(\xi_0)]_{\alpha(\xi_0)}, [\eta(\xi_1)]_{\alpha(\xi_1)}\right). \tag{11}$$

Consider

$$\begin{aligned}
F(\mathscr{D}_b(\xi_2, \xi_1)) &< F\left(\alpha_*(\xi_0, \xi_1) H\left([\phi(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\right) \tau + F(\mathscr{D}_b(\xi_2, \xi_1)) \\
&< \tau + F\left(\alpha_*(\xi_0, \xi_1) H\left([\phi(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\right)\Big) \\
&\leq F\left(a_1 \mathscr{D}_b\left(\xi_1, [\phi(\xi_1)]_{\alpha(\xi_1)}\right) + a_2 \mathscr{D}_b\left(\xi_0, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\right. \\
&\quad \left. + a_3 \mathscr{D}_b\left(\xi_1, [\eta(\xi_0)]_{\alpha(\xi_0)}\right) + a_4 \mathscr{D}_b\left(\xi_0 [\phi(\xi_1)]_{\alpha(\xi_1)}\right) + a_5 \mathscr{D}_b(\xi_0, \xi_1)\right) \\
&\leq F(a_1 \mathscr{D}_b(\xi_1, \xi_2) + a_2 \mathscr{D}_b(\xi_0, \xi_1) + a_3 \mathscr{D}_b(\xi_1, \xi_1)) \\
&\quad + a_4 \mathscr{D}_b(\xi_0, \xi_2) + a_5 \mathscr{D}_b(\xi_0, \xi_1) \\
&\leq F(a_1 \mathscr{D}_b(\xi_1, \xi_2) + a_2 \mathscr{D}_b(\xi_0, \xi_1) + a_4 b(\mathscr{D}_b(\xi_0, \xi_1) \\
&\quad + \mathscr{D}_b(\xi_1, \xi_2)) + a_5 \mathscr{D}_b(\xi_0, \xi_1) \\
&\leq F((a_2 + ba_4 + a_5) \mathscr{D}_b(\xi_0, \xi_1) + (a_1 + ba_4) \mathscr{D}_b(\xi_1, \xi_2)).
\end{aligned} \tag{12}$$

Thus,

$$F(D_b(\xi_1, \xi_2)) \leq F((a_1 + a_3 b + a_5 + a_6) \mathscr{D}_b(\xi_0, \xi_1) + (a_2 + a_3 b) \mathscr{D}_b(\xi_1, \xi_2)). \tag{13}$$

Since $F$ is increasing, one writes

$$D_b(\xi_1, \xi_2) \leq (a_1 + a_3 b + a_5 + a_6) \mathscr{D}_b(\xi_0, \xi_1) + (a_2 + a_3 b) \mathscr{D}_b(\xi_1, \xi_2). \tag{14}$$

We have

$$D_b(\xi_1, \xi_2) \leq \left(\frac{a_1 + a_3 b + a_5 + a_6}{1 - (a_2 + a_3 b)}\right) \mathscr{D}_b(\xi_0, \xi_1). \tag{15}$$

That is,

$$D_b(\xi_1, \xi_2) \leq \mathscr{D}_b(\xi_0, \xi_1) \tag{16}$$

because $a_1 + a_2 + 2ba_3 + a_4 + a_5 + a_6 < 1$. Consequently, $\tau + F(\mathscr{D}_b(\xi_1, \xi_2) \leq F(\mathscr{D}_b(\xi_0, \xi_1))$. Continuing in the same manner, we can define a sequence $\{\xi_n\}$ such that $\xi_n \notin [\eta(\xi_n)]_{\alpha(\xi_n)}, \xi_{n+1} \in [\eta(\xi_n)]_{\alpha(\xi_n)}$ and

$$\tau + F(\mathscr{D}_b(\xi_{n+1}, \xi_{n+2}) \leq F(\mathscr{D}_b(\xi_n, \xi_{n+1})) \forall n \in \mathbb{N} \cup \{0\}. \tag{17}$$

Now, let $d_n = \mathscr{D}_b(\xi_n, \xi_{n+1}) > 0, \forall n \in \mathbb{N} \cup \{0\}$. By (2.2),

$$F(d_n) \leq F(d_{n-1}) - \tau \leq \cdots \leq F(d_0) - n\tau, n \in \mathbb{N} \tag{18}$$

and hence $\lim_{n\longrightarrow+\infty} F(d_n) = -\infty$. By property (Fb2), we get that $d_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, by condition (Fb3), there exists $k \in (0, (1/1 + \log_2 b))$ such that

$$\lim_{n\longrightarrow\infty} d_n^k F(d_n) = 0. \tag{19}$$

Therefore, we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -n d_n^k \tau. \tag{20}$$

Applying limit $n \longrightarrow \infty$, we have $\lim_{n\longrightarrow\infty} d_n^k n = 0$.

Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$d_n^k \leq \frac{1}{n} \Rightarrow d_n \leq \frac{1}{n^{1/k}} \forall, n \geq n_0. \tag{21}$$

Using Lemma 7 given in [23], $\{\xi_n\}$ is a Cauchy sequence in $\Omega$. Since $\Omega$ is complete, there exists $\xi \in \Omega$ such that $\lim_{n\longrightarrow\infty}\xi_n = \xi$. We claim that $\eta$ has an $\alpha -$ ffp. Consider

$$\begin{aligned}
\mathscr{D}_b\left(\xi, [\eta(\xi)]_{\alpha(\xi)}\right) &\leq b\left(\mathscr{D}_b(\xi, \xi_{n+1}) + d\left(\xi_{n+1}, [\eta(\xi)]_{\alpha(\xi)}\right)\right) \\
&\leq b\left(\mathscr{D}_b(\xi, \xi_{n+1}) + H\left([\eta(\xi_n)]_{\alpha(\xi_n)}, [\eta(\xi)]_{\alpha(\xi)}\right)\right) \\
&\leq b\left(\mathscr{D}_b(\xi, \xi_{n+1}) + H\left(\alpha_*(\xi_n, \xi)[\eta(\xi_n)]_{\alpha(\xi_n)}, [\eta(\xi)]_{\alpha(\xi)}\right)\right) \\
&\leq b\Big(\mathscr{D}_b(\xi, \xi_{n+1}) + a_1\mathscr{D}_b\left(\xi_n, [\eta(\xi_n)]_{\alpha(\xi_n)}\right) \\
&\quad + a_2\mathscr{D}_b\left(\xi, [\eta(\xi)]_{\alpha(\xi)}\right) + a_3\mathscr{D}_b\left(\xi_n, [\eta(\xi)]_{\alpha(\xi)}\right) \\
&\quad + a_4\mathscr{D}_b\left(\xi, [\eta(\xi_n)]_{\alpha(\xi_n)}\right) + a_5\mathscr{D}_b(\xi_n, \xi) \\
&\quad + a_6\left(\frac{\mathscr{D}_b\left(\xi_n, [\eta(\xi_n)]_{\alpha(\xi_n)}\left(1 + \mathscr{D}_b\left(\xi_n, [\eta(\xi_n)]_{\alpha(\xi_n)}\right)\right)\right)}{1 + \mathscr{D}_b(\xi_n, \xi)}\right) \\
&\leq b\Big(\mathscr{D}_b(\xi, \xi_{n+1}) + a_1\mathscr{D}_b(\xi_n, \xi_{n+1}) + a_2\mathscr{D}_b\left(\xi, [\eta(\xi)]_{\alpha(\xi)}\right) \\
&\quad + a_3\mathscr{D}_b\left(\xi_n, [\eta(\xi)]_{\alpha(\xi)}\right) + a_4\mathscr{D}_b(\xi, \xi_{n+1}) + a_5\mathscr{D}_b(\xi_n, \xi) \\
&\quad + a_6\left(\frac{\mathscr{D}_b(\xi_n, \xi_{n+1})\left(1 + \mathscr{D}_b\left(\xi_n, [\eta(\xi_n)]_{\alpha(\xi_n)}\right)\right)}{1 + \mathscr{D}_b(\xi_n, \xi)}\right)\Big).
\end{aligned} \tag{22}$$

Taking limit $n \longrightarrow \infty$, we get

$$(1 - ba_2 - ba_3)\mathscr{D}_b\left(\xi, [\eta(\xi)]_{\alpha(\xi)}\right) \leq 0. \tag{23}$$

So, we get $\mathscr{D}_b(\xi, [\eta(\xi)]_{\alpha(\xi)}) = 0$, which implies that $\xi \in [\eta(\xi)]_{\alpha(\xi)}$. Hence, $\xi \in \Omega$ is an $\alpha -$ ffp of $\eta$. □

Letting $b = 1$ in Theorem 10, we obtain the following:

**Corollary 11.** *Let $(\Omega, \mathscr{D})$ be a complete metric space. Let $\eta : \Omega \longrightarrow F(\Omega)$ be a fuzzy $(\alpha_*, F)$-contractive mapping, and then $\eta$ has an $\alpha -$ ffp if following conditions are satisfied:*

*(i) $\eta$ is an $\alpha_*$-admissible mapping*

*(ii) There exists $\xi_0 \in \Omega$ and $\xi_1 \in [\eta(\xi_0)]_{\alpha(\xi_0))}$ such that $\alpha_*(\xi_0, \xi_1) \geq 1$*

*(iii) For any sequence $\{\xi_n\} \subset \Omega$ which converges to $\xi \in \Omega$ with $\alpha_*(\xi_n, \xi_{n+1}) \geq 1 \forall n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\longrightarrow\infty}\alpha_*(\xi_n, \xi) \geq 1$*

If we consider $b = 1$ and $\alpha_*(\xi_1, \xi_2) = 1$ for all $\xi_1, \xi_2 \in \Omega$, then we have the following result proved by Ahmed et al. [11].

**Corollary 12.** *Let $(\Omega, \mathscr{D})$ be a complete metric space, and let $\eta : \Omega \longrightarrow P(\Omega)$. Suppose for each $\xi_1$ and $\xi_2 \in \Omega$, there exists $\alpha(\xi_1)$ and $\alpha(\xi_2) \in (0, 1]$ such that $[\eta(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_2)]_{\alpha(\xi_2)} \in CB(\Omega)$. Assume there exists $F \in \mathscr{F}$ such that*

$$\tau + H\left([\eta(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_2)]_{\alpha(\xi_2)}\right) \leq F(\mathscr{D}(\xi_1, \xi_2)), \tag{24}$$

*then $\eta$ has an $\alpha -$ ffp.*

*Example 13.* Let $\Omega = [0, 1]$ and define $\mathscr{D}_b = (\xi_1 - \xi_2)^2$. Then, $(\Omega, \mathscr{D}_b)$ is a complete bms with $b = 2$. Define a fuzzy mapping $\eta : \Omega \longrightarrow F(\Omega)$ by

$$\eta(\xi)(t) = \begin{cases} 1, \text{for} 0 \leq t \leq \dfrac{\xi}{4}; \\ \dfrac{1}{2}, \text{for} \dfrac{\xi}{4} \leq t \leq \dfrac{\xi}{3}; \\ \dfrac{1}{4}, \text{for} \dfrac{\xi}{3} \leq t \leq \dfrac{\xi}{2}; \\ 0, \text{for} \dfrac{\xi}{2} \leq t \leq 1, \end{cases} \tag{25}$$

and for all $\xi \in \Omega$, there exists $\alpha(\xi) = 1$ such that

$$\begin{aligned}
[\eta\xi]_{\alpha(\xi)} &= \left[0, \frac{\xi}{4}\right]. \\
\alpha_*(\xi_1, \xi_2) &= \begin{cases} 1, \text{if} \xi_1, \xi_2 \in \Omega - \{1\}; \\ 0, \text{otherwise}. \end{cases}
\end{aligned} \tag{26}$$

Let $a_1 = 1/10$, $a_2 = a_3 = 0$, $a_4 = 1/20$, $a_5 = 1/30$, and $a_6 = 1/40$, and then we have

$$\begin{aligned}
&H\left([\eta\xi_1]_{\alpha(\xi_1)}, [\eta\xi_2]_{\alpha(\xi_2)}\right)\left([\eta\xi_2]_{\alpha(\xi_2)}\right) \\
&\quad \leq \frac{1}{10}\left(\xi_1 - \frac{\xi_1}{2}\right)^2 + (0)\left(\xi_2 - \frac{\xi_2}{2}\right)^2 + (0)\left(\xi_1 - \frac{\xi_2}{2}\right)^2 \\
&\qquad + \frac{1}{20}\left(\xi_2 - \frac{\xi_1}{2}\right)^2 + \frac{1}{30}(\xi_1 - \xi_2)^2 \\
&\qquad + \frac{1}{40}\left[\frac{(\xi_1 - \xi_1/2)^2\left[1 + (\xi_1 - \xi_1/2)^2\right]}{1 + (\xi_1 - \xi_2)^2}\right],
\end{aligned} \tag{27}$$

where $\mu = (a_1 + ba_3 + a_5 + a_6)/1 - (a_2 + ba_3) < 1/b$. Therefore,

$$\begin{aligned}
&\tau + F\left(\alpha_*(\xi_1, \xi_2)H\left(\left[\eta(\xi_1)_{\alpha(\xi_1)}, [\eta(\xi_2)]_{\alpha(\xi_2)}\right]\right)\right) \\
&\quad \leq F\Big(a_1\mathscr{D}_b\left(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)}\right) + a_2\mathscr{D}_b\left(\xi_2, [\eta(\xi_2)]_{\alpha(\xi_2)}\right) \\
&\qquad + a_3\mathscr{D}_b\left(\xi_1, [\eta(\xi_2)]_{\alpha(\xi_2)}\right) + a_4\mathscr{D}_b\left(\xi_2, [\eta(\xi_1)]_{\alpha(\xi_1)}\right) \\
&\qquad + a_5\mathscr{D}_b(\xi_1, \xi_2) + a_6\mathscr{D}_b\left(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)}\right)\left(\frac{1 + \mathscr{D}_b\left(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)}\right)}{1 + \mathscr{D}_b(\xi_1, \xi_2)}\right)\Big),
\end{aligned} \tag{28}$$

where $F(x) = \ln x$ for all $x \in (0,\infty)$. It follows that all the conditions of Theorem 10 are satisfied, and 0 is an $\alpha -$ ffp of $\eta$.

*Definition 14.* Let $(\Omega, \mathscr{D}_b)$ be a bms. Let $\alpha_* : \Omega \times \Omega \longrightarrow [0, \infty)$ and $(\phi, \eta)$ be a pair of fuzzy mappings from $\Omega$ into $F(\Omega)$. The pair $(\phi, \eta)$ is said to be $\alpha_*$-admissible if it satisfies the following conditions:

(i) For each $\xi_1 \in \Omega$, there exists $\xi_2 \in [\eta\xi_1]_{\alpha(\xi_1)}$ such that $\alpha_*(\xi_2, \xi_3) \geq 1$ for all $\xi_3 \in [\phi\xi_2]_{\alpha(\xi_2)}$

(ii) For each $\xi_1 \in \Omega$, there exists $\xi_2 \in [\phi\xi_1]_{\alpha(\xi_1)}$ such that $\alpha_*(\xi_2, \xi_3) \geq 1$ for all $\xi_3 \in [\eta\xi_2]_{\alpha(\xi_2)}$

*Definition 15.* Consider a complete bms $(\Omega, \mathscr{D}_b)$. A pair of fuzzy mappings $(\phi, \eta)$ is said to be an $(\alpha_*, F)$-contraction if there exists $F : (0,+\infty) \longrightarrow \mathbb{R}$ so that the condition (F1) holds and the following conditions are satisfied:

(F$'$b2): for each sequence $\{\xi_n\}$ of positive numbers, if $\lim_{n\longrightarrow\infty} F(\xi_n) = -\infty$, then $\lim_{n\longrightarrow\infty} \xi_n = 0$;

(F$'$b3): there exists $k \in (0, (1/1 + \log_2 b))$ such that $\xi^k F(\xi) = 0$;

(F$'$b4): there exists $\tau > 0$, for which

$$\begin{aligned}
&\tau + F\left(\alpha_*(\xi_1, \xi_2) H\left([\eta(\xi_1)]_{\alpha(\xi_1)}, [\phi(\xi_2)]_{\alpha(\xi_2)}\right)\right) \\
&\quad \leq F\left(a_1 \mathscr{D}_b\left(\xi_1, [\eta(\xi_1)]_{\alpha(\xi_1)}\right) + a_2 \mathscr{D}_b\left(\xi_2, [\phi(\xi_2)]_{\alpha(\xi_2)}\right)\right. \\
&\qquad + a_3 \mathscr{D}_b\left(\xi_1, [\phi(\xi_2)]_{\alpha(\xi_2)}\right) + a_4 \mathscr{D}_b\left(\xi_2, [\eta(\xi_1)]_{\alpha(\xi_1)}\right) \\
&\qquad + a_5 \mathscr{D}_b(\xi_1, \xi_2))
\end{aligned} \tag{29}$$

$\forall \xi_1, \xi_2 \in \Omega$, with $a_i \geq 0$, for $1 \leq i \leq 5$, $(a_1 + a_2)(b + 2) + b(a_3 + a_4)(b + 2) + 2ba_5 < 2$.

**Theorem 16.** *Let $(\Omega, \mathscr{D}_b)$ be a complete bms with $b \geq 1$. Let $(\eta, \phi)$ be a pair of $(\alpha_*, F)$ fuzzy contraction mappings, and then $\eta$ and $\phi$ have a common $\alpha -$ ffp if following conditions are satisfied:*

*(i) Both $\eta$ and $\phi$ are $\alpha_*$-admissible mappings*

*(ii) There exists $\xi_0 \in \Omega$ and $\xi_1 \in [\eta(\xi_0)]_{\alpha(\xi_0))}$ such that $\alpha_*(\xi_0, \xi_1) \geq 1$*

*(iii) For any sequence $\{\xi_n\} \subset \Omega$ which converges to $\xi \in \Omega$ with $\alpha_*(\xi_n, \xi_{n+1}) \geq 1 \forall n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\longrightarrow\infty} \alpha_*(\xi_n, \xi) \geq 1$*

*Proof.* Let $\xi_0 \in \Omega$ be an arbitrary point of $\Omega$. Choose $\xi_1 \in [\eta(\xi_0)]_{\alpha(\xi_0)}$; then by Lemma 8, there exists $\xi_2 \in [\phi(\xi_1)]_{\alpha(\xi_1)}$ such that

$$\mathscr{D}_b\left(\xi_1, [\phi(\xi_1)]_{\alpha(\xi_1)}\right) < H\left(\eta(\xi_0)_{\alpha(\xi_0)}, [\phi(\xi_1)]_{\alpha(\xi_1)}\right). \tag{30}$$

Now, by Lemma 8,

$$\mathscr{D}_b(\xi_1, \xi_2) < H\left(\eta(\xi_0)_{\alpha(\xi_0)}, [\phi(\xi_1)]_{\alpha(\xi_1)}\right). \tag{31}$$

By using assumption (ii),

$$\begin{aligned}
F(\mathscr{D}_b(\xi_1, \xi_2)) &< F\left(\alpha_*(\xi_0, \xi_1) H\left([\eta(\xi_0)]_{\alpha(\xi_0)}, [\phi(\xi_1)]_{\alpha(\xi_1)}\right)\right)\tau + F(\mathscr{D}_b(\xi_1, \xi_2)) \\
&< \tau + F\left(\alpha_*(\xi_0, \xi_1) H\left([\eta(\xi_0)]_{\alpha(\xi_0)}, [\phi(\xi_1)]_{\alpha(\xi_1)}\right)\right) \\
&\leq F\left(a_1 \mathscr{D}_b\left(\xi_0, [\eta(\xi_0)]_{\alpha(\xi_0)}\right) + a_2 \mathscr{D}_b\left(\xi_1, [\phi(\xi_1)]_{\alpha(\xi_1)}\right)\right. \\
&\quad + a_3 \mathscr{D}_b\left(\xi_0, [\phi(\xi_1)]_{\alpha(\xi_1)}\right) + a_4 \mathscr{D}_b\left(\xi_1, [\eta(\xi_0)]_{\alpha(\xi_0)}\right) \\
&\quad + a_5 \mathscr{D}_b(\xi_0, \xi_1)) \\
&\leq F(a_1 \mathscr{D}_b(\xi_0, \xi_1) + a_2 \mathscr{D}_b(\xi_1, \xi_2) + a_3 \mathscr{D}_b(\xi_0, \xi_2) \\
&\quad + a_5 \mathscr{D}_b(\xi_0, \xi_1)) \leq F(a_1 \mathscr{D}_b(\xi_0, \xi_1) + a_2 \mathscr{D}_b(\xi_1, \xi_2)) \\
&\quad + a_3 b((\mathscr{D}_b(\xi_0, \xi_1) + \mathscr{D}_b(\xi_1, \xi_2)) + a_5 \mathscr{D}_b(\xi_0, \xi_1)) \\
&\leq F((a_1 + a_3 b + a_5) \mathscr{D}_b(\xi_0, \xi_1) + (a_2 + a_3 b) \mathscr{D}_b(\xi_1, \xi_2). \\
&\Rightarrow F\left(\mathscr{D}_\lfloor(\xi_1, \xi_2)\right) \\
&\leq F((a_1 + a_3 b + a_5) \mathscr{D}_b(\xi_0, \xi_1) + (a_2 + a_3 b) \mathscr{D}_b(\xi_1, \xi_2)).
\end{aligned} \tag{32}$$

Since $F$ is increasing, one writes

$$\mathscr{D}_b(\xi_1, \xi_2) \leq (a_1 + a_3 b + a_5) \mathscr{D}_b(\xi_0, \xi_1) + (a_2 + a_3 b) \mathscr{D}_b(\xi_1, \xi_2). \tag{33}$$

That is,

$$(1 - (a_2 + a_3 b))(\mathscr{D}_b(\xi_1, \xi_2)) \leq ((a_1 + a_3 b) \mathscr{D}_b(\xi_0, \xi_1). \tag{34}$$

So, we have

$$\mathscr{D}_b(\xi_1, \xi_2) \leq \left(\frac{a_1 + a_3 b + a_5}{1 - (a_2 + a_3 b)}\right) \mathscr{D}_b(\xi_0, \xi_1). \tag{35}$$

By using symmetric property of a bm, we have

$$\mathscr{D}_b(\xi_2, \xi_1) < H\left(\phi(\xi_1)_{\alpha(\xi_1)}, [\eta(\xi_0)]_{\alpha(\xi_0)}\right). \tag{36}$$

Then,

$$F(\mathscr{D}_b(\xi_2, \xi_1)) < F\left(H\left(\phi(\xi_1)_{\alpha(\xi_1)}, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\right). \tag{37}$$

Consider

$$\begin{aligned}
F(\mathscr{D}_b(\xi_2, \xi_1)) &< F(\alpha_*(\xi_0, \xi_1) H\left([\phi(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\tau + F(\mathscr{D}_b(\xi_2, \xi_1)) \\
&< \tau + F\left(\alpha_*(\xi_0, \xi_1) H\left([\phi(\xi_1)]_{\alpha(\xi_1)}, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\right) \\
&\leq F\left(a_1 \mathscr{D}_b\left(\xi_1, [\phi(\xi_1)]_{\alpha(\xi_1)}\right) + a_2 \mathscr{D}_b\left(\xi_0, [\eta(\xi_0)]_{\alpha(\xi_0)}\right)\right. \\
&\quad \left. + a_3 \mathscr{D}_b\left(\xi_1, [\eta(\xi_0)]_{\alpha(\xi_0)}\right) + a_4 \mathscr{D}_b\left(\xi_0, [\phi(\xi_1)]_{\alpha(\xi_1)} + a_5 \mathscr{D}_b(\xi_0, \xi_1)\right)\right) \\
&\leq F(a_1 \mathscr{D}_b(\xi_1, \xi_2) + a_2 \mathscr{D}_b(\xi_0, \xi_1) + a_3 \mathscr{D}_b(\xi_1, \xi_1)) + a_4 \mathscr{D}_b(\xi_0, \xi_2) + a_5 \mathscr{D}_b(\xi_0, \xi_1) \\
&\leq F(a_1 \mathscr{D}_b(\xi_1, \xi_2) + a_2 \mathscr{D}_b(\xi_0, \xi_1) + a_4 b(\mathscr{D}_b(\xi_0, \xi_1) + \mathscr{D}_b(\xi_1, \xi_2)) + a_5 \mathscr{D}_b(\xi_0, \xi_1) \\
&\leq F((a_2 + b a_4 + a_5) \mathscr{D}_b(\xi_0, \xi_1) + (a_1 + b a_4) \mathscr{D}_b(\xi_1, \xi_2)).
\end{aligned} \tag{38}$$

Thus,

$$F(\mathscr{D}_b(\xi_1,\xi_2)) \le F((a_2+a_4b+a_5)\mathscr{D}_b(\xi_0,\xi_1)+(a_1+a_4b)\mathscr{D}_b(\xi_1,\xi_2)). \tag{39}$$

Since $F$ is increasing, we have

$$\mathscr{D}_b(\xi_2,\xi_1) \le (a_2+ba_4+a_5)\mathscr{D}_b(\xi_0,\xi_1)+(a_1+ba_4)\mathscr{D}_b(\xi_1,\xi_2). \tag{40}$$

That is,

$$(1-(a_1+ba_4))(\mathscr{D}_b(\xi_1,\xi_2)) \le (a_2+ba_4+a_5)\mathscr{D}_b(\xi_0,\xi_1). \tag{41}$$

Therefore, we obtain

$$\mathscr{D}_b(\xi_2,\xi_1) \le \left(\frac{a_2+a_4b+a_5}{1-(a_1+a_4b)}\right)\mathscr{D}_b(\xi_0,\xi_1). \tag{42}$$

Add (35) and (42) to conclude

$$\mathscr{D}_b(\xi_1,\xi_2) \le \frac{a_1+a_2+ba_3+ba_4+2a_5}{2-2(a_1+a_2+ba_3+ba_4)}\mathscr{D}_b(\xi_0,\xi_1). \tag{43}$$

By using the given condition, we have

$$\mathscr{D}_b(\xi_1,\xi_2) \le \mathscr{D}_b(\xi_0,\xi_1). \tag{44}$$

Consequently, $\tau + F(\mathscr{D}_b(\xi_1,\xi_2)) \le F(\mathscr{D}_b(\xi_0,\xi_1))$. Continuing in the same manner, we can define a sequence $\{\xi_n\}$ such that $\xi_n \notin [\eta(\xi_n)]_{\alpha(\xi_n)}, \xi_{n+1} \in [\eta(\xi_n)]_{\alpha(\xi_n)}$ and

$$\tau + F(\mathscr{D}_b(\xi_{n+1},\xi_{n+2})) \le F(\mathscr{D}_b(\xi_n,\xi_{n+1}))\forall n \in \mathbb{N}\cup\{0\}. \tag{45}$$

Now, let $d_n = \mathscr{D}_b(\xi_n,\xi_{n+1}) > 0, \forall n \in \mathbb{N}\cup\{0\}$. By (2.10),

$$F(d_n) \le F(d_{n-1}) - \tau \le \cdots \le F(d_0) - n\tau, n \in \mathbb{N}, \tag{46}$$

and hence $\lim_{n\longrightarrow+\infty} F(d_n) = -\infty$. By property (F′b2), we obtain $d_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, by condition (F′b3), there exists $k \in (0,(1/1+\log_2 b))$ such that

$$\lim_{n\longrightarrow\infty} d_n^k F(d_n) = 0. \tag{47}$$

From (47),

$$d_n^k F(d_n) - d_n^k F(d_0) \le -n d_n^k \tau. \tag{48}$$

Applying limit $n \longrightarrow \infty$, we have $\lim_{n\longrightarrow\infty} d_n^k n = 0$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$d_n^k \le \frac{1}{n} \Rightarrow d_n \le \frac{1}{n^{1/k}} \forall n \ge n_0. \tag{49}$$

Using Lemma 7 given in [23], $\{\xi_n\}$ is a Cauchy sequence in $\Omega$ which is complete; so, there exists $\xi \in \Omega$ such that $\lim_{n\longrightarrow\infty}\xi_n = \xi$. We shall prove that $\Phi$ has a fixed point. Consider

$$\begin{aligned}\mathscr{D}_b\left(\xi,[\phi(\xi)]_{\alpha(\xi)}\right) &\le b\left(\mathscr{D}_b(\xi,\xi_{n+1}) + \mathscr{D}_b\left(\xi_{n+1},[\phi(\xi)]_{\alpha(\xi)}\right)\right)\\ &\le b\left(\mathscr{D}_b(\xi,\xi_{n+1}) + H\left([\eta(\xi_n)]_{\alpha(\xi_n)},[\phi(\xi)]_{\alpha(\xi)}\right)\right)\\ &\le b\left(\mathscr{D}_b(\xi,\xi_{n+1}) + \alpha_*(\xi_n,\xi)H\left([\eta(\xi_n)]_{\alpha(\xi_n)},[\phi(\xi)]_{\alpha(\xi)}\right)\right).\end{aligned} \tag{50}$$

Since $F$ is increasing and

$$H\left([\eta(\xi_n)]_{\alpha(\xi_n)},[\phi(\xi)]_{\alpha(\xi)}\right) \le \left(\alpha_*(\xi_n,\xi)H\left([\eta(\xi_n)]_{\alpha(\xi_n)},[\phi(\xi)]_{\alpha(\xi)}\right)\right), \tag{51}$$

one writes

$$\begin{aligned}\mathscr{D}_b\left(\xi,[\phi(\xi)]_{\alpha(\xi)}\right) &\le F\left(\alpha_*(\xi_n,\xi)H\left([\eta(\xi_n)]_{\alpha(\xi_n)},[\phi(\xi)]_{\alpha(\xi)}\right)\right)\\ &\le b\Big(\mathscr{D}_b(\xi,\xi_{n+1}) + a_1\mathscr{D}_b\left(\xi_n,[\eta(\xi_n)]_{\alpha(\xi_n)}\right)\\ &\quad + a_2\mathscr{D}_b\left(\xi,[\phi(\xi)]_{\alpha(\xi)}\right) + a_3\mathscr{D}_b\left(\xi_n,[\phi(\xi)]_{\alpha(\xi)}\right)\\ &\quad + a_4\mathscr{D}_b\left(\xi,[\eta(\xi_n)]_{\alpha(\xi_n)}\right) + a_5\mathscr{D}_b(\xi_n,\xi)\Big).\end{aligned} \tag{52}$$

Using (52), we obtain

$$\begin{aligned}\mathscr{D}_b\left(\xi,[\phi(\xi)]_{\alpha(\xi)}\right) &\le F\left(\alpha_*(\xi_n,\xi)H\left([\eta(\xi_n)]_{\alpha(\xi_n)},[\phi(\xi)]_{\alpha(\xi)}\right)\right)\\ &\le b\Big(\mathscr{D}_b(\xi,\xi_{n+1}) + a_1\mathscr{D}_b\left(\xi_n,[\eta(\xi_n)]_{\alpha(\xi_n)}\right)\\ &\quad + a_2\mathscr{D}_b\left(\xi,[\phi(\xi)]_{\alpha(\xi)}\right) + a_3\mathscr{D}_b\left(\xi_n,[\phi(\xi)]_{\alpha(\xi)}\right)\\ &\quad + a_4\mathscr{D}_b\left(\xi,[\eta(\xi_n)]_{\alpha(\xi_n)}\right) + a_5\mathscr{D}_b(\xi_n,\xi)\Big).\end{aligned} \tag{53}$$

Taking limit $n \longrightarrow \infty$, we obtain

$$\mathscr{D}_b\left(\xi,[\phi(\xi)]_{\alpha(\xi)}\right) \le 0. \tag{54}$$

Therefore, $\xi \in [\phi\xi]_{\alpha(\xi)}$. This implies that $\xi$ is an $\alpha -$ ffp of $\phi$. By the same procedure, one can show that $\xi \in [\eta\xi]_{\alpha(\xi)}$. Hence, $\xi \in \Omega$ is a common $\alpha -$ ffp of $\phi$ and $\eta$. □

If we consider $b = 1$ and $\alpha * (\xi_1,\xi_2) = 1$ for all $\xi_1, \xi_2$ in $\Omega$, then we have the following result proved by Ahmed et al. [11].

**Corollary 17.** *Let $(\Omega,\mathscr{D})$ be a complete metric space, and let $\eta,\phi : \Omega \longrightarrow P(\Omega)$. Suppose for each $\xi_1$ and $\xi_2 \in \Omega$, there exists $\alpha(\xi_1)$ and $\alpha(\xi_2) \in (0,1]$ such that $[\eta(\xi_1)]_{\alpha(\xi_1)}$,*

$[\phi(\xi_2)]_{\alpha(\xi_2)} \in CB(\Omega)$. *Assume there exists* $F \in \mathscr{F}$ *such that*

$$\tau + H\left([\eta(\xi_1)]_{\alpha(\xi_1)}, \left[\phi(\xi_2)_{\alpha(\xi_2)}\right]\right) \leq F(\mathscr{D}(\xi_1, \xi_2)), \tag{55}$$

*then* $\eta$ *and* $\phi$ *have a common* $\alpha - ffp$.

*Example 18.* Let $\Omega = \{0, 1, 2\}$ and $\mathscr{D}_b$ be a bm (with $b > 1$) defined by

$$\mathscr{D}_b(\xi_1, \xi_2) = \begin{cases} 0, \text{for} \xi_1 = \xi_2 \text{ and } \xi_1, \xi_2 \in \{0, 1, 2\}; \\ \frac{1}{2}, \text{for} \xi_1 \neq \xi_2 \text{ and } \xi_1, \xi_2 \in \{0, 2\}; \\ \frac{1}{4}, \text{for} \xi_1 \neq \xi_2 \text{ and } \xi_1, \xi_2 \in \{1, 2\}; \\ 1, \text{for} \xi_1 \neq \xi_2 \text{ and } \xi_1, \xi_2 \in \{0, 1\}. \end{cases} \tag{56}$$

It is clear that it is a bms with $b = 4/3$, but it is not a metric space.

Take $\phi, \eta : \Omega \longrightarrow F(\Omega)$ as fuzzy mappings defined by

$$\begin{aligned} (\eta 0)(t) &= \begin{matrix} 0, \text{for} t = 1, 2; \xi = 0; \\ \frac{1}{2}, \text{for} t = 0, \xi = 0; \end{matrix} \\ (\eta 1)(t) &= \begin{cases} 0, \text{for} t = 0, 1; \\ \frac{1}{2}, \text{for} t = 2; \end{cases} \\ (\eta 2)(t) &= \begin{cases} 0, \text{for} t = 0; \\ \frac{1}{2}, \text{for} t = 1; \end{cases} \\ (\phi 0)(t) &= (\phi 1)(t) = (\phi 2)(t) = 0, \text{for} t = 0, 1, 2. \end{aligned} \tag{57}$$

Now, we define $\alpha_{\eta(\xi)} = \alpha_{\phi(\xi)} = \alpha \in (0, 1]$, and we have

$$\begin{aligned} (\eta\xi)_{\alpha_\eta(\xi)} &= \begin{cases} \{0\}, \text{for} \xi = 0; \\ \{1\}, \text{for} \xi = 1; \\ \{2\}, \text{for} \xi = 2; \end{cases} \\ (\phi\xi)_{\alpha_\phi(\xi)} &= \{0\}, \text{for} \xi = 0, 1, 2. \end{aligned} \tag{58}$$

Therefore,

$$\begin{aligned} H\left([\eta\xi]_{\alpha_\eta(\xi)}, [\phi\xi]_{\alpha_\phi(\xi)}\right) &= \begin{cases} H(\{0\}, \{0\}) = 0, \text{for} \xi = 0; \\ H(\{1\}, \{0\}) = 1, \text{for} \xi = 1; \\ H(\{2\}, \{0\}) = 2, \text{for} \xi = 2; \end{cases} \\ \alpha_*(\xi_1, \xi_2) &= \begin{cases} 1, \text{if} \xi_1, \xi_2 \in \Omega - \{1\}; \\ 0, \text{otherwise}. \end{cases} \end{aligned} \tag{59}$$

Let $a_1 = a_2 = 1/4$, $a_3 = a_5 = 0$, and $a_4 = 1/8$, and then we have $b\mu = 4/5 < 1$ and $\mu = (a_1 + a_2 + ba_3 + ba_4 + 2a_5)/2 - (a_1 + a_2 + ba_3 + ba_4) = 3/5$. Further, it can also be seen that $a_i \geq 0$ for $i = 1, 2, \cdots 5$, where $(a_1 + a_2)(b + 1) + b(b + 1)(a_3 + a_4) + 2ba_5 = 189/128 < 2$ and $\sum_{i=1}^{5} a_i < 1$. Further, it is easy to verify that all the conditions of Theorem 10 are satisfied with $F(x) = \ln(x)$ for all $x \in (0,\infty)$. Hence, there exists $0 \in [\eta\xi_1]_{\alpha_\eta(\xi_1)} \cap [\phi\xi_2]_{\alpha_\phi(\xi_2)}$ which is a common fixed point.

## 3. Conclusion

In this paper, the notion of $(\alpha*,F)$ fuzzy contractive mappings has been introduced, and few results have been established for the existence of $\alpha$ fuzzy fixed points of an $(\alpha*,F)$-contraction and a pair of $(\alpha*,F)$-contractions. To elaborate the main results, examples have also been presented. Few corollaries have been established to show that our results generalize and extend many existing classical results available in the literature.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no competing interests.

## Authors' Contributions

All authors contributed equally and significantly in writing this article.

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