The Propagation of Thermoelastic Waves in Different Anisotropic Media Using Matricant Method

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1. Introduction

Thermodynamics of irreversible processes, developed in the last century, also made it possible to solve the problems of the irreversible deformation and gave a unified interpretation of mechanical and thermal processes [1, 2].

Similarly, Nowacki [3, 4] studied the harmonic wave propagation in a thermoelastic layer. Due to the weak coupling of the temperature and strain field, characterized by a thermal and mechanical parameter, the approximate frequency equation is solved by the perturbation method.

The poroelasticity equations formulated by Biot formed the basis for solving wave propagation problems in the poroelasticity region. Biot also presented the relationship between stress and strain in case of anisotropic poroelastic solid. Based on the Biot equations, Sharma [5, 6] investigated the reflection and transmission at the liquid and porous solid interface; however, a porous solid is considered anisotropic with arbitrary symmetry. Similar research has been carried out in the following areas: problems are considered and the solution of three differential equations by introducing the elastic and thermo-elastic potentials [7], the photothermal transport process [8], the effect of variable thermal conductivity of pPhotothermal diffusion (PTD) [9], an electromagnetothermoelastic coupled problem for a homogeneous, isotropic, thermally and electrically conducting half-space solid [10].

The wave propagation in the thermoelastic anisotropic medium have been investigated by using a matricant method, for instance wave propagation waves in liquid crystals and in thermoelastic medium [11–17]. In the paper [18], authors consider the propagation of Rayleigh surface waves in a
2. The Research Method

In this paper, we have used the matricant method [25], which allows to obtain accurate analytical solutions of differential equations describing the related processes in medium with piezoelectric, piezomagnetic, thermoelastic, and thermopiezoelectric properties.

This analytical study is based on the development of matrix methods for analyzing the dynamics of the elastic stratified medium.

The method deals with reducing the initial equations of motion by separation of variable method (representation of the solution in the form of plane waves) to the equivalent system of first order ordinary differential equations with variable coefficients and the constructing the matricant structure, i.e., normalized matrix of fundamental solutions.

The advantage of matricant method is that it allows formulating the wave propagation for wider class of medium. Moreover, another advantage of the method is that the expressions so obtained have a very compact form. This proves to be convenient both for analytical and for numerical calculations.

This method has been tested and the results obtained are in consistent with previously known results in various publications.

The main advantage of the matricant method is the uniform description of wave propagation in various effects for instance, thermoelastic, magnetoeleastic, piezoelectric, and piezomagnetic effects [26–28].

3. Basic Equation and Formulation of Problem

The study of the propagation of thermoelastic waves in anisotropic medium is based on the simultaneous solution of equations of motion in elastic medium [3, 4]:

\[
\frac{\partial \sigma_{XX}}{\partial X} + \frac{\partial \sigma_{XY}}{\partial Y} + \frac{\partial \sigma_{XZ}}{\partial Z} = \rho \frac{\partial^2 U_X}{\partial t^2},
\]

\[
\frac{\partial \sigma_{XY}}{\partial X} + \frac{\partial \sigma_{YY}}{\partial Y} + \frac{\partial \sigma_{YZ}}{\partial Z} = \rho \frac{\partial^2 U_Y}{\partial t^2},
\]

\[
\frac{\partial \sigma_{XZ}}{\partial X} + \frac{\partial \sigma_{YZ}}{\partial Y} + \frac{\partial \sigma_{ZZ}}{\partial Z} = \rho \frac{\partial^2 U_Z}{\partial t^2}.
\]

The equations of heat conductivity proposed by Fourier in case of anisotropic medium is as follows:

\[
\lambda_{ij} \frac{\partial \theta}{\partial x_j} = -q_i,
\]

and the inflow equation without the influence of heat source is given by

\[
\frac{\partial q_i}{\partial x_j} = -i \omega \beta_{ij} \varepsilon_{ij} - i \omega \frac{c_i}{T_0} \theta,
\]

where \(\sigma_{ij}\) represents the components of stress tensor, \(\rho\) is the density of medium, \(\lambda_{ij}\) is the components of the heat conductivity tensor, \(q_i\) are the components of the heat flow vector, \(\omega\) is the angular frequency, \(\beta_{ij}\) are the thermomechanical parameters of medium, \(\varepsilon_{ij}\) are the components of the tensor of small Cauchy deformation, \(c_i\) is the heat capacity under constant deformation, and \(\theta = T - T_0\) is the temperature augments compared with natural state temperature \(T_0\) (\(T_0\) is the temperature of the natural state without deformations). For the case when the deformation is small, \(|\theta/T_0||\ll 1\).

The equations between stress and strain can be described by Duhamel–Neumann relationships as

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta,
\]

where \(c_{ijkl}\) is the elastic constants, \(a = ij, \beta = kl\) and \(\beta_{ij}\) is the thermomechanical parameters of the medium.

Here, equations (1)–(6) show that the relationship between temperature and stress generated in a mechanical process as a function of the heat field and deformation in a medium, whereas they are independent variables.
For the monoclinic system, the matrix of elastic constants $c_{ijkl}$ can be written as

$$
c_{ijkl} = \begin{pmatrix}
  c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16}
  c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26}
  c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36}
  0 & 0 & 0 & c_{44} & c_{45} & 0
  0 & 0 & 0 & c_{54} & c_{55} & 0
  c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66}
\end{pmatrix}.
$$

(7)

The thermomechanical parameters of the body are $\beta_{ij}$, and they depend on both mechanical and heat properties of the body and for an anisotropic medium of a monoclinic system they are given as follows:

$$
\beta_{ij} = \begin{pmatrix}
  \beta_{11} & \beta_{12} & \beta_{13} \\
  \beta_{21} & \beta_{22} & 0 \\
  \beta_{31} & 0 & \beta_{33}
\end{pmatrix}.
$$

(8)

By using separation of variables method, equations (1)–(6) can be reduced to a system of ordinary differential equations, where the heterogeneity of medium is assumed to be along $Z$ axis, i.e., axis $Z||A_2$ where $A_2$ is the second order symmetry axis

$$
d\tilde{W} = B\tilde{W},
$$

(9)

where vector $\tilde{W}$ has the form

$$
\tilde{W}(x, y, z, t) = \left[ u_x(z), u_y(z), u_y(z), u_y(z), \alpha_{zz}, \alpha_{zz}, \beta, \theta, q_z \right]^T \exp(i\omega t - i mx - iny).
$$

(10)

Here, $\tilde{W}$ is a column vector, which includes the boundary conditions of the problem: $u_x(z)$, $u_y(z)$, and $u_y(z)$ represent the projection of displacement vector on the corresponding coordinates, and $m = k_x$, $n = k_y$, and $l = k_z$ shows the $x$, $y$, and $z$ components of a wave vector $k$, respectively.

The coefficients matrix is given as

$$
B = B \left[ c_{ijkl}(z), \beta_{ij}(z), \theta, \omega, m, n, l \right],
$$

(11)

it shows the functional dependence of matrix $B$, for example as $f = f(x, y, z, t)$.

Earlier in work [12], a system of differential equations (9) describing the propagation of coupled elastic and thermal waves in anisotropic medium of rhombic, tetragonal, and hexagonal syngony was formulated.

In the monoclinic system, there is a specific direction or a designated plane, or both. Since the direction determines the plane perpendicular to it so direction is chosen along the horizontal plane. All three axes can be of arbitrary length. An anisotropic monoclinic medium is characterized by a second-order symmetry axis. If the $z||A_2$ inhomogeneity depends on $z$, then the structure of the matrix $B$ in (9) will have the following form:

$$
B = 
\begin{pmatrix}
  0 & b_{12} & b_{13} & 0 & b_{15} & 0 & b_{17} & 0 \\
  b_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  b_{24} & 0 & 0 & 0 & b_{34} & b_{35} & 0 & 0 \\
  0 & 0 & b_{43} & 0 & 0 & b_{45} & 0 & b_{47} \\
  0 & b_{26} & 0 & 0 & b_{36} & b_{37} & 0 & 0 \\
  0 & b_{26} & 0 & 0 & b_{46} & b_{47} & b_{57} & 0 \\
  0 & b_{15} & b_{26} & 0 & b_{65} & b_{67} & b_{77} & b_{77} \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

(12)

where $b_{ij}$ represents the components of coefficient matrix in case of monoclinic syngony and are given as follows:

$$
\begin{align*}
  b_{12} &= \frac{1}{c_{33}}; b_{13} = \frac{c_{34}}{c_{33}} im; b_{15} = \frac{c_{36}}{c_{33}} im; b_{17} = \beta_{33} = \frac{c_{33}}{c_{33}}; \\
  b_{21} &= -\omega^2 \rho; b_{24} = im; b_{26} = in; b_{34} = \frac{c_{44}}{c_{33}} im; b_{35} = \frac{c_{45}}{c_{33}} im; b_{46} = \frac{c_{46}}{c_{33}}; b_{47} = \beta_{47} = \frac{c_{44}}{c_{33}}; \\
  b_{43} &= \frac{c_{44}}{c_{33}} im; b_{45} = \frac{c_{45}}{c_{33}} im; b_{65} = \frac{c_{65}}{c_{33}} im; b_{67} = \frac{c_{67}}{c_{33}} im; b_{77} = \frac{c_{77}}{c_{33}} im; b_{78} = \frac{c_{78}}{c_{33}} im; \\
  b_{45} &= \frac{c_{46}}{c_{33}} im + \frac{c_{45}}{c_{33}} im; b_{47} = \frac{c_{47}}{c_{33}} im; b_{57} = \frac{c_{57}}{c_{33}} im; b_{67} = \frac{c_{67}}{c_{33}} im; b_{77} = \frac{c_{77}}{c_{33}} im; b_{78} = \frac{c_{78}}{c_{33}} im; \\
\end{align*}
$$

(13)

In this paper, the propagation of thermal waves in an anisotropic medium of monoclinic, trigonal, hexagonal, and cubic crystal system have been considered in the presence of even order symmetry axis.
4. Solution of the Problem

The equations of motion as given by (1) for the case of longitudinal elastic wave propagating along one of spatial coordinates in an anisotropic layer can be written as

$$\frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 U_z}{\partial t^2},$$

(14)

where $\sigma_z = c_{33}(\partial U_z/\partial z)$ is the z-component of the stress tensor $\sigma_{ij}$, $\rho$ is the medium density, $U_z$ is the z-component of the displacement vector of medium, and $c_{33}$ is the isotropical elastic modulus.

By using separation of variables method, we get in case of harmonic waves:

$$[U_z; \sigma_z] = [U_\omega(z), \sigma_\omega(z)] e^{i\omega t}. \quad (15)$$

The system of equations (1)–(6) is reduced to a system of differential equations of second order, describing the propagation of harmonic waves (9).

The result is system of first order differential equation (9):

$$\frac{dU_z}{dz} = \frac{1}{c_{33}} \sigma_z, \quad \frac{d\sigma_z}{dz} = -\omega^2 \rho U_z,$$

$$\Longrightarrow \frac{d}{dz} \begin{pmatrix} U_z \\ \sigma_z \end{pmatrix} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix} \begin{pmatrix} U \\ \sigma \end{pmatrix}. \quad (16)$$

Condition for the existence of nontrivial solutions is the vanishing of the following determinant [19]:

$$\det |B - \lambda E| = 0,$$

(17)

where $B$ represents coefficient matrix whose elements contain the parameters of the medium, in which an elastic longitudinal wave propagates. The elements of this matrix are contained in (16) and have the following form:

$$b_{12} = \frac{1}{c_{33}}; \quad b_{21} = -\omega^2 \rho,$$

(18)

this results in obtaining the characteristic equation (17):

$$\lambda^2 = \pm i \omega \sqrt{\frac{\rho}{c_{33}}} \quad (19)$$

The last relation leads to the conclusion that the wave spectrum is equal to:

$$k_{1,2} = \pm i \omega \sqrt{\frac{\rho}{c_{33}}}. \quad (20)$$

This problem can be solved as follows:

$$\phi = Ae^{i k_1 z} + Be^{i k_2 z} \Rightarrow \phi = Ae^{i \omega \sqrt{\rho/c_{33}} z} + Be^{-i \omega \sqrt{\rho/c_{33}} z}. \quad (21)$$

Let us take the abovementioned as an example to consider the heat wave propagation in an anisotropic medium of the monoclinic synony.

Let us assume that harmonic thermal expansion waves with the angular frequency $\omega$ occur in an unlimited thermoelastic medium.

The one-dimension equation of heat conductivity is as follows:

$$c_t \frac{\partial \theta}{\partial t} = \lambda_{33} \frac{\partial^2 \theta}{\partial z^2}, \quad (22)$$

This can be written in matrix form as follows:

$$\frac{d}{dz} \begin{pmatrix} \theta \\ q_z \end{pmatrix} = \begin{pmatrix} 0 & b_{78} \\ b_{87} & 0 \end{pmatrix} \begin{pmatrix} \theta \\ q_z \end{pmatrix}, \quad (23)$$

where $c_t$ shows heat capacity at constant strain, $\theta = T - T_0$ temperature increase compared to the temperature $T_0$ of the natural state, $\lambda_{33}$ is heat conductivity tensor, and $q_z$ represents components of heat vector [2].

The coefficients of the matrix in (23) have the form:

$$b_{78} = \frac{1}{\lambda_{33}}; \quad b_{87} = -i \omega c_t. \quad (24)$$

In this case, the characteristic equation (17) can be represented as follows:

$$\delta^2 - i \omega \frac{c_t}{\lambda_{33}} = 0, \quad (25)$$

hence, it follows that

$$\delta_{1,2} = \pm \sqrt{\frac{i \omega}{a}}, \quad (26)$$

where $a = \lambda/c_t \rho$ is coefficient of heat conductivity, $\lambda$ represents coefficient of thermal conductivity, $c_t$ is specific heat capacity, and $\rho$ shows density of the substance.

The roots of (26) can be represented as follows:

$$\delta_1 = \sqrt{\frac{\omega}{a}} e^{i (\pi/4)}, \quad \delta_2 = \sqrt{\frac{\omega}{a}} e^{i (\pi/4) + \pi}, \quad \delta_1 \delta_2 = \frac{\omega}{2a}(1 + i), \quad (27)$$

$$\delta_2 = -\sqrt{\frac{\omega}{2a}(1 + i)}. \quad (28)$$
the real and the imaginary parts:

\[ \text{Table 1} \]

<table>
<thead>
<tr>
<th>Substance</th>
<th>( \lambda ), coefficient of thermal conductivity (W/(m*K))</th>
<th>( \epsilon_p ), specific heat capacity (J/(kg*K))</th>
<th>( \rho ), density (kg/m(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartz</td>
<td>11.3</td>
<td>750</td>
<td>2650</td>
</tr>
<tr>
<td>Calcite</td>
<td>4.98</td>
<td>800</td>
<td>2710</td>
</tr>
<tr>
<td>Bismuth</td>
<td>6.65</td>
<td>123.5</td>
<td>9750</td>
</tr>
<tr>
<td>Graphite</td>
<td>89</td>
<td>840</td>
<td>2150</td>
</tr>
<tr>
<td>Aluminum</td>
<td>208</td>
<td>897</td>
<td>2700</td>
</tr>
<tr>
<td>Copper</td>
<td>410</td>
<td>385</td>
<td>8950</td>
</tr>
</tbody>
</table>

Subtraction (27) from (28), results in

\[ \delta_2 - \delta_1 = -\sqrt{\frac{\omega}{2a}}(1 + i), \] (29)

then

\[ \delta_1 = -\delta_2 \implies \delta_2 = -\delta_1. \] (30)

The solution of the heat wave propagating in case of one dimension is as follows:

\[ T_m = B - \frac{\delta_2 E}{\delta_1 - \delta_2} e^{\delta_1 z} + B - \frac{\delta_1 E}{\delta_2 - \delta_1} e^{\delta_2 z}. \] (31)

In papers [12, 13], the general form a matricant structure is similar to the (31) and the exact solution of the system of differential equations (9), which describes the propagation of thermoelastic waves in anisotropic medium was formulated. The system of differential equations is as described (23). The formulation of exact solution is given by (31).

Numerator on the right side of (31), by using (29) and (30), becomes

\[ \frac{B - \delta_1 E}{2\delta_2} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{i\omega \epsilon_p}{2\sqrt{2\omega/a}} & \frac{1}{2} \end{pmatrix}. \] (32)

Coefficients of matrix \( B \) can be represented as follows:

\[ B = \begin{pmatrix} \frac{1}{2} & 0 \\ \omega \epsilon_p & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{4\sqrt{2\omega/a}} \\ \omega \epsilon_p & 0 \end{pmatrix}. \] (33)

Consequently, the coefficient matrix \( B \) is separated into the real and the imaginary parts:

\[ B = \text{Re} B + \text{Im} B, \] (34)

this corresponds to heat wave propagation in a solid medium.

For the general case, considering the above relations, the solution of equation (31) can be represented as follows:

\[ T_m = \text{Re} Be^{-\sqrt{\omega/a}z} \cos \sqrt{\frac{\omega}{2a}} + \text{Im} Be^{-\sqrt{\omega/a}z} \sin \sqrt{\frac{\omega}{2a}}. \] (35)

The solution of the heat wave propagation problem in the one-dimensional case coincides with the classic solution, which is as follows [20]:

\[ f = e^{-\sqrt{\omega/a}z} \cos (\omega t - \sqrt{\omega/a}z). \] (36)

For physical reasons, from the two roots \( \delta_1, \delta_2 \), it is necessary to retain the root, which includes the negative real part.

Consequently, the solution for the heat wave is obtained as follows:

\[ \theta = \theta_0 e^{-\sqrt{\omega/a}z} \cos \left( t - \frac{z}{\sqrt{2\omega a}} \right), \] (37)

where \( \nu = \sqrt{2\omega a} \) is the phase velocity and it depends on frequency of the heat wave.

Expression (21) is a purely elastic plane harmonic wave propagating along \( z \)-axis. This wave has neither damping nor dispersion. Expression (37) corresponds to a purely thermal plane harmonic wave, which has an attenuation characterized by the coefficient \( q = \sqrt{\omega/2a} \), and variance due to the fact that the phase velocity is a function of frequency: \( \nu = \sqrt{2\omega a} \).

The attenuation coefficient and the phase velocity of the heat wave have the form \( q = \sqrt{\omega/2a} \) and \( \nu = \sqrt{2\omega a} \), respectively. The coefficient of thermal conductivity is expressed by the ratio \( a = \lambda/c_p\rho \). Consider the following substances and their parameters as given in Table 1 [24].

It can be seen from the Figure 1 that \( q \) and \( \nu \) depend on \( \omega \) in the same way, because \( q \sim \sqrt{\omega} \) and \( \nu \sim \sqrt{\omega} \). It can also be seen that the increase in \( q \) and \( \nu \), depending on the increase in \( \omega \), increase according to a parabolic law.
Figure 1: Continued.
In this paper, the propagation of elastic longitudinal and thermal waves in anisotropic medium of monoclinic, trigonal, hexagonal, and cubic crystal systems is considered on the basis of the matrix method. In particular, the problem of heat wave propagation in the one-dimensional case is solved, the solution of which coincides with the known classical solution.

Moreover, by using the matricant method the solutions of equations of wave propagation in elastic medium are obtained. From these solutions, it is possible to determine the attenuation coefficient and phase velocity of the thermal waves. Finally, the results obtained by the matricant method are in consistent with the models of poroelastic equations obtained by using another analytical solution [5, 6]. It is expected that the results obtained will be helpful for better understanding the thermoelastic wave propagation in

5. Conclusion

In this paper, the propagation of elastic longitudinal and thermal waves in anisotropic medium of monoclinic, trigonal, hexagonal, and cubic crystal systems is considered on the basis of the matrix method. In particular, the problem of heat wave propagation in the one-dimensional case is solved, the solution of which coincides with the known classical solution.
various mediums. In this paper, we got the dependence of the attenuation coefficient and the phase velocity of the heat wave on the angular frequency in different mediums. We analyzed these dependencies.

**Data Availability**

Data is available on request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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