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Research Article

Sub-Lorentzian Geometry of Curves and Surfaces in a Lorentzian Lie Group

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We consider the sub-Lorentzian geometry of curves and surfaces in the Lie group $E(1,1)$. Firstly, as an application of Riemannian approximants scheme, we give the definition of Lorentzian approximants scheme for $E(1,1)$ which is a sequence of Lorentzian manifolds denoted by $E^L_{\lambda_1,\lambda_2}$. By using the Koszul formula, we calculate the expressions of Levi-Civita connection and curvature tensor in the Lorentzian approximants of $E^L_{\lambda_1,\lambda_2}$ in terms of the basis $\{E_1, E_2, E_3\}$. These expressions will be used to define the notions of the intrinsic curvature for curves, the intrinsic geodesic curvature of curves on surfaces, and the intrinsic Gaussian curvature of surfaces away from characteristic points. Furthermore, we derive the expressions of those curvatures and prove two generalized Gauss-Bonnet theorems in $E^L_{\lambda_1,\lambda_2}$.

1. Introduction

In recent years, there has been done a lot of research concerning Gauss-Bonnet theorems in three-dimensional Lie groups with sub-Riemannian geometric structures. One of the reasons for this success is the discovery of the method of Riemannian approximations. Initial work by Balogh et al. proved a Heisenberg version of the Gauss-Bonnet theorem with the help of the method of Riemannian approximations [1, 2]. Hereafter the above work, many research on Gauss-Bonnet theorem have been gained; we refer the monograph [3–6]. In particular, Wang and Wei proved Gauss-Bonnet theorems on the affine group [3], the group of rigid motions of the Minkowski plane [3, 4], the BCV spaces [5], and the Lorentzian Heisenberg group [4, 6]. Inspired by their work, we proved Gauss-Bonnet theorems in the rototranslation group [7, 8], Lorentzian Sasakian space forms, and the group of rigid motions of Minkowski plane with the general left-invariant metric [9, 10]. Riemannian approximations can be extended to the case for any Lie group equipped with left-invariant Lorentzian metric g named Lorentzian approximations. In particular, one can consider a sequence of Lorentzian manifolds denoted by (G, g_L), where a family of metrics $g_L = -\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L\omega \otimes \omega$, for $L > 0$, is essentially obtained as an anisotropic blow-up of the Lorentzian metric g. Then, one can define sub-Lorentzian objects as limits of horizontal objects in (G, g_L), since the intrinsic horizontal geometry does not change with L. Some typical works of Lorentzian approximations in the Lorentzian Heisenberg group are obtained in [4, 6]. For Lorentzian Sasakian space forms, see [9].

In this paper, we consider sub-Lorentzian geometry of curves and surfaces on $E(1,1)$. The group of rigid motions of Minkowski plane $E(1,1)$ has two left-invariant Lorentzian metrics g_1 and g_2 [11, 12]. Onda proved that the metric g_1 is a Lorentz Ricci soliton [12]. In [13], Patrangenaru proved that any left-invariant metric on $E(1,1)$ is isometric to one of the metric $g(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 = 1/\lambda_1\lambda_2$. In [14], the metric $g(\lambda_1, \lambda_2, \lambda_3)$ was denoted by $g(\lambda_1, \lambda_2) = g(\lambda_1, \lambda_2, 1/\lambda_1\lambda_2)$. $E(1,1)$ with the metric $g(\lambda_1, \lambda_2)$ provides a natural 2-parametric deformation family of $\mathrm{Sol}_3 = (E(1,1), g(1,1))$ which is the model space of solve geometry in the eight model geometries of Thurston, which makes $(E(1,1), g(\lambda_1, \lambda_2))$ very interesting and important [14, 15]. However, very little is known about sub-Lorentzian geometry of $E(1,1)$. In this paper, we focus on the general left-invariant

Lorentzian metric $g = -\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega \otimes \omega$ on $E(1,1)$, where the coframe $\omega_1 = 1/\sqrt{\lambda_1\lambda_2}du_3$, $\omega_2 = \lambda_1/\sqrt{2}(-e^{-u_3}du_1 + e^{u_3}du_2)$, $\omega = -\lambda_2/\sqrt{2}(e^{-u_3}du_1 + e^{u_3}du_2)$. By using the method of Lorentzian approximations, we consider a sequence of Lorentzian manifolds $E^L_{\lambda_1,\lambda_2} = (E(1,1), g_L)$, where $g_L = -\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L\omega \otimes \omega$, for $L > 0$, is an anisotropic blow-up of the Lorentzian metric g. Using the Koszul formula, we calculate the expressions of Levi-Civita connection and curvature tensor in the Lorentzian approximants of $E^L_{\lambda_1,\lambda_2}$ in terms of the basis $\{E_1, E_2, E_3\}$. We define the notions of the intrinsic curvature for regular curves, the intrinsic geodesic curvature of regular curves on spacelike surfaces and Lorentzian surfaces, and the intrinsic Gaussian curvature of spacelike surfaces and Lorentzian surfaces away from characteristic points. Furthermore, we derive the expressions of those curvatures and prove two generalized Gauss-Bonnet theorems in $E^L_{\lambda_1,\lambda_2}$.

This paper is organized as follows. In Section 2, we introduce the Lorentzian approximations of $(E(1,1), g)$ and calculate the expressions of corresponding Levi-Civita connection in terms of the basis $\{E_1, E_2, E_3\}$. Furthermore, we define the notions of geodesic curvature and intrinsic geodesic curvature of curves in $E^L_{\lambda_1,\lambda_2}$. We get the expressions of those curvatures and give an example. In Sections 3 and 4, we compute intrinsic geodesic curvatures of regular curves on Lorentzian surfaces and the intrinsic Gaussian curvature of Lorentzian surfaces in $E^L_{\lambda_1,\lambda_2}$. We also give two examples. In Section 5, we get the first Gauss-Bonnet theorem in $E^L_{\lambda_1,\lambda_2}$. In Section 6, we compute intrinsic geodesic curvature of curves on spacelike surfaces and the intrinsic Gaussian curvature of spacelike surfaces in $E^L_{\lambda_1,\lambda_2}$ and we get the second Guass-Bonnet theorem.

2. Curvatures for Curves in Lorentzian Approximations $E^L_{\lambda_1,\lambda_2}$

In this section, some basic notions in the Lorentzian group of rigid motions of the Minkowski plane group will be introduced. Let $E(1,1)$ be the motion group of the Minkowski 2-space. This consists of all matrices of the form

$$E(1,1) = \left\{ \begin{pmatrix} e^{u_3} & 0 & u_1 \\ 0 & e^{-u_3} & u_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| u_1, u_2, u_3 \in \mathbb{R} \right\}. \quad (1)$$

Topologically, $E(1,1)$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$. Its Lie algebra has a basis consisting of

$$E_1 = \lambda_1\lambda_2 \frac{\partial}{\partial u_3}, \quad (2)$$

$$E_2 = \frac{1}{\lambda_1\sqrt{2}}\left(-e^{u_3}\frac{\partial}{\partial u_1} + e^{-u_3}\frac{\partial}{\partial u_2}\right), \quad (3)$$

$$E_3 = -\frac{1}{\lambda_2\sqrt{2}}\left(e^{u_3}\frac{\partial}{\partial u_1} + e^{-u_3}\frac{\partial}{\partial u_2}\right), \quad (4)$$

for which

$$[E_1, E_2] = \lambda_2{}^2 E_3, \quad (5)$$

$$[E_2, E_3] = 0, \quad (6)$$

$$[E_1, E_3] = \lambda_1{}^2 E_2. \quad (7)$$

From the above equations in Equation (2), we get that

$$\frac{\partial}{\partial u_1} = -\frac{\sqrt{2}}{2}e^{-u_3}(\lambda_1 E_2 + \lambda_2 E_3), \quad (8)$$

$$\frac{\partial}{\partial u_2} = \frac{\sqrt{2}}{2}e^{u_3}(\lambda_1 E_2 - \lambda_2 E_3), \quad (9)$$

$$\frac{\partial}{\partial u_3} = \frac{1}{\lambda_1\lambda_2}E_1. \quad (10)$$

We denote $\mathrm{span}\{E_1, E_2, E_3\} = T(E(1,1))$. Let $H = \mathrm{span}\{E_1, E_2\}$ be the horizontal distribution on $E(1,1)$ and

$$\omega_1 = \frac{1}{\sqrt{\lambda_1\lambda_2}}du_3, \quad (11)$$

$$\omega_2 = \frac{\lambda_1}{\sqrt{2}}(-e^{-u_3}du_1 + e^{u_3}du_2), \quad (12)$$

$$\omega = -\frac{\lambda_2}{\sqrt{2}}(e^{-u_3}du_1 + e^{u_3}du_2). \quad (13)$$

Then, $H = \ker\omega$. We consider the left-invariant Lorentzian metrics given by $g = -\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega \otimes \omega$ and its anisotropic blow-up $g_L = -\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L\omega \otimes \omega$, for $L > 0$. We call $(E(1,1), g_L)$ the Lorentzian approximants of $E(1,1)$ and denote by $E^L_{\lambda_1,\lambda_2} = (E(1,1), g_L)$ throughout the paper. It is easy to check that $g = g_1$ is the Lorentzian metric on $E(1,1)$ and $E_1, E_2, \widetilde{E_3} := L^{-1/2}E_3$ are pseudo-orthonormal basis on $T(E(1,1))$ with respect to g_L. Then, the Levi-Civita connection $\nabla^{E,L}$ of $E^L_{\lambda_1,\lambda_2}$ is given by following proposition.

Proposition 1. *The Levi-Civita connection $\nabla^{E,L}$ relative to the coordinate frame $E_1, E_2, \widetilde{E}_3$ of $E^L_{\lambda_1,\lambda_2}$ is described as follows:*

$$
\begin{aligned}
\nabla^{E,L}_{E_j} E_j &= 0,\ 1 \le j \le 3,\\
\nabla^{E,L}_{E_1} E_2 &= \frac{\lambda_2^2 L - \lambda_1^2}{2L} E_3,\\
\nabla^{E,L}_{E_2} E_1 &= \frac{-\lambda_2^2 L - \lambda_1^2}{2L} E_3,\\
\nabla^{E,L}_{E_1} E_3 &= \frac{\lambda_1^2 - \lambda_2^2 L}{2} E_2,\\
\nabla^{E,L}_{E_3} E_1 &= \frac{-\lambda_1^2 - \lambda_2^2 L}{2} E_2,\\
\nabla^{E,L}_{E_2} E_3 = \nabla^{E,L}_{E_3} E_2 &= -\frac{\lambda_1^2 + \lambda_2^2 L}{2} E_1.
\end{aligned}
\tag{14}
$$

Proof. To derive the above expressions, it is useful to recall the following Koszul identity from the famous proof of the unique of Riemannian connection:

$$
\begin{aligned}
2\left\langle \nabla^{E,L}_{E_i} E_j, E_k \right\rangle_L &= \left\langle [E_i, E_j], E_k \right\rangle_L - \left\langle [E_j, E_k], E_i \right\rangle_L\\
&\quad + \left\langle [E_k E_i], E_j \right\rangle_L,
\end{aligned}
\tag{15}
$$

where $i, j, k = 1, 2, 3$. By Equations (3) and (14), the first formula in Equation (11) is as follows:

$$
\begin{aligned}
2\left\langle \nabla^{E,L}_{E_j} E_j, E_k \right\rangle_L &= \left\langle [E_j, E_j], E_k \right\rangle_L - \left\langle [E_j, E_k], E_j \right\rangle_L\\
&\quad + \left\langle [E_k, E_j], E_j \right\rangle_L = -\left\langle [E_j, E_k], E_j \right\rangle_L\\
&\quad + \left\langle [E_k, E_j], E_j \right\rangle_L = 2\left\langle [E_k, E_j], E_j \right\rangle_L.
\end{aligned}
\tag{16}
$$

When $j = 1$, we compute $\langle \nabla^{E,L}_{E_1} E_1, E_k \rangle_L = \langle [E_k, E_1], E_1 \rangle_L$. It follows that $\langle \nabla^{E,L}_{E_1} E_1, E_1 \rangle_L = 0$, $\langle \nabla^{E,L}_{E_1} E_1, E_2 \rangle_L = \langle [E_2, E_1], E_1 \rangle_L = \langle -\lambda_2{}^2 E_3, E_1 \rangle_L = 0$ and $\langle \nabla^{E,L}_{E_1} E_1, E_3 \rangle_L = \langle [E_3, E_1], E_1 \rangle_L = 0$. Hence, $\nabla^{E,L}_{E_1} E_1 = 0$. Similarly, $\nabla^{E,L}_{E_2} E_2 = 0$ and $\nabla^{E,L}_{E_3} E_3 = 0$. Moreover, other expressions follow some similar computations.

A parametrized curve in $E^L_{\lambda_1,\lambda_2}$ is a map $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$, where I is an open interval in $\mathbb{R}$, where $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and each function $\alpha_i(t)$ has derivatives of two orders, for all $t \in I$. Such α is called C^2-smooth. The regular curve α is called regular provided that there does not exist $t \in I$ with $\dot{\alpha}(t) = d\alpha(t)/dt = 0$. We call α a spacelike curve, timelike curve, or null curve if $\dot{\alpha}(t) = d\alpha(t)/dt$ is a spacelike vector, timelike vector, or null vector at any $t \in I$, respectively. □

Definition 2. For an arbitrary C^1-smooth regular curve $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, we say that $\alpha(t)$ is a horizontal point of α provided the following function $\omega(\dot{\alpha}(t))$ satisfies the following:

$$
\begin{aligned}
\omega(\dot{\alpha}(t)) &= -\frac{\lambda_2}{\sqrt{2}}\left(e^{-\alpha_3} du_1 + e^{\alpha_3} du_2\right)\\
&\quad \cdot \left(\dot{\alpha}_1(t)\frac{\partial}{\partial u_1} + \dot{\alpha}_2(t)\frac{\partial}{\partial u_2} + \dot{\alpha}_3(t)\frac{\partial}{\partial u_3}\right)\\
&= -\frac{\lambda_2\sqrt{2}}{2}\left(e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right) = 0.
\end{aligned}
\tag{17}
$$

Definition 3. For an arbitrary C^2-smooth regular curve $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, we define the geodesic curvature $\kappa^{E,L}_\alpha$ of α at t in the following way.

$$
\kappa^{E,L}_\alpha := \sqrt{\frac{\left\|\nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}\right\|_L^2}{\|\dot{\alpha}\|_L^4} - \frac{\left\langle \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_L^2}{\langle \dot{\alpha}, \dot{\alpha}\rangle_L^3}},
\tag{18}
$$

if $\nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}$ is a spacelike vector.

$$
\kappa^{E,L}_\alpha := \sqrt{\frac{\left\|\nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}\right\|_L^2}{\|\dot{\alpha}\|_L^4} + \frac{\left\langle \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_L^2}{\langle \dot{\alpha}, \dot{\alpha}\rangle_L^3}},
\tag{19}
$$

if $\nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}$ is a timelike vector.

Proposition 4. *There are formulae of curvature for C^2-smooth regular curve $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$.*

(1) If $\nabla_{\dot{\alpha}}{}^{E,L}\dot{\alpha}$ is a spacelike vector, then

$$
\begin{aligned}
\kappa^{E,L}_\alpha = \Bigg\{ \Bigg\{ &-\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1{}^3\sqrt{2} + \lambda_1\lambda_2{}^2\sqrt{2}L}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)\right.\right.\\
&\left.\left. + e^{\alpha_3}\dot{\alpha}_2(t)\right)\omega(\dot{\alpha}(t))\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2 e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3 e^{\alpha_3}\right.\right.\\
&\left.\left. - \ddot{\alpha}_1 e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3 e^{-\alpha_3}\right) - L\omega(\dot{\alpha}(t))\dot{\alpha}_3(t)\right]^2\\
&+ L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1{}^2\sqrt{2}}{2\lambda_2 L}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)\right.\right.\\
&\left.\left. + e^{\alpha_3}\dot{\alpha}_2(t)\right)\dot{\alpha}_3(t)\right]^2 \Bigg\} \cdot \Bigg\{ -\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2\\
&+ \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\\
&+ L(\omega(\dot{\alpha}(t)))^2 \Bigg\}^{-2} - \kappa_2 \Bigg\}^{1/2},
\end{aligned}
\tag{20}
$$

where

$$\kappa_2 = -\left\{ \frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) + \frac{\lambda_1^{\ 3}\sqrt{2}+\lambda_1\lambda_2^{\ 2}\sqrt{2}L}{2} \cdot (-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}t)\right] + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1+e^{\alpha_3}\dot{\alpha}_2) \cdot \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\omega(\dot{\alpha})\dot{\alpha}_3\right] + L\omega(\dot{\alpha}(t))\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^{\ 2}\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\}^2 \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2 + L(\omega(\dot{\alpha}(t)))^2\right\}^{-3}. \tag{21}$$

In particular, if $\alpha(t)$ *is a horizontal point of* α,

$$\kappa_\alpha^{E,L} = \left\{\left\{-\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right]^2 + L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^{\ 2}\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]^2\right\} \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2\right\}^{-2} - \left\{\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right] + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right]\right\}^2 \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2\right\}^{-3}\right\}^{1/2}. \tag{22}$$

(2) If $\nabla_{\dot{\alpha}}^{\ E,L}\dot{\alpha}$ *is a timelike vector, then*

$$\kappa_\alpha^{E,L} = \left\{\left\{\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1^{\ 3}\sqrt{2}+\lambda_1\lambda_2^{\ 2}\sqrt{2}L}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}(t))\right]^2 - \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\omega(\dot{\alpha}(t))\dot{\alpha}_3(t)\right]^2 - L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^{\ 2}\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]^2\right\} \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2 + L(\omega(\dot{\alpha}(t)))^2\right\}^{-2} + \left\{\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) + \frac{\lambda_1^{\ 3}\sqrt{2}+\lambda_1\lambda_2^{\ 2}\sqrt{2}L}{2} \cdot (-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}(t))\right] + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1+e^{\alpha_3}\dot{\alpha}_2) \cdot \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\omega(\dot{\alpha})\dot{\alpha}_3\right] + L\omega(\dot{\alpha}(t))\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^{\ 2}\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\}^2 \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2 + L(\omega(\dot{\alpha}(t)))^2\right\}^{-3}\right\}^{1/2}. \tag{23}$$

In particular, if $\alpha(t)$ *is a horizontal point of* α,

$$\kappa_\alpha^{E,L} = \left\{\left\{\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]^2 - \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right]^2 - L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^{\ 2}\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]^2\right\} \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2\right\}^{-2} + \left\{\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right] + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)) \cdot \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right]\right\}^2 \cdot \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\right]^2\right\}^{-3}\right\}^{1/2}. \tag{24}$$

Proof. By Equation (8), we have the following:

$$\begin{aligned}\dot{\alpha}(t) &= \dot{\alpha}_1(t)\frac{\partial}{\partial x} + \dot{\alpha}_2(t)\frac{\partial}{\partial y} + \dot{\alpha}_3(t)\frac{\partial}{\partial z} \\ &= -\frac{\sqrt{2}}{2}e^{-\alpha_3}(\lambda_1E_2+\lambda_2E_3)\dot{\alpha}_1(t) \\ &\quad + \frac{\sqrt{2}}{2}e^{\alpha_3}(\lambda_1E_2-\lambda_2E_3)\dot{\alpha}_2(t) + \frac{1}{\lambda_1\lambda_2}E_1\dot{\alpha}_3(t) \\ &= \frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)E_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) \\ &\quad + e^{\alpha_3}\dot{\alpha}_2(t))E_2 + \omega(\dot{\alpha}(t))E_3.\end{aligned} \tag{25}$$

By Proposition 1 and Equation (25), we get the following:

$$\begin{aligned}\nabla_{\dot\alpha}^{E,L}E_1 &= \frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\nabla_{E_1}^{E,L}E_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t) \\ &\quad + e^{\alpha_3}\dot\alpha_2(t))\nabla_{E_2}^{E,L}E_1 + \omega(\dot\alpha(t))\nabla_{E_3}^{E,L}E_1 \\ &= \frac{-\lambda_1^2-\lambda_2^2L}{2}\omega(\dot\alpha(t))E_2 + \frac{-\lambda_1\lambda_2^2\sqrt{2}L-\lambda_1^3\sqrt{2}}{4L} \\ &\quad \cdot (-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))E_3,\end{aligned} \tag{26}$$

$$\begin{aligned}\nabla_{\dot\alpha}^{E,L}E_2 &= \frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\nabla_{E_1}^{E,L}E_2 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t) \\ &\quad + e^{\alpha_3}\dot\alpha_2(t))\nabla_{E_2}^{E,L}E_2 + \omega(\dot\alpha(t))\nabla_{E_3}^{E,L}E_2 \\ &= -\frac{\lambda_1^2+\lambda_2^2L}{2}\omega(\dot\alpha(t))E_1 + \frac{\lambda_2^2L-\lambda_1^2}{2\lambda_1\lambda_2L}\dot\alpha_3(t)E_3,\end{aligned} \tag{27}$$

$$\begin{aligned}\nabla_{\dot\alpha}^{E,L}E_3 &= \frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\nabla_{E_1}^{E,L}E_3 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t) \\ &\quad + e^{\alpha_3}\dot\alpha_2(t))\nabla_{E_2}^{E,L}E_3 + \omega(\dot\alpha(t))\nabla_{E_3}^{E,L}E_3 \\ &= -\frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{4}(-e^{-\alpha_3}\dot\alpha_1(t) \\ &\quad + e^{\alpha_3}\dot\alpha_2(t))E_1 + \frac{\lambda_1^2-\lambda_2^2L}{2\lambda_1\lambda_2}\dot\alpha_3(t)E_2.\end{aligned} \tag{28}$$

Using Equations (25) and (26), we have the following:

$$\begin{aligned}\nabla_{\dot\alpha}^{E,L}\dot\alpha &= \nabla_{\dot\alpha}^{E,L}\left[\frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)E_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))E_2 + \omega(\dot\alpha(t))E_3\right] \\ &= -\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t) - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\right]E_1 \\ &\quad + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3}) - L\omega(\dot\alpha(t))\dot\alpha_3(t)\right]E_2 \\ &\quad + \left[\frac{d}{dt}(\omega(\dot\alpha(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]E_3.\end{aligned} \tag{29}$$

By Equations (14), (25), and (29), we obtain the following:

$$\begin{aligned}\|\nabla_{\dot\alpha}^{E,L}\dot\alpha\|_L^2 &= -\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t) - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\right]^2 \\ &\quad + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3}) - L\omega(\dot\alpha(t))\dot\alpha_3(t)\right]^2 \\ &\quad + L\left[\frac{d}{dt}(\omega(\dot\alpha(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]^2,\end{aligned}$$

$$\|\dot\alpha\|_L^4 = \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\right]^2 + L(\omega(\dot\alpha(t)))^2\right\}^2,$$

$$\begin{aligned}\langle\nabla_{\dot\alpha}^{E,L}\dot\alpha,\dot\alpha\rangle_L^2 &= \left\{\frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t) - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}\right.\right. \\ &\quad \left.\cdot(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\right] \\ &\quad + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t)) \\ &\quad \cdot\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})\right. \\ &\quad \left. - L\omega(\dot\alpha(t))\dot\alpha_3(t)\right] + L\omega(\dot\alpha(t)) \\ &\quad \left.\cdot\left[\frac{d}{dt}(\omega(\dot\alpha(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]\right\}^2,\end{aligned}$$

$$\langle\dot\alpha,\dot\alpha\rangle_L^3 = \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\right]^2 + L(\omega(\dot\alpha(t)))^2\right\}^3. \tag{30}$$

□

By the definition of $\kappa_\alpha^{E,L}$, we get the desired formulae of curvature.

Definition 5. Let $\alpha : I \longrightarrow E_{\lambda_1,\lambda_2}^L$ be a C^2-smooth regular curve, we define the intrinsic curvature $\kappa_\alpha^{E,\infty}$ of α at $\alpha(t)$ to be

$$\kappa_\alpha^{E,\infty} := \lim_{L\longrightarrow\infty}\kappa_\alpha^{E,L}, \tag{31}$$

if the limit exists.

To derive the expression of intrinsic curvature, we need the following notion: for continuous functions $h_1, h_2 : (0,+\infty) \longrightarrow \mathbb{R}$,

$$h_1(L) \sim h_2(L), \text{as } L \longrightarrow +\infty \Leftrightarrow \lim_{L\longrightarrow\infty}\frac{h_1(L)}{h_2(L)} = 1. \tag{32}$$

Proposition 6. *Let $\alpha : I \longrightarrow E_{\lambda_1,\lambda_2}^L$ be a C^2-smooth regular curve in $E_{\lambda_1,\lambda_2}^L$.*

(1) If $\nabla_{\dot\alpha}{}^{E,L}\dot\alpha$ is a spacelike vector and $(-\lambda_1^2\lambda_2^4/2)(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))^2 + (\dot\alpha_3(t))^2 > 0$, then we have the following formula of $\kappa_\alpha^{E,\infty}$. If $\omega(\dot\alpha(t)) \neq 0$, then

$$\kappa_\alpha^{E,\infty} = \frac{\sqrt{(-\lambda_1^2\lambda_2^4/2)(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))^2 + (\dot\alpha_3(t))^2}}{|\omega(\dot\alpha(t))|}. \tag{33}$$

If $\omega(\dot{\alpha}(t))=0$ and $(d/dt)(\omega(\dot{\alpha}(t)))=0$, then

$$\kappa_{\alpha}^{E,\infty}=\Bigg\{\Bigg\{-\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}\right)\right]^2\Bigg\}\cdot\left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}^{-2}-\Bigg[-\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left(\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right)+\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\cdot\left(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}\right)\Bigg]^2\cdot\left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}^{-3}\Bigg\}^{1/2}. \tag{34}$$

If $\omega(\dot{\alpha}(t))=0$ and $d/dt(\omega(\dot{\alpha}(t)))\neq 0$, then

$$\lim_{L\longrightarrow\infty}\frac{\kappa_{\alpha}^{E,L}}{\sqrt{L}}=\frac{|(d/dt)(\omega(\dot{\alpha}(t)))|}{\left|-[(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)]^2+\left[\left(\lambda_1\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right|}. \tag{35}$$

(2) If $\nabla_{\dot{\alpha}}^{E,L}\dot{\alpha}$ is a timelike vector and $(\lambda_1^2\lambda_2^4/2)(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))^2-(\dot{\alpha}_3(t))^2>0$, then we have the following formula of $\kappa_{\alpha}^{E,\infty}$. If $\omega(\dot{\alpha}(t))\neq 0$, then

$$\kappa_{\alpha}^{E,\infty}=\frac{\sqrt{\left(\lambda_1^2\lambda_2^4/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)^2-(\dot{\alpha}_3(t))^2}}{|\omega(\dot{\alpha}(t))|}. \tag{36}$$

If $\omega(\dot{\alpha}(t))=0$ and $(d/dt)(\omega(\dot{\alpha}(t)))=0$, then

$$\kappa_{\alpha}^{E,\infty}=\Bigg\{-\Bigg\{-\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}\right)\right]^2\Bigg\}\cdot\left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}^{-2}+\Bigg[-\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left(\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right)+\frac{\lambda_1\sqrt{2}}{2}\cdot\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\left(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}\right)\Bigg]^2\cdot\left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}^{-3}\Bigg\}^{1/2}. \tag{37}$$

Proof.

(1) If $\nabla_{\dot{\alpha}}^{L}\dot{\alpha}$ is a spacelike vector, when $\omega(\dot{\alpha}(t))\neq 0$, we have the following:

$$\left\langle\nabla_{\dot{\alpha}}^{E,L}\dot{\alpha},\nabla_{\dot{\alpha}}^{E,L}\dot{\alpha}\right\rangle\sim\left[-\frac{\lambda_1^2\lambda_2^4}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)^2+(\dot{\alpha}_3(t))^2\right]\cdot(\omega(\dot{\alpha}(t)))^2L^2\text{ as }L\longrightarrow+\infty,$$
$$\langle\dot{\alpha},\dot{\alpha}\rangle_L\sim L(\omega(\dot{\alpha}(t)))^2,\ \left\langle\nabla_{\dot{\alpha}}^{E,L}\dot{\alpha},\dot{\alpha}\right\rangle_L^2\sim O\left(L^2\right)\text{ as }L\longrightarrow+\infty. \tag{38}$$

Therefore,

$$\frac{\left\|\nabla_{\dot{\alpha}}^{E,L}\dot{\alpha}\right\|_L^2}{\|\dot{\alpha}\|_L^4}\longrightarrow\frac{\left(-\lambda_1^2\lambda_2^4/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)^2+(\dot{\alpha}_3(t))^2}{(\omega(\dot{\alpha}(t)))^2}\cdot\text{ as }L\longrightarrow+\infty,$$
$$\frac{\left\langle\nabla_{\dot{\alpha}}^{E,L}\dot{\alpha},\dot{\alpha}\right\rangle_L^2}{\langle\dot{\alpha},\dot{\alpha}\rangle_L^3}\longrightarrow 0\text{ as }L\longrightarrow+\infty. \tag{39}$$

If $\omega(\dot{\alpha}(t))\neq 0$, by Equation (7), we have the following:

$$\kappa_{\alpha}^{E,\infty}=\frac{\sqrt{\left(-\lambda_1^2\lambda_2^4/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)^2+(\dot{\alpha}_3(t))^2}}{|\omega(\dot{\alpha}(t))|}. \tag{40}$$

By Equation (20) and $(d/dt)(\omega(\dot{\alpha}(t)))=0$, we have the following:

$$\kappa_{\alpha}^{E,\infty}=\Bigg\{\Bigg\{-\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}\right)\right]^2\Bigg\}\cdot\left\{\left[-\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}^{-2}-\Bigg[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\left(\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right)+\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\left(\ddot{\alpha}_2e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}-\ddot{\alpha}_1e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}\right)\Bigg]^2\cdot\left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}^{-3}\Bigg\}^{1/2}. \tag{41}$$

When $\omega(\dot{\alpha}(t)) = 0$ and $(d/dt)(\omega(\dot{\alpha}(t))) \neq 0$, we have the following:

$$\langle \nabla^{E,L}_{\dot{\alpha}}\alpha, \nabla^{E,L}_{\dot{\alpha}}\alpha\rangle_L \sim L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t)))\right]^2 \text{ as } L \longrightarrow +\infty,$$
$$\langle \dot{\alpha}, \dot{\alpha}\rangle_L = -\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2, \tag{42}$$

$\langle \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\rangle^2_L = O(1)$ as $L \longrightarrow +\infty$. If $\omega(\dot{\alpha}(t)) = 0$ and $(d/dt)(\omega(\dot{\alpha}(t))) \neq 0$, by (Equation (15)), we get the following:

$$\lim_{L\longrightarrow\infty} \frac{\kappa^{E,L}_{\alpha}}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{\alpha}(t)))|}{\left|-[(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)]^2 + \left[\left(\lambda_1\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right|}. \tag{43}$$

(2) By some similar computations, we get (2).

□

Example 7. Let $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$ be a C^2-smooth regular curve, where I is an open interval in $\mathbb{R}$ and $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$. We compute the following:

$$\omega(\dot{\alpha}(t)) = -\frac{\lambda_2\sqrt{2}}{2}\left(e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right). \tag{44}$$

By Equation (29), we have the following:

$$\begin{aligned}\nabla^{E,L}_{\dot{\alpha}}\dot{\alpha} = &-\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)\right.\right.\\ &\left.\left. + e^{\alpha_3}\dot{\alpha}_2(t)\right)\omega(\dot{\alpha}(t))\right]E_1 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2 e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3 e^{\alpha_3}\right.\right.\\ &\left.\left. - \ddot{\alpha}_1 e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3 e^{-\alpha_3}\right) - L\omega(\dot{\alpha}(t))\dot{\alpha}_3(t)\right]E_2\\ &+ \left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2 L}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)\right.\right.\\ &\left.\left. + e^{\alpha_3}\dot{\alpha}_2(t)\right)\dot{\alpha}_3(t)\right]E_3,\\ g^L(E_1, E_1) &= -1,\\ g^L(E_2, E_2) &= 1,\\ g^L(E_3, E_3) &= L.\end{aligned} \tag{45}$$

Suppose that $\alpha(t) = (1, t, 0)$, $\lambda_1 = \lambda_2 = 1$ be a C^2-smooth regular curve, and $\alpha_1(t) = 1, \alpha_2(t) = t, \alpha_3(t) = 0$. Then, we have the following:

$$\begin{aligned}\omega(\dot{\alpha}(t)) &= -\frac{\sqrt{2}}{2}\left(e^0 \times 0 + e^0 \times 1\right) = -\frac{\sqrt{2}}{2} \neq 0,\\ \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha} &= -\left[0 - \frac{\sqrt{2} + \sqrt{2}L}{2}\left(-e^0 \times 0 + e^0 \times 1\right) \times \left(-\frac{\sqrt{2}}{2}\right)\right]E_1\\ &+ \left[\frac{\sqrt{2}}{2}\left(0 \times e^0 + 1 \times 0 \times e^0 - 0 \times e^0 + 0 \times 0 \times e^0\right)\right.\\ &\left. - L \times \left(-\frac{\sqrt{2}}{2}\right) \times 0\right]E_2 + \left[\frac{d}{dt}\left(-\frac{\sqrt{2}}{2}\right)\right.\\ &\left. - \frac{\sqrt{2}}{2L}\left(-e^0 \times 0 + e^0 \times 1\right) \times 0\right]E_3 = -\frac{1 + L}{2}E_1.\end{aligned} \tag{46}$$

Therefore, we have the following:

$$\langle \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}\rangle = -\frac{(1 + L)^2}{4} < 0. \tag{47}$$

It implies that $\nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}$ is a timelike vector and $(\lambda_1^2\lambda_2^4/2)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))^2 - (\dot{\alpha}_3(t))^2 = 1/2 > 0$. By Proposition 6, we have the following:

$$\kappa^{E,\infty}_{\alpha} = \frac{\sqrt{1/2}}{\sqrt{2}/2} = 1. \tag{48}$$

We can use the above example to illustrate the meaning of Definition 5. We assume that $\alpha(t) = (1, t, 0)$, $\lambda_1 = 1$, $\lambda_2 = 2$. Then, we have the following:

$$\begin{aligned}\omega(\dot{\alpha}(t)) &= \sqrt{2}\left(e^0 \times 0 + e^0 \times 1\right) = -\sqrt{2} \neq 0,\\ \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha} &= -\left[-\frac{1}{2} \times 0 - \frac{\sqrt{2} + 4\sqrt{2}L}{2}\left(-e^0 \times 0 + e^0 \times 1\right)\right.\\ &\left. \times \left(-\sqrt{2}\right)\right]E_1 + \left[\frac{\sqrt{2}}{2}\left(0 \times e^0 + 1 \times 0 \times e^0 - 0\right.\right.\\ &\left.\left. \times e^0 + 0 \times 0 \times e^0\right) - L \times \left(-\sqrt{2}\right) \times 0\right]E_2\\ &+ \left[\frac{d}{dt}\left(-\sqrt{2}\right) - \frac{\sqrt{2}}{4L}\left(-e^0 \times 0 + e^0 \times 1\right) \times 0\right]E_3\\ &= -(1 + 4L)E_1.\end{aligned} \tag{49}$$

Therefore, we have the following:

$$\langle \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{E,L}_{\dot{\alpha}}\dot{\alpha}\rangle = -(1 + 4L)^2 < 0. \tag{50}$$

It implies that $\nabla^{E,L}_{\dot\alpha}\dot\alpha$ is a timelike vector and $(\lambda_1^2\lambda_2^4/2)(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))^2-(\dot\alpha_3(t))^2=8>0$. By Proposition 6, we have the following:

$$\kappa^{E,\infty}_{\alpha}=\frac{\sqrt{8}}{\sqrt{2}}=2. \tag{51}$$

3. Geodesic Curvatures for Curves on Lorentzian Surfaces in Lorentzian Approximations $E^L_{\lambda_1,\lambda_2}$

In this section, we will compute geodesic curvatures for curves on Lorentzian surfaces in Lorentzian approximations $E^L_{\lambda_1,\lambda_2}$. We consider a regular surface $S\subset E^L_{\lambda_1,\lambda_2}$ such that S is a C^2-smooth compact and oriented surface. Supposed that

$$S=\left\{(u_1,u_2,u_3)\in E^L_{\lambda_1,\lambda_2}:F(u_1,u_2,u_3)=0\right\}, \tag{52}$$

where $F:E^L_{\lambda_1,\lambda_2}\longrightarrow\mathbb{R}$ is a C^2-smooth function and $F_{u_1}\partial_{u_1}+F_{u_2}\partial_{u_2}+F_{u_3}\partial_{u_3}\neq 0$. A point $u\in S$ is called characteristic if $\nabla_H F(u)=0$, where $\nabla_H F=-E_1(F)E_1+E_2(F)E_2$. We call the set $C(S):=\{u_1,u_2,u_3)\in S|\nabla_H F(u_1,u_2,u_3)=0\}$ characteristic set of S. We will give some symbols away from characteristic points of S. We define $p_1:=E_1F$, $q_1:=E_2F$, and $r:=\widetilde{E}_3F$. If $-p_1^2+q_1^2>0$, we say that $S\subset E^L_{\lambda_1,\lambda_2}$ is a horizontal spacelike surface. Under this assumption, $-p_1^2+q_1^2+r^2>0$ for $L\longrightarrow+\infty$. Therefore, we can define the following functions:

$$l:=\sqrt{-p_1^2+q_1^2}, \tag{53}$$

$$l_L:=\sqrt{-p_1^2+q_1^2+r_1^2}, \tag{54}$$

$$\bar p_1:=\frac{p_1}{l}, \tag{55}$$

$$\bar q_1:=\frac{q_1}{l}, \tag{56}$$

$$\bar p_{1,L}:=\frac{p_1}{l_L}, \tag{57}$$

$$\bar q_{1,L}:=\frac{q_1}{l_L}, \tag{58}$$

$$\bar r_{1,L}:=\frac{r_1}{l_L}, \tag{59}$$

which will be used to define a frame of S. We construct the following:

$$\begin{aligned}N_L&=-\bar p_{1,L}E_1+\bar q_{1,L}E_2+\bar r_{1,L}\tilde E_3,\\ T_1&=\bar q_1T_1-\bar p_1E_2,\\ T_2&=\bar r_{1,L}\bar p_1T_1-\bar r_{1,L}\bar q_1E_2+\frac{l}{l_L}\tilde E_3,\end{aligned} \tag{60}$$

where N_L is the unit spacelike normal vector to S, T_1 is the unit timelike vector, and T_2 is the unit spacelike vector. One can check that $\{T_1,T_2\}$ are a pseudo-orthonomal basis for tangent space of S. We call S a Lorentzian surface in $E^L_{\lambda_1,\lambda_2}$. For the case that $\alpha:I\longrightarrow E^L_{\lambda_1,\lambda_2}$ is a C^2-smooth timelike curve and $\dot\alpha=m\,T_1+nT_2$, we define the following:

$$J_L(\dot\alpha):=mT_2+nT_1. \tag{61}$$

For the case that $\alpha:I\longrightarrow E^L_{\lambda_1,\lambda_2}$ is a C^2-smooth spacelike curve, we define the following:

$$J_L(\dot\alpha):=-mT_2-nT_1. \tag{62}$$

That makes $\langle\dot\alpha,J_L(\dot\alpha)\rangle=0$ and $(\dot\alpha,J_L(\dot\alpha))$ have the same orientation with $\{T_1,T_2\}$. For every $X,Y\in TS$, we define the Levi-Civita connection with respect to the metric g_L on S by $\nabla_X^{S,E,L}Y=\pi\nabla_X^{E,L}Y$, where $\pi:T(E^L_{\lambda_1,\lambda_2})\longrightarrow TS$ is the projection. Furthermore,

$$\nabla^{S,E,L}_{\dot\alpha}\dot\alpha=-\left\langle\nabla^{E,L}_{\dot\alpha}\dot\alpha,T_1\right\rangle_L T_1+\left\langle\nabla^{E,L}_{\dot\alpha}\dot\alpha,T_2\right\rangle_L T_2. \tag{63}$$

In particular, we get the following:

$$\begin{aligned}\nabla^{S,E,L}_{\dot\alpha}\dot\alpha=&-\Bigg\{-\bar q_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)-\frac{\lambda_1^3\sqrt2+\lambda_1\lambda_2^2\sqrt2L}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\right]\\&-\bar p_1\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})-L\dot\alpha_3(t)\omega(\dot\alpha(t))\right]\Bigg\}T_1\\&+\Bigg\{-\bar r_{1,L}\bar p_1\Bigg[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)-\frac{\lambda_1^3\sqrt2+\lambda_1\lambda_2^2\sqrt2L}{2}\\&\cdot(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\Bigg]\\&-\bar r_{1,L}\bar q_1\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})-L\dot\alpha_3(t)\omega(\dot\alpha(t))\right]\\&+\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot\alpha(t)))-\frac{\lambda_1^2\sqrt2}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]\Bigg\}T_2.\end{aligned} \tag{64}$$

Moreover, if $\omega(\dot\alpha(t))=0$, then

$$\begin{aligned}\nabla^{S,E,L}_{\dot\alpha}\dot\alpha=&-\Bigg\{-\bar q_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)\right]-\bar p_1\Bigg[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}\\&-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})\Bigg]\Bigg\}T_1+\Bigg\{-\bar r_{1,L}\bar p_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)\right]\\&-\bar r_{1,L}\bar q_1\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})\right]\\&+\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot\alpha(t)))-\frac{\lambda_1^2\sqrt2}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]\Bigg\}T_2.\end{aligned} \tag{65}$$

Definition 8. Let $\alpha:I\longrightarrow S$ be a C^2-smooth regular curve, where $S\subset E^L_{\lambda_1,\lambda_2}$ is a regular Lorentzian surface. We define the geodesic curvature as follows.

(1) If $\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}$ is spacelike vector, the geodesic curvature $\kappa^{E,L}_{\alpha,S}$ of α at $\alpha(t)$ is defined by the following:

$$\kappa^{E,L}_{\alpha,S} := \sqrt{\frac{\left\|\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\|^2_{S,L}}{\|\dot{\alpha}\|^4_{S,L}} - \frac{\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha},\dot{\alpha}\right\rangle^2_{S,L}}{\langle\dot{\alpha},\dot{\alpha}\rangle^3_{S,L}}}. \tag{66}$$

(2) If $\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}$ is timelike vector, the geodesic curvature $\kappa^{E,L}_{\alpha,S}$ of α at $\alpha(t)$ is defined by the following:

$$\kappa^{E,L}_{\alpha,S} := \sqrt{\frac{\left\|\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\|^2_{S,L}}{\|\dot{\alpha}\|^4_{S,L}} + \frac{\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha},\dot{\alpha}\right\rangle^2_{S,L}}{\langle\dot{\alpha},\dot{\alpha}\rangle^3_{S,L}}}. \tag{67}$$

Definition 9. Let $\alpha : I \longrightarrow S$ be a C^2-smooth regular curve, where $S \subset E^L_{\lambda_1,\lambda_2}$ is a regular Lorentzian surface. We define the intrinsic geodesic curvature $\kappa^{E,\infty}_{\alpha,S}$ of α at $\alpha(t)$ to be

$$\kappa^{E,\infty}_{\alpha,S} := \lim_{L\longrightarrow+\infty}\kappa^{E,L}_{\alpha,S}, \tag{68}$$

if the limit exists.

Proposition 10. *Let $\alpha : I \longrightarrow S$ be a C^2-smooth regular curve, where $S \subset E^L_{\lambda_1,\lambda_2}$ is a regular Lorentzian surface.*

(1) If $\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}$ is a timelike vector, then

$$\kappa^{E,\infty}_{\alpha,S} = \frac{\sqrt{\left(\lambda_1^2\lambda_2^4/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)^2+\bar{p}_1{}^2\left(\dot{\alpha}_3(t)\right)^2}}{|\omega(\dot{\alpha}(t))|},\ \text{if } \omega(\dot{\alpha}(t)) \neq 0, \tag{69}$$

$$\kappa^{E,\infty}_{\alpha,S} = 0,\ \text{if } \omega(\dot{\alpha}(t)) = 0, \tag{70}$$

$$\frac{d}{dt}(\omega(\dot{\alpha}(t))) = 0. \tag{71}$$

(2) If $\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}$ is a spacelike vector, then

$$\lim_{L\longrightarrow+\infty}\frac{\kappa^{E,L}_{\alpha,S}}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{\alpha}(t)))|}{\left[(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1+\left(\lambda_1\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right]^2},\ \text{if } \omega(\dot{\alpha}(t)) = 0, \quad \frac{d}{dt}(\omega(\dot{\alpha}(t))) \neq 0. \tag{72}$$

Proof. On the one hand, by Equation (25), we get the following:

$$\dot{\alpha}(t) = \frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)E_1 + \frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)E_2 + \omega(\dot{\alpha}(t))E_3. \tag{73}$$

On the other hand, since $\dot{\alpha} \in TS$, we denote the following:

$$\begin{aligned}\dot{\alpha}(t) &= mT_1 + nT_2 = m(\bar{q}_1E_1 - \bar{p}_1E_2) + n\left(\bar{r}_{1,L}\bar{p}_1E_1 - \bar{r}_{1,L}\bar{q}_1E_2 + \frac{l}{l_L}\widetilde{E}_3\right)\\ &= (m\bar{q}_1 + n\bar{r}_{1,L}\bar{p}_1)E_1 - (m\bar{p}_1 + n\bar{r}_{1,L}\bar{q}_1)E_2 + \frac{nl}{l_L}L^{-1/2}E_3.\end{aligned} \tag{74}$$

The above equations yield the following:

$$\begin{cases} m\bar{q}_1 + n\bar{r}_{1,L}\bar{p}_1 = \dfrac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t),\\ -m\bar{p}_1 - n\bar{r}_{1,L}\bar{q}_1 = \dfrac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right),\\ \dfrac{nl}{l_L}L^{-1/2} = \omega(\dot{\alpha}(t)).\end{cases} \tag{75}$$

By solving Equation (75), we get the following:

$$\begin{cases} m = \dfrac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \dfrac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1,\\ n = \dfrac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t)).\end{cases} \tag{76}$$

Therefore, $\dot{\alpha}$ takes the following form:

$$\dot{\alpha} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]T_1 + \frac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t))T_2. \tag{77}$$

We denote the following:

$$\begin{aligned}C = &-\left\{-\bar{q}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}(t))\right]\right.\\
&\left.-\bar{p}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\dot{\alpha}_3(t)\omega(\dot{\alpha}(t))\right]\right\}^2\\
&+\left\{-\bar{r}_{1,L}\bar{p}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}(t))\right]\right.\\
&-\bar{r}_{1,L}\bar{q}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\dot{\alpha}_3(t)\omega(\dot{\alpha}(t))\right]\\
&\left.+\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\}^2.\end{aligned} \tag{78}$$

Then, by Equation (64), we get $\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\rangle_{S,L} = C$, where

$$C \sim -\left[\frac{\lambda_1^2\lambda_2^4}{2}\bar{q}_1{}^2(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))^2 + (\bar{p}_1\dot{\alpha}_3(t))^2\right]\cdot(\omega(\dot{\alpha}(t)))^2L^2 \text{ as } L \longrightarrow +\infty. \tag{79}$$

In case $\omega(\dot{\alpha}(t)) \neq 0$, we obtain the following:

$$\langle\dot{\alpha}, \dot{\alpha}\rangle_{S,L} = \left\{-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]^2 + L\left(\frac{l}{l_L}\omega(\dot{\alpha}(t))\right)^2\right\}^2 \sim L^2(\omega(\dot{\alpha}(t)))^2 \text{ as } L \longrightarrow +\infty. \tag{80}$$

By Equations (64) and (77), we obtain the following:

$$\begin{aligned}\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_{S,L} = D \sim &\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]\\
&\cdot\left[-\bar{q}_1\frac{\lambda_1\lambda_2^2\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) - \bar{p}_1\dot{\alpha}_3(t)\right]\omega(\dot{\alpha}(t))L\\
&+\frac{d}{dt}(\omega(\dot{\alpha}(t)))\omega(\dot{\alpha}(t))L \sim N_0L \text{ as } L \longrightarrow +\infty,\end{aligned} \tag{81}$$

where

$$\begin{aligned}D = &\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]\\
&\cdot\left\{\bar{q}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3 - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\alpha_3}\dot{\alpha}_1 + e^{\alpha_3}\dot{\alpha}_2)\omega(\dot{\alpha})\right]\right.\\
&\left.+\bar{p}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\dot{\alpha}_3(t)\omega(\dot{\alpha}(t))\right]\right\}\\
&+\frac{l}{l_L}L^{1/2}\omega(\dot{\alpha})\cdot\left\{-\bar{r}_{1,L}\bar{p}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3 - \frac{\lambda_1^3\sqrt{2}+\lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\alpha_3}\dot{\alpha}_1 + e^{\alpha_3}\dot{\alpha}_2)\omega(\dot{\alpha})\right]\right.\\
&-\bar{r}_{1,L}\bar{q}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3}) - L\dot{\alpha}_3(t)\omega(\dot{\alpha}(t))\right]\\
&\left.+\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\},\end{aligned} \tag{82}$$

and N_0 does not depend on L. By Equation (64), we have the following:

$$\begin{aligned}\kappa^{E,\infty}_{\alpha,S} &= \lim_{L\longrightarrow+\infty}\kappa^{E,L}_{\alpha,S}\\
&= \frac{\sqrt{(\lambda_1^2\lambda_2^4/2)\bar{q}_1{}^2(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))^2 + \bar{p}_1{}^2(\dot{\alpha}_3(t))^2}}{|\omega(\dot{\alpha}(t))|} \text{ if } \omega(\dot{\alpha}(t)) \neq 0.\end{aligned} \tag{83}$$

If $\omega(\dot{\alpha}(t)) = 0$ and $(d/dt)(\omega(\dot{\alpha}(t))) = 0$, we have the following:

$$\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\rangle_{S,L} \sim -\left\{\bar{q}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right] + \bar{p}_1E\right\}^2 \text{ as } L \longrightarrow +\infty, \tag{84}$$

where $E = [(\lambda_1\sqrt{2}/2)(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})]$, and

$$\langle\dot{\alpha}, \dot{\alpha}\rangle_{S,L} = -\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]^2, \tag{85}$$

$$\begin{aligned}\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_{S,L} = &-\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]\\
&\cdot\left[\bar{q}_1\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) + \bar{p}_1\frac{\lambda_1\sqrt{2}}{2}\right.\\
&\left.\cdot(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right] := -A_1A_2.\end{aligned} \tag{86}$$

By Equations (79)–(86) and Equation (65), we get the following:

$$\kappa^{E,\infty}_{\alpha,S} = \sqrt{\frac{A_2^2}{A_1^4} - \frac{A_1^2A_2^2}{A_1^6}} = 0. \tag{87}$$

When $\omega(\dot{\alpha}(t))=0$, and $(d/dt)(\omega(\dot{\alpha}(t)))\neq 0$, we have the following:

$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\rangle_{S,L} \sim L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t)))\right]^2 \text{ as } L\longrightarrow +\infty,$$
$$\langle\dot{\alpha},\dot{\alpha}\rangle_{S,L} = -\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]^2,$$
$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_{S,L} = O(1). \tag{88}$$

It follows that

$$\lim_{L\longrightarrow+\infty}\frac{\kappa^{E,L}_{\alpha,S}}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{\alpha}(t)))|}{\left[(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1 + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]^2}, \tag{89}$$

if $\omega(\dot{\alpha}(t))=0$, and $(d/dt)(\omega(\dot{\alpha}(t)))\neq 0$. This completes the proof. □

Example 11. We assume that there exists a C^2-smooth function $F=y: E^L_{\lambda_1,\lambda_2}\longrightarrow \mathbb{R}$ such that

$$S=\left\{(u_1,u_2,u_3)\in E^L_{\lambda_1,\lambda_2} : y=0\right\}. \tag{90}$$

Then, $F_{u_1}\partial_{u_1}+F_{u_2}\partial_{u_2}+F_{u_3}\partial_{u_3}=\partial_y\neq 0$. Let

$$E_1=\lambda_1\lambda_2\frac{\partial}{\partial u_3},$$
$$E_2=\frac{1}{\lambda_1\sqrt{2}}\left(-e^{u_3}\frac{\partial}{\partial u_1}+e^{-u_3}\frac{\partial}{\partial u_2}\right), \tag{91}$$
$$E_3=-\frac{1}{\lambda_2\sqrt{2}}\left(e^{u_3}\frac{\partial}{\partial u_1}+e^{-u_3}\frac{\partial}{\partial u_2}\right).$$

So, we have the following:

$$p_1 := E_1F=\lambda_1\lambda_2\frac{\partial}{\partial u_3}(y)=0,$$
$$q_1 := E_2F=\frac{1}{\lambda_1\sqrt{2}}\left(-e^{u_3}\frac{\partial}{\partial u_1}+e^{-u_3}\frac{\partial}{\partial u_2}\right)(y) = \frac{1}{\lambda_1\sqrt{2}}e^{-u_3},$$
$$r_1 := \tilde{E}_3F=-L^{(-1/2)}\frac{1}{\lambda_2\sqrt{2}}\left(e^{u_3}\frac{\partial}{\partial u_1}+e^{-u_3}\frac{\partial}{\partial u_2}\right)(y) = -L^{(-1/2)}\frac{1}{\lambda_2\sqrt{2}}e^{-u_3}. \tag{92}$$

Therefore, $-p_1^2+q_1^2=((1/\lambda_1\sqrt{2})e^{-u_3})^2>0$, so $S\subset E^L_{\lambda_1,\lambda_2}$ is a horizontal spacelike surface. By Equation (53), we have the following:

$$l:=\sqrt{-p_1^2+q_1^2}=\frac{1}{\lambda_1\sqrt{2}}e^{-u_3},$$
$$l_L:=\sqrt{-p_1^2+q_1^2+r_1^2}=\frac{e^{-u_3}}{\sqrt{2}}\sqrt{\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2L}},$$
$$\bar{p}_1:=\frac{p_1}{l}=0,$$
$$\bar{q}_1:=\frac{q_1}{l}=1, \tag{93}$$
$$\bar{p}_{1,L}:=\frac{p_1}{l_L}=0,$$
$$\bar{q}_{1,L}:=\frac{q}{l_L}=\frac{1/\lambda_1}{\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)}},$$
$$\bar{r}_{1,L}:=\frac{r_1}{l_L}=\frac{-L^{-1/2}(1/\lambda_2)}{\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)}}.$$

By Equation (60), we have the following:

$$N_L=\frac{1/\lambda_1}{\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)}}E_2-\frac{-L^{-1/2}1/\lambda_2}{\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)}}\tilde{E}_3,$$
$$T_1=E_1,$$
$$T_2=\frac{L^{-1/2}1/\lambda_2}{\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)}}E_2+\frac{1/\lambda_1}{\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)}}\tilde{E}_3. \tag{94}$$

Then, $\{T_1,T_2\}=\{E_1(L^{-1/2}1/\lambda_2/\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)})E_2 + (1/\lambda_1/\sqrt{(1/\lambda_1^2)+(1/\lambda_2^2L)})\tilde{E}_3\}$. Thus, it is concluded that S is a Lorentzian surface in $E^L_{\lambda_1,\lambda_2}$.

Let

$$\alpha:[0,2\pi]\longrightarrow S, \theta\longrightarrow(\cos\theta,0,\sin\theta), \lambda_1=\lambda_2=1 \tag{95}$$

be the circle centered at the origin on $y=0$. By

$$\omega(\dot{\alpha}(\theta))=-\frac{\lambda_2\sqrt{2}}{2}(e^{-\alpha_3}\dot{\alpha}_1(\theta)+e^{\alpha_3}\dot{\alpha}_2(\theta)) \tag{96}$$

and $\alpha_1(\theta)=\cos\theta, \alpha_2(\theta)=0, \alpha_3(\theta)=\sin\theta$, we have the following:

$$\omega(\dot{\alpha}(\theta))=\frac{\sqrt{2}}{2}e^{-\sin\theta}\sin\theta. \tag{97}$$

By Equation (64), we have the following:

$$\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha} = -\left(\sin\theta - \frac{1+L}{2}e^{-2\sin\theta}\sin^2\theta\right)T_1 + \left[\frac{\sqrt{2}}{\sqrt{L+1}}(L-1)\left(\frac{1}{2}\cos\theta e^{-\sin\theta}\right)\right]T_2. \tag{98}$$

Then,

$$\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\rangle = -\left(\sin\theta - \frac{1+L}{2}e^{-2\sin\theta}\sin^2\theta\right)^2 + \left[\frac{\sqrt{2}}{\sqrt{L+1}}(L-1)\left(\frac{1}{2}\cos\theta e^{-\sin\theta}\right)\right]^2. \tag{99}$$

If $\cos\theta = 0$, then $\sin\theta = \pm 1$. In this case, $\omega(\dot{\alpha}(\theta)) \neq 0$. Then, we have $\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}$ is a timelike vector. By Proposition 10 and Equation (69), we have at the point θ which satisfies $\cos\theta = 0$:

$$\kappa^{E,\infty}_{\alpha,S} = \frac{\sqrt{\left(\lambda_1^2\lambda_2^4/2\right)\bar{q}_1{}^2\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)^2 + \bar{p}_1{}^2\left(\dot{\alpha}_3(t)\right)^2}}{|\omega(\dot{\alpha}(t))|} = \frac{\sqrt{2}}{2}. \tag{100}$$

When we assume that $\alpha : [0, 2\pi] \longrightarrow S, \theta \longrightarrow (\cos\theta, 0, \sin\theta), \lambda_1 = 1, \lambda_2 = 2$ be a circle centered at the origin on $y = 0$. We get the following:

$$\omega(\dot{\alpha}(\theta)) = -\sqrt{2}e^{-\sin\theta}\sin\theta. \tag{101}$$

By Equation (64), we have the following:

$$\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha} = \left(-\frac{1}{2}\sin\theta + (1+4L)e^{-2\sin\theta}\sin^2\theta\right)T_1 + \left(\frac{\sqrt{2}}{2\sqrt{4L+1}}\cos\theta e^{-\sin\theta}(1+4L) - \cos\theta\sin\theta e^{-\sin\theta}(2+6L)\right)T_2. \tag{102}$$

So,

$$\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\rangle = -\left(-\frac{1}{2}\sin\theta + (1+4L)e^{-2\sin\theta}\sin^2\theta\right)^2 + \left(\frac{\sqrt{2}}{2\sqrt{4L+1}}\cos\theta e^{-\sin\theta}(1+4L) - \cos\theta\sin\theta e^{-\sin\theta}(2+6L)\right)^2. \tag{103}$$

If $\cos\theta = 0$, then $\sin\theta = \pm 1$. In this case, $\omega(\dot{\alpha}(\theta)) \neq 0$. Then, we have $\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}$ is a timelike vector. By Proposition 10 and Equation (69), we have at the point θ which satisfies $\cos\theta = 0$.

$$\kappa^{E,\infty}_{\alpha,S} = \frac{\sqrt{\left(\lambda_1^2\lambda_2^4/2\right)\bar{q}_1{}^2\left(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)^2 + \bar{p}_1{}^2\left(\dot{\alpha}_3(t)\right)^2}}{|\omega(\dot{\alpha}(t))|} = 2. \tag{104}$$

Definition 12. Let $\alpha : I \longrightarrow S$ be a C^2-smooth regular curve, where $S \subset E^L_{\lambda_1,\lambda_2}$ is a regular Lorentzian surface. The signed geodesic curvature $\kappa^{E,L,s}_{\alpha,S}$ of α at $\alpha(t)$ is defined as follows:

$$\kappa^{E,L,s}_{\alpha,S} := \frac{\left\langle\nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{S,L}}{\|\dot{\alpha}\|^3_{S,L}}, \tag{105}$$

where J_L is defined by Equations (61) and (62).

Definition 13. Let $\alpha : I \longrightarrow S$ be a C^2-smooth regular curve, where $S \subset E^L_{\lambda_1,\lambda_2}$ is a regular Lorentzian surface. We define the intrinsic geodesic curvature $\kappa^{E,\infty,s}_{\alpha,S}$ of α at the noncharacteristic point $\alpha(t)$ to be

$$\kappa^{E,\infty,s}_{\alpha,S} := \lim_{L\longrightarrow+\infty}\kappa^{E,L,s}_{\alpha,S}, \tag{106}$$

if the limit exists.

Proposition 14. *Let $\alpha : I \longrightarrow S$ be a C^2-smooth regular curve, where $S \subset E^L_{\lambda_1,\lambda_2}$ is a regular Lorentzian surface.*

(1) If $\alpha : I \longrightarrow S$ be a spacelike C^2-smooth curve, then $\omega(\dot{\alpha}(t)) \neq 0$ and

$$\kappa_{\alpha,S}^{E,\infty,s} = -\frac{\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) + \bar{p}_1\dot{\alpha}_3(t)}{|\omega(\dot{\alpha}(t))|}. \tag{107}$$

(2) If $\alpha : I \longrightarrow S$ be a timelike C^2-smooth curve, then $\omega(\dot{\alpha}(t)) = 0$ and

$$\kappa_{\alpha,S}^{E,\infty,s} = 0, \text{ if } \frac{d}{dt}(\omega(\dot{\alpha}(t))) = 0,$$
$$\lim_{L\longrightarrow+\infty}\frac{\kappa_{\alpha,S}^{E,L,s}}{\sqrt{L}} = \frac{(d/dt)(\omega(\dot{\alpha}(t)))}{\left|(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1 + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right|^2}, \tag{108}$$

if $(d/dt)(\omega(\dot{\alpha}(t))) \neq 0$.

Proof. By Equations (61), (62), and (77), we get the following:

$$\begin{aligned} J_L(\dot{\alpha}) &= \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]J_L(T_1) \\ &\quad - \frac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t))J_L(T_2) = \frac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t))T_1 \\ &\quad + \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]T_2. \end{aligned} \tag{109}$$

By Equation (64) and the above equation, we have the following:

$$\begin{aligned} \left\langle \nabla_{\dot{\alpha}}^{S,E,L}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle &= \frac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t)) \\ &\quad\cdot\left\{-\bar{q}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}\right.\right. \\ &\quad\left.\cdot(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}(t))\right] \\ &\quad - \bar{p}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right. \\ &\quad\left.\left. - L\dot{\alpha}_3(t)\omega(\dot{\alpha}(t))\right]\right\} \\ &\quad + \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right] \\ &\quad\cdot\left\{-\bar{r}_{1,L}\bar{p}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t) - \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}\right.\right. \\ &\quad\left.\cdot(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\omega(\dot{\alpha}(t))\right] \\ &\quad - \bar{r}_{1,L}\bar{q}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right. \\ &\quad\left. - L\dot{\alpha}_3(t)\omega(\dot{\alpha}(t))\right] + \frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}\right. \\ &\quad\left.\left.\cdot(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\} \\ &\sim \left[\bar{q}_1\frac{\lambda_1\lambda_2^2\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) + \bar{p}_1\dot{\alpha}_3(t)\right] \\ &\quad\cdot(\omega(\dot{\alpha}(t)))^2L^{3/2} \text{ as } L \longrightarrow +\infty. \end{aligned} \tag{110}$$

So, we get the following:

$$\begin{aligned} \kappa_{\alpha,S}^{E,L,s} &= \frac{\left\langle \nabla_{\dot{\alpha}}^{S,E,L}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{S,L}}{\|\dot{\alpha}\|_{S,L}^3} \\ &= \frac{L^{3/2}\left[\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) + \bar{p}_1\dot{\alpha}_3(t)\right](\omega(\dot{\alpha}(t)))^2}{L^{3/2}|\omega(\dot{\alpha}(t))|^3}. \end{aligned} \tag{111}$$

Furthermore,

$$\kappa_{\alpha,S}^{E,\infty,s} = \lim_{L\longrightarrow+\infty}\kappa_{\alpha,S}^{E,L,s} = -\frac{\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) + \bar{p}_1\dot{\alpha}_3(t)}{|\omega(\dot{\alpha}(t))|}. \tag{112}$$

When $\omega(\dot{\alpha}(t)) = 0$, and $(d/dt)(\omega(\dot{\alpha}(t))) = 0$, we get the following:

$$\begin{aligned} \left\langle \nabla_{\dot{\alpha}}^{S,E,L}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{L,S} &= \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right] \\ &\quad\cdot\left\{-\bar{r}_{1,L}\bar{p}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]\right. \\ &\quad - \bar{r}_{1,L}\bar{q}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3}\right. \\ &\quad\left. - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right] \\ &\quad\left. + \frac{l}{l_L}L^{1/2}\left[-\frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\} \\ &\sim N_1L^{-1/2} \text{ as } L \longrightarrow +\infty, \end{aligned} \tag{113}$$

where N_1 does not depend on L. So, $\kappa_{\alpha,S}^{E,\infty,s} = 0$. When $\omega(\dot{\alpha}(t)) = 0$, and $(d/dt)(\omega(\dot{\alpha}(t))) \neq 0$, we have the following:

$$\begin{aligned} \left\langle \nabla_{\dot{\alpha}}^{S,E,L}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{L,S} &= \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right] \\ &\quad\cdot\left\{-\bar{r}_{1,L}\bar{p}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right]\right. \\ &\quad - \bar{r}_{1,L}\bar{q}_1\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\alpha}_2e^{\alpha_3} + \dot{\alpha}_2\dot{\alpha}_3e^{\alpha_3} - \ddot{\alpha}_1e^{-\alpha_3} + \dot{\alpha}_1\dot{\alpha}_3e^{-\alpha_3})\right] \\ &\quad\left. + \frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot{\alpha}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\dot{\alpha}_3(t)\right]\right\} \\ &\sim L^{1/2}\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]\frac{d}{dt} \\ &\quad\cdot(\omega(\dot{\alpha}(t))) \text{ as } L \longrightarrow +\infty. \end{aligned} \tag{114}$$

We get the following:

$$\kappa_{\alpha,S}^{E,\infty,s} = \lim_{L\longrightarrow+\infty} \frac{\kappa_{\alpha,S}^{E,L}}{\sqrt{L}} = \lim_{L\longrightarrow+\infty} \frac{\left[(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1 + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right](d/dt)(\omega(\dot{\alpha}(t)))L^{1/2}}{\left|(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1 + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right|^3\sqrt{L}}$$
$$= \frac{(d/dt)(\omega(\dot{\alpha}(t)))}{\left|(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1 + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right|^2}. \quad (115)$$

□

Example 15. We take $F = y : E_{\lambda_1,\lambda_2}^L \longrightarrow \mathbb{R}$ and $\alpha(\theta) = (\cos\theta, 0, \sin\theta)$, $\lambda_1 = \lambda_2 = 1$ as in Example 11. Then,

$$\omega(\dot{\alpha}(\theta)) = \frac{\sqrt{2}}{2} e^{-\sin\theta} \sin\theta. \quad (116)$$

By Equation (77), we have the following:

$$\dot{\alpha} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t))\bar{p}_1\right]T_1 + \frac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t))T_2 = \cos\theta T_1 + \sqrt{L+1}\frac{\sqrt{2}}{2}e^{-\sin\theta}\sin\theta T_2. \quad (117)$$

Then,

$$\langle\dot{\alpha}, \dot{\alpha}\rangle = -\cos^2\theta + \frac{L+1}{2}e^{-2\sin\theta}\sin^2\theta. \quad (118)$$

So, we have when $\sin\theta \neq 0$, then $|\dot{\alpha}|^2 > 0$ for the large L and we have $\dot{\alpha}$ is a spacelike vector. If $\sin\theta \neq 0$, by Proposition 14 (1), we have the following:

$$\kappa_{\alpha,S}^{E,\infty,s} = -\frac{\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) + \bar{p}_1\dot{\alpha}_3(t)}{|\omega(\dot{\alpha}(t))|} = -1. \quad (119)$$

When we take $\lambda_1 = 1$, $\lambda_2 = 2$, we have the following:

$$\omega(\dot{\alpha}(\theta)) = -\sqrt{2}e^{-\sin\theta}\sin\theta. \quad (120)$$

By Equation (77), we have the following:

$$\dot{\alpha} = \frac{1}{2}\cos\theta T_1 - \frac{\sqrt{2}\sqrt{4L+1}}{2}e^{-\sin\theta}\sin\theta T_2. \quad (121)$$

Then,

$$\langle\dot{\alpha}, \dot{\alpha}\rangle = -\frac{1}{4}\cos^2\theta + \frac{4L+1}{2}e^{-2\sin\theta}\sin^2\theta. \quad (122)$$

So, we have when $\sin\theta \neq 0$, then $|\dot{\alpha}|^2 > 0$ for the large L and we have $\dot{\alpha}$ is a spacelike vector. If $\sin\theta \neq 0$, by Proposition 14 (1), we have the following:

$$\kappa_{\alpha,S}^{E,\infty,s} = -\frac{\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)) + \bar{p}_1\dot{\alpha}_3(t)}{|\omega(\dot{\alpha}(t))|} = -2. \quad (123)$$

4. Curvatures for Lorentzian Surfaces in Lorentzian Approximations $E_{\lambda_1,\lambda_2}^L$

In this section, we will compute the intrinsic Gaussian curvature of Lorentzian surfaces in $E_{\lambda_1,\lambda_2}^L$. We define the second fundamental form $II^{E,L}$ of the embedding of S into $E_{\lambda_1,\lambda_2}^L$ by the following:

$$II^{E,L} = \begin{pmatrix} \left\langle\nabla_{T_1}^{E,L}v_L, T_1\right\rangle_L & \left\langle\nabla_{T_1}^{E,L}v_L, T_2\right\rangle_L \\ \left\langle\nabla_{T_2}^{E,L}v_L, T_1\right\rangle_L & \left\langle\nabla_{T_2}^{E,L}v_L, T_2\right\rangle_L \end{pmatrix}. \quad (124)$$

We have the following theorem.

Theorem 16. *The second fundamental form $II^{E,L}$ of the embedding of S into $E_{\lambda_1,\lambda_2}^L$ is given by the following:*

$$II^{E,L} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \quad (125)$$

where

$$h_{11} = \frac{l}{l_L}(E_1(\bar{p}_1) - E_2(\bar{q}_1)) - \lambda_1^2 \bar{r}_L \bar{p}_1 \bar{q}_1 L^{-1/2},$$
$$h_{21} = h_{12} = \frac{l_L}{l}\langle T_1, \nabla_H(\bar{r}_{1,L})\rangle_L + \frac{\sqrt{L}}{2}\lambda_2^2 - \frac{\lambda_1^2}{2\sqrt{L}}(\bar{q}_{1,L}^2 + \bar{p}_{1,L}^2) - \frac{\bar{r}_L^2\lambda_1^2}{2\sqrt{L}}({\bar{q}_1}^2 + {\bar{p}_1}^2),$$
$$h_{22} = \frac{l^2}{l_L^2}\left\langle T_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L + \widetilde{E}_3(\bar{r}_{1,L}) - \frac{\bar{r}_{1,L}\bar{p}_{1,L}\bar{q}_{1,L}}{\sqrt{L}}\lambda_1^2 - \frac{\bar{r}_L^3\bar{p}_1\bar{q}_1}{\sqrt{L}}\lambda_1^2. \tag{126}$$

Proof. Since $\langle T_1, N_L\rangle_L = 0, \langle T_2, N_L\rangle_L = 0$, we have the following:

$$\left\langle \nabla_{T_1}^{E,L} N_L, T_1\right\rangle_L = -\left\langle \nabla_{T_1}^{E,L} T_1, N_L\right\rangle_L, \left\langle \nabla_{T_2}^{E,L} N_L, T_2\right\rangle_L = -\left\langle \nabla_{T_2}^{E,L} T_2, N_L\right\rangle_L. \tag{127}$$

Using the definition of the connection, identities in Equation (11), and grouping terms, we have the following:

$$\begin{aligned}\nabla_{T_1}^{E,L} T_1 &= \nabla_{\bar{q}_1E_1 - \bar{p}_1E_2}^{E,L} \bar{q}_1 E_1 - \bar{p}_1 E_2 \\ &= \bar{q}_1\left(E_1\bar{q}_1E_1 - E_1\bar{p}_1E_2 - \bar{p}_1\nabla_{E_1}^{E,L}E_2\right) \\ &\quad - \bar{p}_1\left(E_2\bar{q}_1E_1 - E_2\bar{p}_1E_2 + \bar{q}_1\nabla_{E_2}^{E,L}E_1\right) \\ &= (\bar{q}_1E_1\bar{q}_1 - \bar{p}_1E_2\bar{q}_1)E_1 - (\bar{q}_1E_1\bar{p}_1 - \bar{p}_1E_2\bar{p}_1)E_2 + \frac{\bar{p}_1\bar{q}_1\lambda_1^2}{L}E_3.\end{aligned} \tag{128}$$

Since ${\bar{p}_1}^2 - {\bar{q}_1}^2 = -1$, we have $\bar{p}_1E_i\bar{p}_1 - \bar{q}_1E_i\bar{q}_1 = 0$, $i = 1, 2, 3$. Furthermore, we obtain the following:

$$\begin{aligned}h_{11} &= -\left\langle \nabla_{T_1}^{E,L} T_1, N_L\right\rangle_L \\ &= -\left[\bar{p}_{1,L}(\bar{q}_1E_1\bar{q}_1 - \bar{p}_1E_2\bar{q}_1) - \bar{q}_{1,L}(\bar{q}_1E_1\bar{p}_1 - \bar{p}_1E_2\bar{p}_1) + \bar{r}_{1,L}\frac{\bar{p}_1\bar{q}_1\lambda_1^2}{\sqrt{L}}\right] \\ &= \frac{l}{l_L}(E_1(\bar{p}_1) - E_2(\bar{q}_1)) - \lambda_1^2\bar{r}_{1,L}\bar{p}_1\bar{q}_1L^{-1/2}.\end{aligned} \tag{129}$$

To compute h_{12} and h_{21}, using the properties of connection, we compute the following:

$$\begin{aligned}\nabla_{T_1}^{E,L} T_2 &= \nabla_{\bar{q}_1E_1 - \bar{p}_1E_2}^{E,L} \bar{r}_{1,L}\bar{p}_1E_1 - \bar{r}_{1,L}\bar{q}_1E_2 + \frac{l}{l_L}L^{-1/2}E_3 \\ &= \bar{q}_1\Big(E_1(\bar{r}_{1,L}\bar{p}_1)E_1 - E_1(\bar{r}_{1,L}\bar{q}_1)E_2 - \bar{r}_{1,L}\bar{q}_1\nabla_{E_1}^{E,L}E_2 \\ &\quad + L^{-1/2}E_1\left(\frac{l}{l_L}\right)E_3 + \frac{l}{l_L}L^{-1/2}\nabla_{E_1}^{E,L}E_3\Big) \\ &\quad - \bar{p}_1\Big(E_2(\bar{r}_{1,L}\bar{p}_1)E_1 + \bar{r}_{1,L}\bar{p}_1\nabla_{E_2}^{E,L}E_1 - E_2(\bar{r}_{1,L}\bar{q}_1)E_2 \\ &\quad + L^{-1/2}E_2\left(\frac{l}{l_L}\right)E_3 + \frac{l}{l_L}L^{-1/2}\nabla_{E_2}^{E,L}E_3\Big) \\ &= \left(\bar{q}_1E_1(\bar{r}_{1,L}\bar{p}_1) - \bar{p}_1E_2(\bar{r}_{1,L}\bar{p}_1) + \bar{p}_1\frac{l}{l_L}L^{-1/2}\frac{\lambda_1^2 + \lambda_2^2L}{2}\right)E_1 \\ &\quad + \left(-\bar{q}_1E_1(\bar{r}_{1,L}\bar{q}_1) + \bar{p}_1E_2(\bar{r}_{1,L}\bar{q}_1) + \bar{q}_1\frac{l}{l_L}L^{-1/2}\frac{\lambda_1^2 - \lambda_2^2L}{2}\right)E_2 \\ &\quad + \Bigg[-\bar{p}_1E_2L^{-1/2}\left(\frac{l}{l_L}\right) + \bar{q}_1E_1L^{-1/2}\left(\frac{l}{l_L}\right) \\ &\quad + \frac{\bar{r}_{1,L}}{2}\left(-\lambda_2^2 + \frac{\lambda_1^2}{L}({\bar{p}_1}^2 + {\bar{q}_1}^2)\right)\Bigg]E_3.\end{aligned} \tag{130}$$

Next, we compute the inner product of this with N_L. Using the product rule and the identity $\bar{q}_{1,L}\bar{p}_1 = \bar{p}_{1,L}\bar{q}_1$, we obtain the following:

$$\begin{aligned}\left\langle \nabla_{T_1}^{E,L} T_2, N_L\right\rangle_L &= \bar{p}_{1,L}\bar{q}_1(\bar{p}_1E_1\bar{r}_{1,L} + \bar{r}_{1,L}E_1\bar{p}_1) \\ &\quad - \bar{p}_{1,L}\bar{p}_1(\bar{p}_1E_2\bar{r}_{1,L} + \bar{r}_{1,L}E_2\bar{p}_1) \\ &\quad - \bar{q}_{1,L}\bar{q}_1(\bar{q}_1E_1\bar{r}_{1,L} + \bar{r}_{1,L}E_1\bar{q}_1) \\ &\quad + \bar{q}_{1,L}\bar{p}_1(\bar{q}_1E_2\bar{r}_{1,L} + \bar{r}_{1,L}E_2\bar{q}_1) \\ &\quad - \frac{\sqrt{L}}{2}\lambda_2^2(-\bar{p}_{1,L}^2 + \bar{q}_{1,L}^2 + \bar{r}_{1,L}^2) \\ &\quad - \bar{r}_{1,L}\bar{p}_1E_2\left(\frac{l}{l_L}\right) + \bar{r}_{1,L}\bar{q}_1E_1\left(\frac{l}{l_L}\right) \\ &\quad + \frac{\lambda_1^2}{2\sqrt{L}}(\bar{p}_{1,L}^2 + \bar{q}_{1,L}^2) + \frac{\bar{r}_{1,L}^2}{2}\frac{\lambda_1^2}{\sqrt{L}}({\bar{p}_1}^2 + {\bar{q}_1}^2).\end{aligned} \tag{131}$$

The identities $-\bar{p}_{1,L}^2 + \bar{q}_{1,L}^2 + {\bar{r}_L}^2 = 1$ and $-{\bar{p}_1}^2 + {\bar{q}_1}^2 = 1$ yield the following:

$$\begin{aligned}\left\langle \nabla_{T_1}^{E,L} T_2, N_L\right\rangle_L &= -\frac{l}{l_L}\langle E_1, \nabla_H\bar{r}_{1,L}\rangle_L + \bar{r}_{1,L}\left\langle E_1, \nabla_H\left(\frac{l}{l_L}\right)\right\rangle_L \\ &\quad - \frac{\sqrt{L}}{2}\lambda_2^2 + \frac{\lambda_1^2}{2\sqrt{L}}(\bar{p}_{1,L}^2 + \bar{q}_{1,L}^2) \\ &\quad + \frac{\bar{r}_{1,L}^2}{2}\frac{\lambda_1^2}{\sqrt{L}}({\bar{p}_1}^2 + {\bar{q}_1}^2).\end{aligned} \tag{132}$$

Finally, by using the identity $(l/l_L - l_L/l)\nabla_H \bar{r}_{1,L} = \bar{r}_{1,L}\nabla_H (l/l_L)$, we obtain the following:

$$\left\langle \nabla^{E,L}_{T_1} T_2, N_L \right\rangle_L = -\frac{l_L}{l}\left\langle T_1, \nabla_H(\bar{r}_{1,L})\right\rangle_L - \frac{\sqrt{L}}{2}\lambda_2^2 + \frac{\lambda_1^2}{\sqrt{L}}\left(\bar{p}_{1,L}^2 + \bar{q}_{1,L}^2\right) + \frac{\bar{r}_{1,L}^2}{2}\frac{\lambda_1^2}{\sqrt{L}}\left(\bar{p}_1{}^2 + \bar{q}_1{}^2\right). \tag{133}$$

Therefore,

$$\begin{aligned} h_{21} = h_{12} &= -\left\langle \nabla^{E,L}_{T_1} T_2, N_L \right\rangle_L \\ &= \frac{l_L}{l}\left\langle T_1, \nabla_H(\bar{r}_{1,L})\right\rangle_L \\ &\quad + \frac{\sqrt{L}}{2}\lambda_2^2 - \frac{\lambda_1^2}{2\sqrt{L}}\left(\bar{q}_{1,L}^2 + \bar{p}_{1,L}^2\right) \\ &\quad - \frac{\bar{r}_{1,L}^2\lambda_1^2}{2\sqrt{L}}\left(\bar{q}_1{}^2 + \bar{p}_1{}^2\right). \end{aligned} \tag{134}$$

Since $\langle \nabla^{E,L}_{T_2} N_L, T_2\rangle_L = -\langle \nabla^{E,L}_{T_2} T_2, N_L\rangle_L$, by similar computation as above, we get the following:

$$\begin{aligned} \nabla^{E,L}_{T_2} T_2 &= \nabla^{E,L}_{\bar{r}_{1,L}\bar{p}_1 E_1 - \bar{r}_{1,L}\bar{q}_1 E_2 + (l/l_L)\widetilde{E}_3} \bar{r}_{1,L}\bar{p}_1 E_1 - \bar{r}_{1,L}\bar{q}_1 E_2 + \frac{l}{l_L}\widetilde{E}_3 \\ &= \bar{r}_{1,L}\bar{p}_1\Big(E_1(\bar{r}_{1,L}\bar{p}_1)E_1 - E_1(\bar{r}_{1,L}\bar{q}_1)E_2 - \bar{r}_{1,L}\bar{q}_1\nabla^{E,L}_{E_1}E_2 \\ &\quad + L^{-1/2}E_1\left(\frac{l}{l_L}\right)E_3 + L^{-1/2}\frac{l}{l_L}\nabla^{E,L}_{E_1}E_3\Big) \\ &\quad - \bar{r}_{1,L}\bar{q}_1\Big(E_2(\bar{r}_{1,L}\bar{p}_1)E_1 - E_2(\bar{r}_{1,L}\bar{q}_1)E_2 + \bar{r}_{1,L}\bar{p}_1\nabla^{E,L}_{E_2}E_1 \\ &\quad + L^{-1/2}E_2\left(\frac{l}{l_L}\right)E_3 + L^{-1/2}\frac{l}{l_L}\nabla^{E,L}_{E_2}E_3) + \frac{l}{l_L}L^{-1/2} \\ &\quad \cdot\Big(E_3(\bar{r}_{1,L}\bar{p}_1)E_1 + \bar{r}_{1,L}\bar{p}_1\nabla^{E,L}_{E_3}E_1 - E_3(\bar{r}_{1,L}\bar{q}_1)E_2 \\ &\quad - \bar{r}_{1,L}\bar{q}_1\nabla^{E,L}_{E_3}E_2 + L^{-1/2}E_3\left(\frac{l}{l_L}\right)E_3\Big) \\ &= \Big(\bar{r}_{1,L}\bar{p}_1E_1\bar{r}_{1,L}\bar{p}_1 - \bar{r}_{1,L}\bar{q}_1E_2\bar{r}_{1,L}\bar{p}_1 + L^{-1/2}\frac{l}{l_L}E_3(\bar{r}_{1,L}\bar{p}_1) \\ &\quad + \bar{r}_{1,L}\bar{q}_1L^{-1/2}\frac{l}{l_L}\left(\lambda_1^2 + \lambda_2^2L\right)\Big)E_1 \\ &\quad + \Big(-\bar{r}_{1,L}\bar{p}_1E_1\bar{r}_{1,L}\bar{q}_1 + \bar{r}_{1,L}\bar{q}_1E_2\bar{r}_{1,L}\bar{q}_1 - L^{-1/2}\frac{l}{l_L}E_3(\bar{r}_{1,L}\bar{q}_1) \\ &\quad + \bar{r}_{1,L}\bar{p}_1L^{-1/2}\frac{l}{l_L}(-\lambda_2^2L))E_2 + \Big(\bar{r}_{1,L}\bar{p}_1L^{-1/2}E_1\frac{l}{l_L} \\ &\quad - \bar{r}_{1,L}\bar{q}_1E_2L^{-1/2}E_2\frac{l}{l_L} + L^{-1}\frac{l}{l_L}E_3\left(\frac{l}{l_L}\right) + \bar{r}_{1,L}^2\bar{p}_1\bar{q}_1\frac{\lambda_1^2}{L}\Big)E_3. \end{aligned} \tag{135}$$

Taking the inner product with N_L yields the following:

$$\begin{aligned} \left\langle \nabla^{E,L}_{T_2} T_2, N_L \right\rangle_L &= \bar{p}_{1,L}(\bar{r}_{1,L}\bar{p}_1E_1\bar{r}_{1,L}\bar{p}_1 - \bar{r}_{1,L}\bar{q}_1E_2\bar{r}_{1,L}\bar{p}_1 \\ &\quad + L^{-1/2}\frac{l}{l_L}E_3(\bar{r}_{1,L}\bar{p}_1) + \bar{r}_{1,L}\bar{q}_1L^{-1/2}\frac{l}{l_L}\left(\lambda_1^2 + \lambda_2^2L\right)) \\ &\quad + \bar{q}_{1,L}\Big(-\bar{r}_{1,L}\bar{p}_1E_1\bar{r}_{1,L}\bar{q}_1 + \bar{r}_{1,L}\bar{q}_1E_2\bar{r}_{1,L}\bar{q}_1 - L^{-1/2}\frac{l}{l_L}E_3(\bar{r}_{1,L}\bar{q}_1) \\ &\quad + \bar{r}_{1,L}\bar{p}_1L^{-1/2}\frac{l}{l_L}\left(-\lambda_2^2L\right)\Big) \\ &\quad + \bar{r}_{1,L}\Big(\bar{r}_{1,L}\bar{p}_1E_1\frac{l}{l_L} - \bar{r}_{1,L}\bar{q}_1E_2E_2\frac{l}{l_L} \\ &\quad + L^{-1/2}\frac{l}{l_L}E_3\left(\frac{l}{l_L}\right) + \bar{r}_{1,L}^2\bar{p}_1\bar{q}_1\frac{\lambda_1^2}{\sqrt{L}}\Big). \end{aligned} \tag{136}$$

Under some simplifications, one can get the following:

$$h_{22} = -\left\langle \nabla^{E,L}_{T_2} T_2, N_L \right\rangle_L = \frac{l^2}{l_L^2}\left\langle T_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L + \widetilde{E}_3(\bar{r}_{1,L}) - \frac{\bar{r}_{1,L}\bar{p}_{1,L}\bar{q}_{1,L}}{\sqrt{L}}\lambda_1^2 - \frac{\bar{r}_{1,L}^3\bar{p}_1\bar{q}_1}{\sqrt{L}}\lambda_1^2. \tag{137}$$

□

The Riemannian mean curvature $\mathscr{H}_{E,L}$ of S is defined by the following:

$$\mathscr{H}_{E,L} := tr\left(II^{E,L}\right) = \frac{l}{l_L}\left(E_1(\bar{p}_1) - E_2(\bar{q}_1)\right) - \lambda_1^2\bar{r}_{1,L}\bar{p}_1\bar{q}_1L^{-1/2} + \frac{l^2}{l_L^2}\left\langle T_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L + \widetilde{E}_3(\bar{r}_{1,L}) - \frac{\bar{r}_{1,L}\bar{p}_{1,L}\bar{q}_{1,L}}{\sqrt{L}}\lambda_1^2 - \frac{\bar{r}_{1,L}^3\bar{p}_1\bar{q}_1}{\sqrt{L}}\lambda_1^2. \tag{138}$$

Proposition 17. *Away from characteristic point, the horizontal mean curvature $\mathscr{H}_{E,\infty}$ of $S \in E^L_{\lambda_1,\lambda_2}$ is given by the following:*

$$\mathscr{H}_{E,\infty} = \lim_{L\longrightarrow\infty}\mathscr{H}_{E,L} = E_1(\bar{p}_1) - E_2(\bar{q}_1). \tag{139}$$

Proof. By

$$\begin{aligned} &\frac{l^2}{l_L^2}\left\langle T_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L = \frac{\bar{p}_1 r}{l}E_1(\bar{r}_{1,L}) + \frac{\bar{q}_1 r}{l}E_2(\bar{r}_{1,L}) = O\left(L^{-1}\right), \\ &\frac{l}{l_L}\left(E_1(\bar{p}_1) - E_2(\bar{q}_1)\right) \longrightarrow E_1(\bar{p}_1) - E_2(\bar{q}_1), \\ &\widetilde{E}_3(\bar{r}_{1,L}) \longrightarrow 0, \lambda_1^2\bar{r}_{1,L}\bar{p}_1\bar{q}_1L^{-1/2} \longrightarrow 0, \\ &-\frac{\bar{r}_{1,L}\bar{p}_{1,L}\bar{q}_{1,L}}{\sqrt{L}}\lambda_1^2 - \frac{\bar{r}_{1,L}^3\bar{p}_1\bar{q}_1}{\sqrt{L}}\lambda_1^2 \longrightarrow 0, \end{aligned} \tag{140}$$

we get Equation (139). □

Recalling the definition of curvature tensor for a connection ∇ is defined by the following:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{141}$$

By Proposition 1 and Equation (141), we have the following lemma.

Lemma 18. *The curvature tensor of $E^L_{\lambda_1,\lambda_2}$ is given by the following:*

$$R^{E,L}(E_1,E_2)E_1 = \frac{-\lambda_1^4 + 2\lambda_1^2\lambda_2^2 L + 3\lambda_2^4 L^2}{4L}E_2,\ R^{E,L}(E_1,E_2)E_2 = \frac{-\lambda_1^4 + 2\lambda_1^2\lambda_2^2 L + 3\lambda_2^4 L^2}{4L}E_1,$$

$$R^{E,L}(E_1,E_2)E_3 = 0,\ R^{E,L}(E_1,E_3)E_1 = \frac{3\lambda_1^4 + 2\lambda_1^2\lambda_2^2 L - \lambda_2^4 L^2}{4L}E_3,$$

$$R^{E,L}(E_1,E_3)E_2 = 0,\ R^{E,L}(E_1,E_3)E_3 = \frac{3\lambda_1^4 + 2\lambda_1^2\lambda_2^2 L - \lambda_2^4 L^2}{4}E_1,$$

$$R^{E,L}(E_2,E_3)E_1 = 0,\ R^{E,L}(E_2,E_3)E_2 = \frac{\lambda_1^4 + 2\lambda_1^2\lambda_2^2 L + \lambda_2^4 L^2}{4L}E_3,$$

$$R^{E,L}(E_2,E_3)E_3 = -\frac{\lambda_1^4 + 2\lambda_1^2\lambda_2^2 L + \lambda_2^4 L^2}{4}E_2. \tag{142}$$

Let

$$\begin{aligned}\mathcal{K}^{S,E,L}(T_1,T_2) &= -\left\langle R^{S,E,L}(T_1,T_2)T_1, T_2\right\rangle_{S,L},\\ \mathcal{K}^{E,L}(T_1,T_2) &= -\left\langle R^{E,L}(T_1,T_2)T_1, T_2\right\rangle_L.\end{aligned} \tag{143}$$

By the Gauss equation, we have the following:

$$\mathcal{K}^{S,E,L}(T_1,T_2) = \mathcal{K}^{E,L}(T_1,T_2) + \det\left(II^{E,L}\right). \tag{144}$$

Proposition 19. *Away from characteristic points, we have the following:*

$$\mathcal{K}^{S,E,\infty}(T_1,T_2) = -\left\langle T_1, \nabla_H\left(\frac{E_3F}{|\nabla_H F|}\right)\right\rangle_L - \lambda_2{}^4\frac{(E_3F)^2}{l^2},\ as\ L \longrightarrow \infty. \tag{145}$$

Proof. We compute the following:

$$\begin{aligned}
&R^{E,L}(T_1,T_2)T_1\\
&= R^{E,L}\left(\bar q_1E_1 - \bar p_1E_2, \bar r_{1,L}\bar p_1E_1 - \bar r_{1,L}\bar q_1E_2 + \frac{l}{l_L\sqrt{L}}E_3\right)(\bar q_1E_1 - \bar p_1E_2)\\
&= \bar r_{1,L}\bar p_1\bar q_1{}^2R^{E,L}(E_1,E_1)E_1 - \bar r_{1,L}\bar q_1{}^3R^{E,L}(E_1,E_2)E_1\\
&\quad + \frac{l\bar q_1{}^2}{l_L\sqrt{L}}R^{E,L}(E_1,E_3)E_1 - \bar r_{1,L}\bar p_1{}^2\bar q_1R^{E,L}(E_2,E_1)E_1\\
&\quad + \bar r_{1,L}\bar p_1\bar q_1{}^2R^{E,L}(E_2,E_2)E_1 - \frac{l\bar p_1\bar q_1}{l_L\sqrt{L}}R^{E,L}(E_2,E_3)E_1\\
&\quad - \bar r_{1,L}\bar p_1{}^2\bar q_1R^{E,L}(E_1,E_1)E_2 + \bar r_{1,L}\bar p_1\bar q_1{}^2R^{E,L}(E_1,E_2)E_2\\
&\quad - \frac{l\bar p_1\bar q_1}{l_L\sqrt{L}}R^{E,L}(E_1,E_3)E_2 + \bar r_{1,L}\bar p_1{}^3R^{E,L}(E_2,E_1)E_2\\
&\quad - \bar r_{1,L}\bar p_1{}^2\bar q_1R^{E,L}(E_2,E_2)E_2 + \frac{l\bar p_1{}^2}{l_L\sqrt{L}}R^{E,L}(E_2,E_3)E_2\\
&= -\bar r_{1,L}\bar q_1{}^3R^{E,L}(E_1,E_2)E_1 + \bar q_1{}^2\frac{l}{l_L\sqrt{L}}R^{E,L}(E_1,E_3)E_1\\
&\quad - \bar r_{1,L}\bar p_1{}^2\bar q_1R^{E,L}(E_2,E_1)E_1 + \bar r_{1,L}\bar q_1{}^2\bar p_1R^{E,L}(E_1,E_2)E_2\\
&\quad + \bar r_{1,L}\bar p_1{}^3R^{E,L}(E_2,E_1)E_2 + \bar p_1{}^2\frac{l}{l_L\sqrt{L}}R^{E,L}(E_2,E_3)E_2\\
&= \bar r_{1,L}\bar p_1\frac{-\lambda_1^4 + 2\lambda_1^2\lambda_2^2L + 3\lambda_2^4L^2}{4L}E_1\\
&\quad - \bar r_{1,L}\bar q_1\frac{-\lambda_1^4 + 2\lambda_1^2\lambda_2^2L + 3\lambda_2^4L^2}{4L}E_2\\
&\quad + \frac{l}{l_L\sqrt{L}}\left(\bar q_1{}^2\frac{3\lambda_1^4 + 2\lambda_1^2\lambda_2^2L - \lambda_2^4L^2}{4L} + \bar p_1{}^2\frac{\lambda_1^4 + 2\lambda_1^2\lambda_2^2L + \lambda_2^4L^2}{4L}\right)E_3,
\end{aligned} \tag{146}$$

$$\begin{aligned}
\mathcal{K}^{E,L}(T_1,T_2) &= -\left\langle R^{E,L}(T_1,T_2)T_1,T_2\right\rangle_L\\
&= -\bar r_{1,L}^2\left(\frac{\lambda_1^2\lambda_2^2}{2} - \frac{\lambda_1^4}{4L} + \frac{3\lambda_2^4L}{4}\right) - \left(\frac{l}{l_L}\bar q_1\right)^2\\
&\quad\cdot\left(\frac{\lambda_1^2\lambda_2^2}{2} - \frac{\lambda_2^4L}{4} + \frac{3\lambda_1^4}{4L}\right) - \left(\frac{l}{l_L}\bar p_1\right)^2\\
&\quad\cdot\left(\frac{\lambda_1^2\lambda_2^2}{2} + \frac{\lambda_1^4}{4L} + \frac{\lambda_2^4L}{4}\right) \sim \left(\frac{l}{l_L}\right)^2\frac{\lambda_2^4L}{4}\\
&\quad - \frac{3\lambda_2^4}{4}\frac{(E_3u)^2}{l^2} - \lambda_1{}^2\lambda_2{}^2\bar q_1{}^2\\
&\quad + \frac{\lambda_1^2\lambda_2^2}{2}\ \text{as}\ L \longrightarrow \infty.
\end{aligned} \tag{147}$$

By Theorem 16 and $\nabla_H(\bar{r}_{1,L}) = L^{-1/2}\nabla_H(E_3F/|\nabla_H F|) + O(L^{-1})$ as $L \longrightarrow +\infty$, we get the following:

$$\det\left(II^{E,L}\right) = h_{11}h_{22} - h_{12}^2 = -\frac{\lambda_2^4 L}{4} - \left\langle T_1, \nabla_H\left(\frac{E_3F}{|\nabla_H F|}\right)\right\rangle_L + \frac{\lambda_1^2\lambda_2^2}{2}\left(\bar{q}_1^{\,2} + \bar{p}_1^{\,2}\right) + O\left(L^{-1/2}\right) \text{ as } L \longrightarrow \infty. \tag{148}$$

By Equations (144), (147), and (148), we get the desired equation. □

5. The First Gauss-Bonnet Theorem in $E^L_{\lambda_1,\lambda_2}$

In this section, we will prove Gauss-Bonnet theorem in $E^L_{\lambda_1,\lambda_2}$. We consider a spacelike curve $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$, and define the Riemannian length measure by $ds_{E,L} = \|\dot{\alpha}\|_L dt$.

Lemma 20. *Let $\alpha : I \longrightarrow E^L_{\lambda_1,\lambda_2}$ be a spacelike C^2-smooth. Let $ds_E := |\omega(\dot{\alpha}(t))|dt$,*

$$d\bar{s}_E := \frac{1}{2}\frac{1}{|\omega(\dot{\alpha}(t))|}\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\}dt. \tag{149}$$

Then,

$$\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\int_\alpha ds_{E,L} = \int_a^b ds_E. \tag{150}$$

When $\omega(\dot{\alpha}(t)) \neq 0$, we have the following:

$$\frac{1}{\sqrt{L}}ds_{E,L} = ds_E + d\bar{s}_E L^{-1} + O\left(L^{-2}\right) \text{ as } L \longrightarrow +\infty. \tag{151}$$

When $\omega(\dot{\alpha}(t)) = 0$, we have the following:

$$\frac{1}{\sqrt{L}}ds_{E,L} = \frac{1}{\sqrt{L}}\sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2}\,dt. \tag{152}$$

Proof. Since

$$\|\dot{\alpha}(t)\|_L = \sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2 + L(\omega(\dot{\alpha}(t)))^2}, \tag{153}$$

we get the following:

$$\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\int_\alpha \|\dot{\alpha}(t)\|_L dt = \int_a^b \lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\|\dot{\alpha}(t)\|_L dt = \int_a^b \lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2 + L(\omega(\dot{\alpha}(t)))^2}\,dt = \int_a^b |\omega(\dot{\alpha}(t))|dt = \int_a^b ds_E. \tag{154}$$

When $\omega(\dot{\alpha}(t)) \neq 0$, we have the following:

$$\frac{1}{\sqrt{L}}ds_{E,L} = \sqrt{L^{-1}\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2\right\} + \omega(\dot{\alpha}(t))^2}\,dt. \tag{155}$$

Using the Taylor expansion, we can prove the following:

$$\frac{1}{\sqrt{L}}ds_{E,L} = ds_E + d\bar{s}_E L^{-1} + O\left(L^{-2}\right) \text{ as } L \longrightarrow +\infty. \tag{156}$$

From the definition of $ds_{E,L}$ and $\omega(\dot{\alpha}(t)) = 0$, we get the following:

$$\frac{1}{\sqrt{L}}ds_{E,L} = \frac{1}{\sqrt{L}}\sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\alpha_3}\dot{\alpha}_1(t) + e^{\alpha_3}\dot{\alpha}_2(t)\right)\right]^2}\,dt. \tag{157}$$

□

Proposition 21. *Let $S \subset E^L_{\lambda_1,\lambda_2}$ be a Euclidean C^2-smooth surface and $S = \{F = 0\}$ and $d\sigma_{S,E,L}$ denote the surface measure on S with respect to the Lorentzian metric g_L. Let*

$$d\sigma_{S,E} := -(\bar{p}_1\omega_2 - \bar{q}_1\omega_1)\wedge\omega,\; d\bar{\sigma}_{S,E} := -\frac{E_3F}{l}\omega_1\wedge\omega_2 + \frac{(E_3F)^2}{2l^2}(\bar{p}_1\omega_2 - \bar{q}_1\omega_1)\wedge\omega. \tag{158}$$

Then,

$$\frac{1}{\sqrt{L}}d\sigma_{S,E,L} = d\sigma_{S,E} + d\bar{\sigma}_{S,E}L^{-1} + O\left(L^{-2}\right) \text{ as } L \longrightarrow +\infty. \tag{159}$$

If $S = f(D)$ with $f = f(x_1, x_2) = (f_1, f_2, f_3): D \subset \mathbb{R}^2 \longrightarrow E^L_{\lambda_1,\lambda_2}$, then

$$\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\int_S d\sigma_{S,E,L} = \int_D A(x_1, x_2, \lambda_1, \lambda_1)dx_1dx_2, \tag{160}$$

where $A(x_1, x_2, \lambda_1, \lambda_1) = \{[(f_1)_{x_1}(f_2)_{x_2}\lambda_1\lambda_2 - (f_2)_{x_1}(f_1)_{x_2}\lambda_1\lambda_2]^2 - \{[(f_3)_{x_2}(f_1)_{x_1} - (f_3)_{x_1}(f_1)_{x_2}]\sqrt{2}/2e^{-u_3}1/\lambda_1 + [(f_3)_{x_2}(f_2)_{x_1} - (f_3)_{x_1}(f_2)_{x_2}]\sqrt{2}/2e^{u_3}1/\lambda_1\}^2\}^{1/2}$.

Proof. Let $g_L(E_1,\cdot) = -\omega_1$, $g_L(E_2,\cdot) = \omega_2$, $g_L(E_3,\cdot) = L\omega$. We define $T_1{}^* \coloneqq g_L(T_1,\cdot)$, $T_2{}^* \coloneqq g_L(T_2,\cdot)$; then,

$$\begin{aligned} T_1^* &= \bar{q}_1\omega_1 - \bar{p}_1\omega_2, \\ T_2^* &= \bar{r}_L\bar{p}_1\omega_1 - \bar{r}_L\bar{q}_1\omega_2 + \frac{l}{l_L}L^{1/2}\omega. \end{aligned} \tag{161}$$

Therefore,

$$\frac{1}{\sqrt{L}}d\sigma_{S,E,L} = \frac{1}{\sqrt{L}}T_1^*\wedge T_2^* = -\frac{l}{l_L}(\bar{p}_1\omega_2 - \bar{q}_1\omega_1)\wedge\omega - \frac{1}{\sqrt{L}}\bar{r}_L\omega_1\wedge\omega_2. \tag{162}$$

Recalling

$$\bar{r}_{1,L} = \frac{(E_3F)L^{1/2}}{\sqrt{-p^2 + q^2 + L^{-1}(E_1F)^2}} \tag{163}$$

and the Taylor expansion

$$\frac{1}{l_L} = \frac{1}{l} - \frac{1}{2l^3}(E_3F)^2L^{-1} + O\left(L^{-2}\right) \text{ as } L \longrightarrow +\infty, \tag{164}$$

we get Equation (159). By Equation (8), we have the following:

$$\begin{aligned} f_{x_1} &= (f_1)_{x_1}\partial_{u_1} + (f_2)_{x_1}\partial_{u_2} + (f_3)_{x_1}\partial_{u_3} \\ &= a_{11}E_1 + a_{12}E_2 + a_{13}\widetilde{E}_3, \\ f_{x_2} &= (f_1)_{x_2}\partial_{u_1} + (f_2)_{x_2}\partial_{u_2} + (f_3)_{x_2}\partial_{u_3} \\ &= a_{21}E_1 + a_{22}E_2 + a_{23}\widetilde{E}_3, \end{aligned} \tag{165}$$

where $a_{11} = (f_3)_{x_1}1/\lambda_1\lambda_2$, $a_{12} = ((f_1)_{x_1}(-\sqrt{2}/2e^{-u_3}\lambda_1) + (f_2)_{x_1}\sqrt{2}/2e^{u_3}\lambda_1)$, $a_{21} = (f_3)_{x_2}(1/\lambda_1\lambda_2)$, $a_{13} = \sqrt{L}((f_1)_{x_1}(-\sqrt{2}/2e^{-u_3}\lambda_2) + (f_2)_{x_1}(-\sqrt{2}/2e^{u_3}\lambda_2))$. $a_{22} = ((f_1)_{x_2}(-\sqrt{2}/2e^{-u_3}\lambda_1) + (f_2)_{x_2}\sqrt{2}/2e^{u_3}\lambda_1)$ and $a_{23} = \sqrt{L}((f_1)_{x_2}(-\sqrt{2}/2e^{-u_3}\lambda_2) + (f_2)_{x_2}(-\sqrt{2}/2e^{u_3}\lambda_2))$. Let

$$\begin{aligned} \bar{N}_L &= \begin{vmatrix} -E_1 & E_2 & \widetilde{E}_3 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = -\sqrt{L}\Big((f_1)_{x_1}(f_2)_{x_2}\lambda_1\lambda_2 \\ &\quad - (f_2)_{x_1}(f_1)_{x_2}\lambda_1\lambda_2\Big)E_1 + \sqrt{L}\bigg[\Big((f_3)_{x_2}(f_1)_{x_1} \\ &\quad - (f_3)_{x_1}(f_1)_{x_2}\Big)\frac{\sqrt{2}}{2}e^{-u_3}\frac{1}{\lambda_1} + \Big((f_3)_{x_2}(f_2)_{x_1} \\ &\quad - (f_3)_{x_1}(f_2)_{x_2}\Big)\frac{\sqrt{2}}{2}e^{u_3}\frac{1}{\lambda_1}\bigg]E_2 + \bigg[\Big((f_3)_{x_2}(f_1)_{x_1} \\ &\quad - (f_3)_{x_1}(f_1)_{x_2}\Big)\frac{\sqrt{2}}{2}e^{-u_3}\frac{1}{\lambda_2} + \Big((f_3)_{x_1}(f_2)_{x_2} \\ &\quad - (f_3)_{x_2}(f_2)_{x_1}\Big)\frac{\sqrt{2}}{2}e^{u_3}\frac{1}{\lambda_2}\bigg]\widetilde{E}_3. \end{aligned} \tag{166}$$

We know that $d\sigma_{S,E,L} = \sqrt{\det(g_{ij})}dx_1dx_2$, $g_{ij} = g_L(f_{x_i}, f_{x_j})$, and

$$\begin{aligned} \det\left(g_{ij}\right) &= \|\bar{N}_L\|_L^2 = -\langle\bar{N}_L, \bar{N}_L\rangle = L\Big((f_1)_{x_1}(f_2)_{x_2}\lambda_1\lambda_2 \\ &\quad - (f_2)_{x_1}(f_1)_{x_2}\lambda_1\lambda_2\Big)^2 - L\bigg[\Big((f_3)_{x_2}(f_1)_{x_1} \\ &\quad - (f_3)_{x_1}(f_1)_{x_2}\Big)\frac{\sqrt{2}}{2}e^{-u_3}\frac{1}{\lambda_1} + \Big((f_3)_{x_2}(f_2)_{x_1} \\ &\quad - (f_3)_{x_1}(f_2)_{x_2}\Big)\frac{\sqrt{2}}{2}e^{u_3}\frac{1}{\lambda_1}\bigg]^2 - \bigg[\Big((f_3)_{x_2}(f_1)_{x_1} \\ &\quad - (f_3)_{x_1}(f_1)_{x_2}\Big)\frac{\sqrt{2}}{2}e^{-u_3}\frac{1}{\lambda_2} + \Big((f_3)_{x_1}(f_2)_{x_2} \\ &\quad - (f_3)_{x_2}(f_2)_{x_1}\Big)\frac{\sqrt{2}}{2}e^{u_3}\frac{1}{\lambda_2}\bigg]^2, \end{aligned} \tag{167}$$

so by the dominated convergence theorem, we get the following:

$$\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\int_S d\sigma_{S,E,L} = \int_D A(x_1, x_2, \lambda_1, \lambda_1)dx_1dx_2. \tag{168}$$

□

Similar to the proof of Theorem 4.3 in [3], we get a Gauss-Bonnet theorem in $E^L_{\lambda_1,\lambda_2}$ as follows:

Theorem 22. *Let* $S \subset E^L_{\lambda_1,\lambda_2}$ *be a regular Lorentzian surface with finitely many boundary components* $(\partial S)_i$, $i \in \{1,\cdots,n\}$, *given by* C^2*-smooth regular and closed spacelike curve* $\alpha_i : [0,$

$2\pi] \longrightarrow (\partial S)_i$. Let $\mathscr{K}^{S,E,\infty}$ be intrinsic Gaussian curvature of S in Proposition 19 and $\kappa_{\alpha_i,S}^{E,\infty,s}$ the intrinsic signed geodesic curvature of α_i relative to S in Proposition 14. Suppose that the characteristic set $C(S)$ satisfies $\mathscr{H}^1(C(S)) = 0$ and that $\|\nabla_H F\|_H^{-1}$ is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set $C(S)$. Then,

$$\int_S \mathscr{K}^{S,E,\infty} d\sigma_{S,E} + \sum_{i=1}^{n} \int_{\alpha_i} \kappa_{\alpha_i,S}^{E,\infty,s} ds_E = 0. \tag{169}$$

Proof. Using the discussions in [1, 2], we may assume that all points satisfy the following:

$$\omega(\dot{\alpha}_i(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\alpha}_i(t))) \neq 0. \tag{170}$$

Then, by Lemma 20, we obtain the following:

$$\kappa_{\alpha_i,S}^{E,L,s} = \kappa_{\alpha_i,S}^{E,\infty,s} + O\left(L^{-1/2}\right). \tag{171}$$

By the Gauss-Bonnet theorem, we have the following:

$$\int_S \mathscr{K}^{S,E,L} \frac{1}{\sqrt{L}} d\sigma_{S,E,L} + \sum_{i=1}^{n} \int_{\alpha_i} \kappa_{\alpha_i,S}^{E,\infty,s} \frac{1}{\sqrt{L}} ds_{E,L} = 2\pi \frac{\chi(S)}{\sqrt{L}}. \tag{172}$$

So, by Equations (171), (172), (145), (159), and (160) and Lemma 20, we get the following:

$$\left(\int_S A d\sigma_{S,E} + \sum_{i=1}^{n} \int_{\alpha_i} \kappa_{\alpha_i,S}^{E,\infty,s} ds_{E,L} \right) + O\left(L^{-1/2}\right) = 2\pi \frac{\chi(S)}{\sqrt{L}}, \tag{173}$$

where $A = -\langle T_1, \nabla_H (E_3 F / |\nabla_H F|) \rangle_L - \lambda_2^4 (E_3 F)^2 / l^2$. Let L go to the infinity, and by using the dominated convergence theorem, we get the desired result. □

6. Curvatures for Spacelike Surfaces and the Second Gauss-Bonnet Theorem

The geodesic curvature of spacelike curves on spacelike surface and intrinsic Gaussian curvature of spacelike surfaces in $E^L_{\lambda_1,\lambda_2}$ will be investigated in this section. For a regular surface $S \subset E^L_{\lambda_1,\lambda_2}$ and regular curves $\alpha \subset S$, suppose that there is a C^2-smooth function $F : E^L_{\lambda_1,\lambda_2} \longrightarrow \mathbb{R}$ such that

$$S = \left\{ (u_1, u_2, u_3) \in E^L_{\lambda_1,\lambda_2} : F(u_1, u_2, u_3) = 0 \right\}. \tag{174}$$

Similar to Section 4, we give $p_1, q_1, r_1, l, l_L, \bar{p}_1, \bar{q}_1, \bar{p}_{1,L}, \bar{q}_{1,L}, \bar{r}_{1,L}, N_L, T_1, T_2, J_L, \kappa_{\alpha,S}^{E,L}, \kappa_{\alpha,S}^{\infty}, \kappa_{\alpha,S}^{E,L,s}, \kappa_{\alpha,S}^{E,\infty,s}$. We consider the case that S is a spacelike surface in $E^L_{\lambda_1,\lambda_2}$. In particular, let $p_1 \coloneqq E_1 F$, $q_1 \coloneqq E_2 F$, and $r_1 \coloneqq \widetilde{E}_3 F$. Let $p_1^2 - q_1^2 > 0$, when $L \longrightarrow +\infty$, we have $p_1^2 - q_1^2 - r_1^2 > 0$. We then define the following:

$$\begin{aligned} l &\coloneqq \sqrt{p_1^2 - q_1^2}, \\ l_L &\coloneqq \sqrt{p_1^2 - q_1^2 - r_1^2}, \\ \bar{p}_1 &\coloneqq \frac{p_1}{l}, \\ \bar{q}_1 &\coloneqq \frac{q_1}{l}, \\ \bar{p}_{1,L} &\coloneqq \frac{p_1}{l_L}, \\ \bar{q}_{1,L} &\coloneqq \frac{q_1}{l_L}, \\ \bar{r}_{1,L} &\coloneqq \frac{r_1}{l_L}, \\ l &\coloneqq \sqrt{p_1^2 - q_1^2}. \end{aligned} \tag{175}$$

In particular, $\bar{p}_1^2 - \bar{q}_1^2 = 1$. These functions are well defined at every noncharacteristic point. Let

$$N_L = -\bar{p}_{1,L} E_1 + \bar{q}_{1,L} E_2 + \bar{r}_{1,L} \widetilde{E}_3, \tag{176}$$

$$T_1 = \bar{q}_1 E_1 - \bar{p}_1 E_2, \tag{177}$$

$$T_2 = \bar{r}_{1,L} \bar{p}_1 E_1 - \bar{r}_{1,L} \bar{q}_1 E_2 - \frac{l}{l_L} \widetilde{E}_3, \tag{178}$$

then N_L is the unit timelike normal vector to S and T_1, T_2 is the unit spacelike vector. $\{T_1, T_2\}$ are the orthonormal basis of S. We call S a spacelike surface in the Lorentzian group of rigid motions of the Minkowski plane. We define a linear transformation on TS by $J_L : TS \longrightarrow TS$,

$$\begin{aligned} J_L(T_1) &\coloneqq T_2, \\ J_L(T_2) &\coloneqq -T_1. \end{aligned} \tag{179}$$

For every $X, Y \in TS$, we define $\nabla_X^{S,E,L} Y = \pi \nabla_U^{E,L} V$, where $\pi : TG \longrightarrow TS$ is the projection. Then, $\nabla^{S,E,L}$ is the Levi-Civita connection on S with respect to the metric g_L. In particular,

$$\nabla_{\dot{\alpha}}^{S,E,L} \dot{\alpha} = \left\langle \nabla_{\dot{\alpha}}^{E,L} \dot{\alpha}, T_1 \right\rangle_L T_1 + \left\langle \nabla_{\dot{\alpha}}^{E,L} \dot{\alpha}, T_2 \right\rangle_L T_2. \tag{180}$$

A simple computation shows that

$$\nabla^{S,E,L}_{\dot\alpha}\dot\alpha=\Bigg\{-\bar q_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)-\frac{\lambda_1^3\sqrt2+\lambda_1\lambda_2^2\sqrt2L}{2}\cdot(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\right]-\bar p_1\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})-L\dot\alpha_3(t)\omega(\dot\alpha(t))\right]\Bigg\}T_1+\Bigg\{-\bar r_{1,L}\bar p_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)-\frac{\lambda_1^3\sqrt2+\lambda_1\lambda_2^2\sqrt2L}{2}\cdot(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\omega(\dot\alpha(t))\right]-\bar r_{1,L}\bar q_1\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})-L\dot\alpha_3(t)\omega(\dot\alpha(t))\right]-\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot\alpha(t)))-\frac{\lambda_1^2\sqrt2}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]\Bigg\}T_2. \tag{181}$$

Moreover, if $\omega(\dot\alpha(t))=0$, then

$$\nabla^{S,E,L}_{\dot\alpha}\dot\alpha=\Bigg\{-\bar q_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)\right]-\bar p_1\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})\right]\Bigg\}T_1+\Bigg\{-\bar r_{1,L}\bar p_1\left[\frac{1}{\lambda_1\lambda_2}\ddot\alpha_3(t)\right]-\bar r_{1,L}\bar q_1\cdot\left[\frac{\lambda_1\sqrt2}{2}(\ddot\alpha_2e^{\alpha_3}+\dot\alpha_2\dot\alpha_3e^{\alpha_3}-\ddot\alpha_1e^{-\alpha_3}+\dot\alpha_1\dot\alpha_3e^{-\alpha_3})\right]-\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot\alpha(t)))-\frac{\lambda_1^2\sqrt2}{2\lambda_2L}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\dot\alpha_3(t)\right]\Bigg\}T_2. \tag{182}$$

Definition 23. Let $S\subset E^L_{\lambda_1,\lambda_2}$ be a regular spacelike surface, $\alpha:I\longrightarrow S$ be a regular C^2-smooth spacelike curve. The geodesic curvature $\kappa^{E,L}_{\alpha,S}$ of α at $\alpha(t)$ is defined as follows:

$$\kappa^{E,L}_{\alpha,S}:=\sqrt{\frac{\|\nabla^{S,E,L}_{\dot\alpha}\dot\alpha\|^2_{S,L}}{\|\dot\alpha\|^4_{S,L}}-\frac{\left\langle\nabla^{S,E,L}_{\dot\alpha}\dot\alpha,\dot\alpha\right\rangle^2_{S,L}}{\|\dot\alpha\|^6_{S,L}}}. \tag{183}$$

Definition 24. Let $S\subset E^L_{\lambda_1,\lambda_2}$ be a regular spacelike surface and $\alpha:I\longrightarrow S$ be a regular C^2-smooth spacelike curve. The intrinsic geodesic curvature $\kappa^{E,\infty,s}_{\alpha,S}$ of α at $\alpha(t)$ is defined to be

$$\kappa^{E,\infty,s}_{\alpha,S}:=\lim_{L\longrightarrow+\infty}\kappa^{E,L}_{\alpha,S}, \tag{184}$$

if the limit exists.

Proposition 25. *Let $S\subset E^L_{\lambda_1,\lambda_2}$ be a regular spacelike surface and $\alpha:I\longrightarrow S$ be a regular C^2-smooth spacelike curve. Then,*

$$\kappa^{E,\infty,s}_{\alpha,S}=\frac{\sqrt{(\lambda_1^2\lambda_2^4/2)\bar q_1{}^2(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))^2+\bar p_1{}^2(\dot\alpha_3(t))^2}}{|\omega(\dot\alpha(t))|}\ \text{if } \omega(\dot\alpha(t))\neq0, \tag{185}$$

$$\kappa^{E,\infty,s}_{\alpha,S}=0,\ \text{if } \omega(\dot\alpha(t))=0, \tag{186}$$

$$\frac{d}{dt}(\omega(\dot\alpha(t)))=0, \tag{187}$$

$$\lim_{L\longrightarrow+\infty}\frac{\kappa^{E,L}_{\alpha,S}}{\sqrt L}=\frac{|(d/dt)(\omega(\dot\alpha(t)))|}{\left[(1/\lambda_1\lambda_2)\dot\alpha_3(t)\bar q_1+\left(\lambda_1\sqrt2/2\right)(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\bar p_1\right]^2},\ \text{if } \omega(\dot\alpha(t))=0,\ \frac{d}{dt}(\omega(\dot\alpha(t)))\neq0. \tag{188}$$

Proof. By Equation (25) and $\dot\alpha\in TS$, we get the following:

$$\dot\alpha=-\left[\frac{1}{\lambda_1\lambda_2}\dot\alpha_3(t)\bar q_1+\frac{\lambda_1\sqrt2}{2}(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))\bar p_1\right]T_1-\frac{l_L}{l}L^{1/2}\omega(\dot\alpha(t))T_2. \tag{189}$$

By Equation (181), we have the following:

$$\left\langle\nabla^{S,E,L}_{\dot\alpha}\dot\alpha,\nabla^{S,E,L}_{\dot\alpha}\dot\alpha\right\rangle_{S,L}\sim\left[\frac{\lambda_1^2\lambda_2^4}{2}\bar q_1{}^2(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))^2+\bar p_1{}^2(\dot\alpha_3(t))^2\right]L^2(\omega(\dot\alpha(t)))^2\ \text{as } L\longrightarrow+\infty. \tag{190}$$

Similarly, we have that when $\omega(\dot\alpha(t))\neq0$,

$$\langle\dot\alpha,\dot\alpha\rangle_{S,L}\sim L[\omega(\dot\alpha(t))]^2\ \text{as } L\longrightarrow+\infty. \tag{191}$$

By Equations (181) and (189), we have the following:

$$\left\langle\nabla^{S,E,L}_{\dot\alpha}\dot\alpha,\dot\alpha\right\rangle_{S,L}\sim M_0L\ \text{as } L\longrightarrow+\infty, \tag{192}$$

where M_0 does not depend on L. By Equation (183), we have the following:

$$\kappa^{E,\infty}_{\alpha,S}=\lim_{L\longrightarrow+\infty}\kappa^{E,L}_{\alpha,S}=\frac{\sqrt{(\lambda_1^2\lambda_2^4/2)\bar q_1{}^2(-e^{-\alpha_3}\dot\alpha_1(t)+e^{\alpha_3}\dot\alpha_2(t))^2+\bar p_1{}^2(\dot\alpha_3(t))^2}}{|\omega(\dot\alpha(t))|}\ \text{if } \omega(\dot\alpha(t))\neq0. \tag{193}$$

When $\omega(\dot{\alpha}(t))=0$, and $(d/dt)(\omega(\dot{\alpha}(t)))=0$, we have the following:

$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\rangle_{S,L} \sim \left\{-\bar{q}_1\left[\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)\right] - \bar{p}_1\left[\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2 e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3 e^{\alpha_3}-\ddot{\alpha}_1 e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3 e^{-\alpha_3}\right)\right]\right\}^2 \text{as } L\longrightarrow+\infty, \tag{194}$$

$$\langle\dot{\alpha},\dot{\alpha}\rangle_{S,L} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right]^2, \tag{195}$$

$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_{S,L} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right] \cdot\left[\bar{q}_1\frac{1}{\lambda_1\lambda_2}\ddot{\alpha}_3(t)+\bar{p}_1\frac{\lambda_1\sqrt{2}}{2}\left(\ddot{\alpha}_2 e^{\alpha_3}+\dot{\alpha}_2\dot{\alpha}_3 e^{\alpha_3} - \ddot{\alpha}_1 e^{-\alpha_3}+\dot{\alpha}_1\dot{\alpha}_3 e^{-\alpha_3}\right)\right] \coloneqq B_1B_2. \tag{196}$$

By Equations (194)–(196) and Equation (182), we get the following:

$$\kappa^{E,\infty,s}_{\alpha,S} = \sqrt{\frac{B_2^2}{B_1^4}-\frac{B_1^2B_2^2}{B_1^6}} = 0. \tag{197}$$

When $\omega(\dot{\alpha}(t))=0$ and $d/dt(\omega(\dot{\alpha}(t)))\neq 0$, we have the following:

$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}\right\rangle_{S,L} \sim L\left[\frac{d}{dt}(\omega(\dot{\alpha}(t)))\right]^2 \text{as } L\longrightarrow+\infty,$$
$$\langle\dot{\alpha},\dot{\alpha}\rangle_{S,L} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right]^2,$$
$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\right\rangle_{S,L} = O(1),$$
$$\lim_{L\longrightarrow+\infty}\frac{\kappa^{E,L}_{\alpha,S}}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{\alpha}(t)))|}{\left[(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1+\left(\lambda_1\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right]^2}, \tag{198}$$

if $\omega(\dot{\alpha}(t))=0$, and $(d/dt)(\omega(\dot{\alpha}(t)))\neq 0$, so we get Equation (188). □

Proposition 26. *Let $S\subset E^L_{\lambda_1,\lambda_2}$ be a regular spacelike surface. Let $\alpha: I\longrightarrow S$ be a spacelike C^2-smooth regular curve. Then,*

$$\kappa^{E,\infty,s}_{\alpha,S} = \frac{\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)+\bar{p}_1\dot{\alpha}_3(t)}{|\omega(\dot{\alpha}(t))|}, \text{ if } \omega(\dot{\alpha}(t))\neq 0,$$
$$\kappa^{E,\infty,s}_{\alpha,S} = 0, \text{ if } \omega(\dot{\alpha}(t))=0,$$
$$\frac{d}{dt}(\omega(\dot{\alpha}(t)))=0,$$
$$\lim_{L\longrightarrow+\infty}\frac{\kappa^{E,L,s}_{\alpha,S}}{\sqrt{L}} = \frac{(d/dt)(\omega(\dot{\alpha}(t)))}{\left|(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1+\left(\lambda_1\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right|^2}, \tag{199}$$

if $\omega(\dot{\alpha}(t))=0$, and $d/dt(\omega(\dot{\alpha}(t)))\neq 0$.

Proof. By Equations (176) and (188), we get the following:

$$J_L(\dot{\alpha}) = \frac{l_L}{l}L^{1/2}\omega(\dot{\alpha}(t))T_1 - \left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1 + \frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right]T_2. \tag{200}$$

By Equation (180) and the above equation, we have the following:

$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{S,L} \sim \left[\bar{q}_1\frac{\lambda_1\lambda_2^2\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right) + \bar{p}_1\dot{\alpha}_3(t)\right](\omega(\dot{\alpha}(t)))^2L^{3/2} \text{ as } L\longrightarrow+\infty. \tag{201}$$

So, we get the following:

$$\kappa^{E,L,s}_{\alpha,S} = \frac{\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{S,L}}{\|\dot{\alpha}\|^3_{S,L}} = \frac{L^{3/2}\left[\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)+\bar{p}_1\dot{\alpha}_3(t)\right](\omega(\dot{\alpha}(t)))^2}{L^{3/2}|\omega(\dot{\alpha}(t))|^3}. \tag{202}$$

Furthermore,

$$\kappa^{E,\infty,s}_{\alpha,S} = \lim_{L\longrightarrow+\infty}\kappa^{E,L,s}_{\alpha,S} = \frac{\bar{q}_1\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)+\bar{p}_1\dot{\alpha}_3(t)}{|\omega(\dot{\alpha}(t))|}. \tag{203}$$

When $\omega(\dot{\alpha}(t))=0$, and $(d/dt)(\omega(\dot{\alpha}(t)))=0$, we get the following:

$$\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{S,L} \sim M_1 L^{-1/2} \text{ as } L \longrightarrow +\infty, \tag{204}$$

where M_1 does not depend on L. So, $\kappa^{E,\infty,s}_{\alpha,S}=0$. When $\omega(\dot{\alpha}(t))=0$, and $(d/dt)(\omega(\dot{\alpha}(t)))\neq 0$, we have the following:

$$\begin{aligned}&\left\langle \nabla^{S,E,L}_{\dot{\alpha}}\dot{\alpha}, J_L(\dot{\alpha})\right\rangle_{S,L}\\ &\sim L^{1/2}\left[\frac{1}{\lambda_1\lambda_2}\dot{\alpha}_3(t)\bar{q}_1+\frac{\lambda_1\sqrt{2}}{2}\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right]\frac{d}{dt}(\omega(\dot{\alpha}(t)))\end{aligned} \tag{205}$$

as $L\longrightarrow +\infty$. We get the following:

$$\kappa^{E,\infty,s}_{\alpha,S}=\lim_{L\longrightarrow+\infty}\frac{\kappa^{E,L}_{\alpha,S}}{\sqrt{L}}=\frac{(d/dt)(\omega(\dot{\alpha}(t)))}{\left|(1/\lambda_1\lambda_2)\dot{\alpha}_3(t)\bar{q}_1+\left(\lambda_1\sqrt{2}/2\right)\left(-e^{-\alpha_3}\dot{\alpha}_1(t)+e^{\alpha_3}\dot{\alpha}_2(t)\right)\bar{p}_1\right|^2}. \tag{206}$$

□

In the following, we compute the intrinsic Gaussian curvature of spacelike surfaces in $E^L_{\lambda_1,\lambda_2}$. Similar to Theorem 4.3 in [16], we have the following:

Theorem 27. *The second fundamental form $II^{E,L}$ of the embedding of S into $E^L_{\lambda_1,\lambda_2}$ is given by the following:*

$$II^{E,L}=\begin{pmatrix} h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}, \tag{207}$$

where

$$\begin{aligned}h_{11}&=-\frac{l}{l_L}\left(E_1(\bar{p}_1)-E_2(\bar{q}_1)\right)-\lambda_1^2\bar{r}_{1,L}\bar{p}_1\bar{q}_1L^{-1/2},\\ h_{12}=h_{21}&=-\frac{l}{l_L}\langle T_1,\nabla_H(\bar{r}_{1,L})\rangle_L+\frac{\sqrt{L}}{2}\lambda_2^2+\frac{\lambda_1^2}{2\sqrt{L}}\left(\bar{q}^2_{1,L}+\bar{p}^2_{1,L}\right)-\frac{\bar{r}^2_{1,L}\lambda_1^2}{2\sqrt{L}}\left(\bar{p}_1{}^2+\bar{q}_1{}^2\right),\\ h_{22}&=-\frac{l^2}{l_L^2}\left\langle T_2,\nabla_H\left(\frac{r}{l}\right)\right\rangle_L+\widetilde{E}_3(\bar{r}_{1,L})+\frac{\bar{r}_{1,L}\bar{p}_{1,L}\bar{q}_{1,L}}{\sqrt{L}}\lambda_1^2-\frac{\bar{r}^3_{1,L}\bar{p}_1\bar{q}_1}{\sqrt{L}}\lambda_1^2.\end{aligned} \tag{208}$$

Proof. Since $\langle T_1,N_L\rangle_L=0, \langle T_2,N_L\rangle_L=0$, we have the following:

$$\begin{aligned}\left\langle \nabla^{E,L}_{T_1}N_L,T_1\right\rangle_L&=-\left\langle \nabla^{E,L}_{T_1}T_1,N_L\right\rangle_L,\left\langle \nabla^{E,L}_{T_2}N_L,T_2\right\rangle_L\\ &=-\left\langle \nabla^{E,L}_{T_2}T_2,N_L\right\rangle_L.\end{aligned} \tag{209}$$

Using the definition of the connection, the identities in Equation (11), and grouping terms, we have the following:

$$\begin{aligned}\nabla^{E,L}_{T_1}T_1&=\nabla^{E,L}_{\bar{q}_1E_1-\bar{p}_1E_2}\bar{q}_1E_1-\bar{p}_1E_2\\ &=\bar{q}_1\left(E_1\bar{q}_1E_1-E_1\bar{p}_1E_2-\bar{p}_1\nabla^{E,L}_{E_1}E_2\right)\\ &\quad-\bar{p}_1\left(E_2\bar{q}_1E_1-E_2\bar{p}_1E_2+\bar{q}_1\nabla^{E,L}_{E_2}E_1\right)\\ &=(\bar{q}_1E_1\bar{q}_1-\bar{p}_1E_2\bar{q}_1)E_1-(\bar{q}_1E_1\bar{p}_1-\bar{p}_1E_2\bar{p}_1)E_2\\ &\quad+\frac{\bar{p}_1\bar{q}_1\lambda_1^2}{L}E_3.\end{aligned} \tag{210}$$

Since $\bar{p}_1{}^2-\bar{q}_1{}^2=1$, we have $\bar{p}_1E_i\bar{p}_1-\bar{q}_1E_i\bar{q}_1=0, i=1,2,3$. We have the following:

$$\begin{aligned}h_{11}&=-\left\langle\nabla^{E,L}_{T_1}T_1,N_L\right\rangle_L=-\left[\bar{p}_{1,L}(\bar{q}_1E_1\bar{q}_1-\bar{p}_1E_2\bar{q}_1)\right.\\ &\quad\left.-\bar{q}_{1,L}(\bar{q}_1E_1\bar{p}_1-\bar{p}_1E_2\bar{p}_1)+\bar{r}_{1,L}\frac{\bar{p}_1\bar{q}_1\lambda_1^2}{\sqrt{L}}\right]\\ &=-\frac{l}{l_L}\left(E_1(\bar{p}_1)-E_2(\bar{q}_1)\right)-\lambda_1^2\bar{r}_{1,L}\bar{p}_1\bar{q}_1L^{-1/2}.\end{aligned} \tag{211}$$

Similarly, we have the following:

$$\begin{aligned}h_{12}&=-\frac{l}{l_L}\langle T_1,\nabla_H(\bar{r}_{1,L})\rangle_L+\frac{\sqrt{L}}{2}\lambda_2^2\\ &\quad+\frac{\lambda_1^2}{2\sqrt{L}}\left(\bar{q}^2_{1,L}+\bar{p}^2_{1,L}\right)-\frac{\bar{r}^2_{1,L}\lambda_1^2}{2\sqrt{L}}\left(\bar{p}_1{}^2+\bar{q}_1{}^2\right),\\ h_{21}&=-\frac{l}{l_L}\langle T_1,\nabla_H(\bar{r}_{1,L})\rangle_L+\frac{\sqrt{L}}{2}\lambda_2^2\\ &\quad+\frac{\lambda_1^2}{2\sqrt{L}}\left(\bar{q}^2_{1,L}+\bar{p}^2_{1,L}\right)-\frac{\bar{r}^2_{1,L}\lambda_1^2}{2\sqrt{L}}\left(\bar{p}_1{}^2+\bar{q}_1{}^2\right),\\ h_{22}&=-\frac{l^2}{l_L^2}\left\langle T_2,\nabla_H\left(\frac{r}{l}\right)\right\rangle_L+\widetilde{E}_3(\bar{r}_{1,L})\\ &\quad+\frac{\bar{r}_{1,L}\bar{p}_{1,L}\bar{q}_{1,L}}{\sqrt{L}}\lambda_1^2-\frac{\bar{r}^3_{1,L}\bar{p}_1\bar{q}_1}{\sqrt{L}}\lambda_1^2.\end{aligned} \tag{212}$$

□

Similar to Proposition 17, we get the expression of the horizontal mean curvature of the spacelike surface.

Proposition 28. *Away from characteristic point, the horizontal mean curvature $\mathscr{H}_{E,\infty}$ of $S\subset E^L_{\lambda_1,\lambda_2}$ is given by the following:*

$$\mathscr{H}_{E,\infty}=-E_1(\bar{p}_1)+E_2(\bar{q}_1). \tag{213}$$

Proposition 29. *Away from characteristic points, we have the following:*

$$\mathscr{K}^{S,E,\infty}(T_1, T_2) = -\left\langle T_1, \nabla_H\left(\frac{E_3F}{|\nabla_H F|}\right)\right\rangle_L - {\lambda_2}^4 \frac{(E_3F)^2}{l^2}. \quad (214)$$

Proof. By Equation (141) and Lemma 18, we have the following:

$$\begin{aligned}\mathscr{K}^{E,L}(T_1, T_2) &= -\left\langle R^{E,L}(T_1, T_2)T_1, T_2\right\rangle_L \\ &= -\bar{r}_{1,L}^2\left(\frac{\lambda_1^2\lambda_2^2}{2} - \frac{\lambda_1^4}{4L} + \frac{3\lambda_2^4 L}{4}\right) - \left(\frac{l}{l_L}\bar{q}_1\right)^2 \\ &\cdot \left(\frac{\lambda_1^2\lambda_2^2}{2} - \frac{\lambda_2^4 L}{4} + \frac{3\lambda_1^4}{4L}\right) - \left(\frac{l}{l_L}\bar{p}_1\right)^2 \\ &\cdot \left(\frac{\lambda_1^2\lambda_2^2}{2} + \frac{\lambda_1^4}{4L} + \frac{\lambda_2^4 L}{4}\right) \sim -\left(\frac{l}{l_L}\right)^2 \frac{\lambda_2^4 L}{4} \\ &- \frac{3\lambda_2^4}{4}\frac{(E_3u)^2}{l^2} - {\lambda_1}^2{\lambda_2}^2{\bar{q}_1}^2 \\ &- \frac{\lambda_1^2\lambda_2^2}{2} \text{ as } L \longrightarrow \infty.\end{aligned} \quad (215)$$

Similar to Equation (148), we obtain the following:

$$\begin{aligned}\det\left(II^{E,L}\right) = h_{11}h_{22} - h_{12}^2 &= -\frac{\lambda_2^4 L}{4} + \left\langle T_1, \nabla_H\left(\frac{E_3F}{|\nabla_H F|}\right)\right\rangle_L \\ &- \frac{\lambda_1^2\lambda_2^2}{2}\left({\bar{p}_1}^2 + {\bar{q}_1}^2\right) + O\left(L^{-1/2}\right) \text{ as } L \longrightarrow \infty.\end{aligned} \quad (216)$$

By Equations (215) and (216) and $\mathscr{K}^{S,E,L} = \mathscr{K}^{E,L} - \det(II^{E,L})$, we get the desired equation.

Similar to Lemma 20 and Proposition 21, for spacelike curve and spacelike surface, we obtain the following:

$$\lim_{L\longrightarrow\infty} \frac{1}{\sqrt{L}} ds_{E,L} = ds_E, \quad (217)$$

$$\lim_{L\longrightarrow\infty} \frac{1}{\sqrt{L}} d\sigma_{S,E,L} = d\sigma_{S,E}. \quad (218)$$

Combining Equations (214) and (217) and Proposition 26, similar to the proof of Theorem 22, we have the second Gauss-Bonnet theorem. □

Theorem 30. *Let $S \subset E^L_{\lambda_1,\lambda_2}$ be a regular spacelike surface with finitely many boundary components $(\partial S)_i, i \in \{1, \cdots, n\}$, given by C^2-smooth closed and regular spacelike curves $\alpha_i : [0, 2\pi] \longrightarrow (\partial S)_i$. Let $\mathscr{K}^{S,E,\infty}$ be intrinsic Gaussian curvature of S in Proposition 29 and $\kappa^{E,\infty,s}_{\alpha_i,S}$ the intrinsic signed geodesic curvature of α_i relative to S in Proposition 26. Suppose that the characteristic set $C(S)$ satisfies $\mathscr{H}^1(C(S)) = 0$ and that $\|\nabla_H F\|_H^{-1}$ is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set $C(S)$. Then,*

$$\int_S \mathscr{K}^{S,E,\infty} d\sigma_{S,E} + \sum_{i=1}^n \int_{\alpha_i} \kappa^{E,\infty,s}_{\alpha_i,S} ds_E = 0. \quad (219)$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interests in this work.

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