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Research Article

An Improved Version of Residual Power Series Method for Space-Time Fractional Problems

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The task of present research is to establish an enhanced version of residual power series (RPS) technique for the approximate solutions of linear and nonlinear space-time fractional problems with Dirichlet boundary conditions by introducing new parameter λ. The parameter λ allows us to establish the best numerical solutions for space-time fractional differential equations (STFDE). Since each problem has different Dirichlet boundary conditions, the best choice of the parameter λ depends on the problem. This is the major contribution of this research. The illustrated examples also show that the best approximate solutions of various problems are constructed for distinct values of parameter λ. Moreover, the efficiency and reliability of this technique are verified by the numerical examples.

1. Introduction

The well-established tool fractional differential equations (FDEs) attract growing attention of scientists in a wide range of scientific areas such as biology, physics, and engineering since it is very effective and accurate for modelling many processes in real life [1–11].

In FDEs, noninteger derivatives are taken into account to include memory of the system. Fractional mathematical models have results which are very close to the experimental data [12]. Since FDEs are applicable and variable, the progress of them is very rapid in diverse fields of science and engineering. Therefore, studies on solving FDEs produce efficient and reliable techniques to construct numerical and analytical solutions of them [13–20].

Diverse methods such as a semianalytical approach, a modified wavelet approach, linearized novel operational matrix-based scheme, hybrid spectral linearized scheme, finite difference/spectral algorithm, innovative operational matrix-based computational scheme, and a Chelyshkov polynomial-based algorithm are utilized to construct the solutions of FDEs [20–22].

RPS technique, which can be applied directly to the problem, is one of the most convenient ones to establish approximate solutions with high precision for linear and nonlinear FDEs [21–28]. Moreover, RPS technique allows us to construct the exact solution of the initial value problems whose solutions are polynomial.

Various mathematical models including differential equations such as model of vibration equation of large membranes, fractional Black–Scholes option pricing equations, and time–fractional fuzzy vibration equation of large membranes are analyzed by the RPS technique [27, 28].

Modified versions of the RPS technique are developed to acquire better approximate solutions of FDEs such as the solutions of STFDE in [29–34].

The motivation of this study is to employ a new version of RPS technique to establish better approximate solutions of fractional mathematical models including STFDE in Caputo sense. In the RPS technique, determining a suitable estimation for the initial guess approximations plays an important role. Therefore, in this study, we try to construct best initial guess approximation by introducing a new parameter. The novelty of this method is that new parameter λ is introduced to get the best approximate solution for the fractional mathematical problems including STFDE with Dirichlet boundary conditions. When we are given Dirichlet boundary conditions, we utilize the parameter λ to determine the initial rate of change which enables us to acquire the best approximate solution by the RPS technique.

2. Preliminaries

Now, the basic concepts of fractional calculus are presented [1–5].

Definition 1. The Riemann-Liouville (R-L) fractional derivative operator D^α of order α is described as [2, 5]

$$D^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\dfrac{d}{dt^n}\displaystyle\int_0^t\dfrac{f(x)}{(t-x)^{\alpha-n+1}}dx, & n-1<\alpha<n,\\ \dfrac{\partial^n}{\partial t^n}f(t), & \alpha=n,\end{cases}\tag{1}$$

where $n\in Z^+$ and $\alpha\in R^+$.

Definition 2. The R-L fractional-order integration operator J^α is described as [2, 5]

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-x)^{\alpha-1}f(x)dx,\quad x>0,\alpha>0.\tag{2}$$

Following Podlubny [2], we may have

$$\begin{aligned}J^\alpha t^n&=\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1}t^{n+\alpha},\\ D^\alpha t^n&=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1}t^{n-\alpha}.\end{aligned}\tag{3}$$

Definition 3. The α^{th}-order derivative of $f(t)$ in Caputo sense is defined as [2, 5]

$$^{C}D^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\dfrac{f^n(x)}{(t-x)^{\alpha-n+1}}dx, & n-1<\alpha<n,\\ \dfrac{\partial^n f(t)}{\partial t^n}, & \alpha=n.\end{cases}\tag{4}$$

Definition 4 (see [2, 5]).

(1) $D^\alpha J^\alpha f(t)=f(t)$

(2) $J^\alpha D^\alpha f(t)=f(t)-\sum_{k=0}^n f^{(k)}(0^+)(t^k/k!),\ t>0$

Table 1: The approximate values of $u_{31}(x,t)$ for various values of λ at $\alpha=1$ and $\beta=2$.

x	t	Exact	$\lambda=-1$	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
	0.3	0.21893	0.01731	0.10942	0.20153	0.29364	0.56997
0.3	0.6	0.16218	0.01214	0.08038	0.14862	0.21686	0.42157
	0.9	0.12015	0.00611	0.05667	0.10722	0.15777	0.30942
	0.3	0.41830	0.03307	0.20906	0.38505	0.56104	1.08902
0.6	0.6	0.30988	0.02320	0.15358	0.28396	0.41434	0.80548
	0.9	0.22957	0.01168	0.10827	0.20486	0.30144	0.59120
	0.3	0.58030	0.04587	0.29003	0.53418	0.77833	1.51080
0.9	0.6	0.42990	0.03219	0.21306	0.39394	0.57481	1.11743
	0.9	0.31848	0.01621	0.15020	0.28420	0.41819	0.82017

Definition 5. (k,l)-truncated series $u_{kl}(x,t)$ of the RPS method is given as

$$\begin{aligned}u_{kl}(x,t)&=\sum_{n=0}^{i-1}\frac{a_n(x)}{n!}+\sum_{n=1}^{k}\sum_{m=0}^{l}f_{nm}(x)\frac{t^{n\alpha+m}}{\Gamma(1+n\alpha+m)},\quad n-1<\alpha\le n,t>0,\\ u_{kl}(x,t)&=\sum_{n=0}^{i-1}\frac{b_n(t)}{n!}+\lambda\sum_{n=1}^{k}\sum_{m=0}^{l}(g_{nm}(t)-g_{nm-1}(t))\frac{x^{n\beta+m}}{\Gamma(1+n\beta+m)},\quad n-1<\beta\le n,x>0.\end{aligned}\tag{5}$$

3. RPS Technique for Fractional Mathematical Models with Dirichlet Boundary Conditions

We provide an efficient and reliable method for the solution of STFDE with Dirichlet boundary conditions. Let us take the following initial-boundary value problem into consideration:

$$D_t^\alpha u=D_x^\beta u+F(u),\tag{6}$$

$$u(x,0)=f_0(x),\tag{7}$$

$$\begin{aligned}u(0,t)&=g_0(t),\\ u(1,t)&=g_1(t).\end{aligned}\tag{8}$$

The $(k,0)$-truncated series $u_{k0}(x,t)$ is established by taking Equations (6) and (7) into account as follows:

$$u_{k0}(x,t)=f_{00}(x)+\sum_{n=1}^{k}f_{n0}(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)},\quad t>0,0<\alpha\le 1.\tag{9}$$

The use of Equations (1) and (3) leads to the kl-truncated series $u_{kl}(x,t)$ as follows:

$$u_{kl}(x,t)=g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x+\lambda\sum_{n=1}^{k}\sum_{m=1}^{l}(g_{nm}(t)-g_{nm-1}(t))\cdot\frac{x^{n\beta+m}}{\Gamma(1+n\beta+m)},\quad x>0,1<\beta\le 2.\tag{10}$$

Table 2: The approximate values of $u_{31}(x,t)$ with the parameter $\lambda = 1.22$ and various orders α and β.

x	t	Exact	$\alpha=1$ $\beta=2$	$\alpha=1$ $\beta=1.9$	$\alpha=1$ $\beta=1.8$	$\alpha=0.9$ $\beta=2$	$\alpha=0.9$ $\beta=1.9$	$\alpha=0.9$ $\beta=1.8$
	0.3	0.21893	0.22179	0.22642	0.23048	0.21694	0.22205	0.22659
0.3	0.6	0.16218	0.16363	0.17129	0.17831	0.16014	0.16806	0.17524
	0.9	0.12015	0.11834	0.12841	0.13752	0.11581	0.12663	0.13603
	0.3	0.41830	0.42377	0.42683	0.42944	0.41489	0.41853	0.42175
0.6	0.6	0.30988	0.31264	0.31944	0.32588	0.30551	0.31252	0.31909
	0.9	0.22957	0.22611	0.23596	0.24531	0.22045	0.23117	0.24104
	0.3	0.58030	0.58789	0.58576	0.58351	0.57654	0.57492	0.57319
0.9	0.6	0.42990	0.43373	0.43516	0.43667	0.42266	0.42409	0.42558
	0.9	0.31848	0.31368	0.31807	0.32267	0.30382	0.30869	0.31383

An approximation for (6)–(8) is achieved by employing the new RPS technique as follows:

$$u(x,t)=\frac{1}{2}\left(f_{00}(x)+g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x+\sum_{n=1}^{\infty}f_{n0}(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}+\sum_{n=1}^{\infty}g_{n0}(t)\frac{x^{n\beta}}{\Gamma(1+n\beta)}+\lambda\sum_{n=1}^{\infty}(g_{n1}(t)-g_{n0}(t))\frac{x^{n\beta+1}}{\Gamma(2+n\beta)}\right). \quad (11)$$

Notice that we introduce a new parameter λ in this approximate solutions to get a better approximation of the problems (6)–(8). Hence, the (k,l)-truncated series of $u_{kl}(x,t)$ of $u(x,t)$ is achieved as follows:

$$u_{kl}(x,t)=\frac{1}{2}\left(f_{00}(x)+g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x+\sum_{n=1}^{k}f_{n0}(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}+\sum_{n=1}^{k}g_{n0}(t)\frac{x^{n\beta}}{\Gamma(1+n\beta)}+\lambda\sum_{n=1}^{k}\sum_{m=1}^{l}(g_{nm}(t)-g_{nm-1}(t))\frac{x^{n\beta+m}}{\Gamma(1+n\beta+m)}\right). \quad (12)$$

Equation (13) leads to the approximate solution $u_{00}(x,t)$ to become

$$u_{00}(x,t)=\frac{1}{2}(f_{00}(x)+g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x). \quad (13)$$

The RPS technique leads to the following definition of the kl residual function $\operatorname{Re}s_{kl}(x,t)$ as

$$\operatorname{Re}s_{kl}(x,t)=D_t^{\alpha}u_{kl}-D_x^{\beta}u_{kl}-F(u_{kl}). \quad (14)$$

In order to establish the coefficients $g_{ij}(t), i,j=1,2,3,\cdots,k$, plunge Equation (12) into Equation (14), and solve the following equation

$$D_x^{(k-1)\beta}D_x^{l}\operatorname{Re}s_{kl}(0,t)=0. \quad (15)$$

To acquire $g_{10}(t)$, taking $k=1$ and $l=0$ in Equation (14)

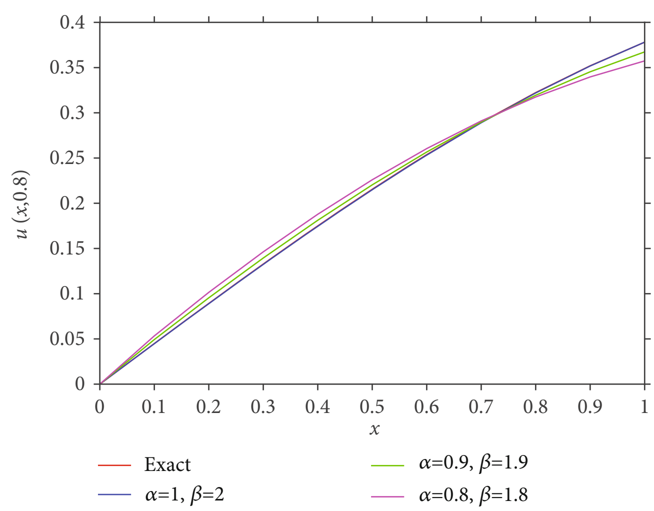

Figure 1: The graphics of approximate solution $u_{31}(x,t)$ at $t=0.8$ for $\lambda=1.22$ and various orders α and β.

leads to

$$\operatorname{Re}s_{10}(x,t)=f_{10}(x)+D_t^{\alpha}g_{00}(t)+\lambda(D_t^{\alpha}g_{01}(t)-D_t^{\alpha}g_{00}(t))x-D_x^{\beta}f_{00}(x)-D_x^{\beta}f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}-g_{10}(t)-F(u_{10}), \quad (16)$$

where

$$u_{10}(x,t)=\frac{1}{2}\left(f_{00}(x)+g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x+f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+g_{10}(t)\frac{x^{\beta}}{\Gamma(1+\beta)}\right). \quad (17)$$

Employing Equation (15) allows us to have

$$g_{10}(t)=f_{10}(0)+D_t^{\alpha}g_{00}(t)-D_x^{\beta}f_{00}(0)-D_x^{\beta}f_{10}(0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}-F(u_{10}(0,t)). \quad (18)$$

TABLE 3: The approximate values of $u_{21}(x,t)$ for various values of λ at $\alpha = 1$ and $\beta = 2$.

x	t	Exact	$\lambda=-1$	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
	0.3	-0.53847	-0.03648	-0.26301	-0.48954	-0.71608	-1.39567
0.3	0.6	-0.98116	-0.01907	-0.43146	-0.84385	-1.25624	-2.49342
	0.9	-1.78779	0.09664	-0.65310	-1.40284	-2.15258	-4.40181
	0.3	-1.02885	-0.06983	-0.50253	-0.93524	-1.36794	-2.66605
0.6	0.6	-1.87468	-0.03878	-0.82438	-1.60998	-2.39558	-4.75238
	0.9	-3.41589	0.17085	-1.24786	-2.66657	-4.08528	-8.34142
	0.3	-1.42731	-0.09709	-0.69716	-1.29723	-1.89730	-3.69751
0.9	0.6	-2.60074	-0.05891	-1.14366	-2.22840	-3.31315	-6.56740
	0.9	-4.73885	0.20655	-1.73115	-3.66885	-5.60655	-11.41964

TABLE 4: The approximate values of $u_{21}(x,t)$ with the parameter $\lambda = 1.47$ for various orders α and β.

x	t	Exact	$\alpha=1$ $\beta=2$	$\alpha=1$ $\beta=1.9$	$\alpha=1$ $\beta=1.8$	$\alpha=0.9$ $\beta=2$	$\alpha=0.9$ $\beta=1.9$	$\alpha=0.9$ $\beta=1.8$
	0.3	-0.53847	-0.59601	-0.60551	-0.61355	-0.62399	-0.63650	-0.64718
0.3	0.6	-0.98116	-1.03768	-1.06509	-1.08905	-1.08716	-1.12080	-1.15039
	0.9	-1.78779	-1.75522	-1.80729	-1.85298	-1.81910	-1.87986	-1.93342
	0.3	-1.02885	-1.13861	-1.14310	-1.14629	-1.18552	-1.19207	-1.19710
0.6	0.6	-1.87468	-1.97921	-1.99778	-2.01391	-2.06547	-2.08925	-2.11030
	0.9	-3.41589	-3.33337	-3.36702	-3.39611	-3.44429	-3.48526	-3.52129
	0.3	-1.42731	-1.57926	-1.56996	-1.56013	-1.62855	-1.61841	-1.60792
0.9	0.6	-2.60074	-2.73824	-2.72758	-2.71682	-2.83788	-2.82787	-2.81811
	0.9	-4.73885	-4.57957	-4.55689	-4.53486	-4.70650	-4.68487	-4.66435

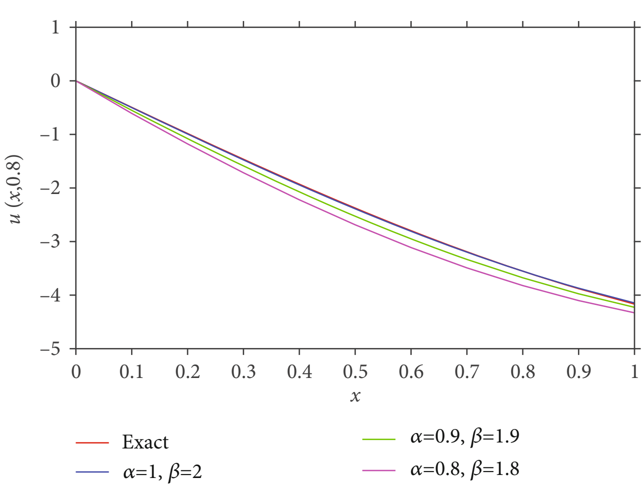

FIGURE 2: The graphics of approximate solution $u_{21}(x,t)$ at $t = 0.8$ and $\lambda = 1.47$ for various orders α and β.

Similarly, we take $k = 2$ and $l = 0$ in Equation (14) to establish the coefficient $g_{20}(t)$,

$$\operatorname{Re} s_{20}(x,t) = D_t^\alpha u_{20} - D_x^\beta u_{20} - F(u_{20}), \tag{19}$$

where

$$u_{20}(x,t) = \frac{1}{2}\left(f_{00}(x) + g_{00}(t) + \lambda(g_{01}(t) - g_{00}(t))x + f_{10}(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_{20}(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + g_{10}(t)\frac{x^\beta}{\Gamma(1+\beta)} + g_{20}(t)\frac{x^{2\beta}}{\Gamma(1+2\beta)}\right). \tag{20}$$

Taking Equation (15) into account leads to

$$g_{20}(t) = D_x^\beta f_{10}(0) + D_x^\beta f_{20}(0)\frac{t^\alpha}{\Gamma(1+\alpha)} + D_t^\alpha g_{10} - D_x D_x^\beta f_{00}(0) - D_x D_x^\beta f_{10}(0)\frac{t^\alpha}{\Gamma(1+\alpha)} - D_x D_x^\beta f_{20}(0)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - D_x^\beta F(u_{20}(0,t)). \tag{21}$$

As in the previous steps, taking $k = 1$ and $l = 1$ in Equation (14) leads to

$$\operatorname{Re} s_{11}(x,t) = D_t^\alpha u_{11} - D_x^\beta u_{11} - F(u_{11}), \tag{22}$$

Table 5: The approximate values of $u_{21}(x,t)$ for various values of λ at $\alpha=1$ and $\beta=2$.

x	t	Exact	$\lambda=-1$	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=5$
0.1	0.1	0.25000	-0.00069	0.12385	0.24736	0.36982	0.73096
	0.2	0.33333	-0.01045	0.15568	0.31996	0.48238	0.95855
	0.3	0.50000	-0.05424	0.19548	0.44103	0.68242	1.38158
	0.4	1.00000	-0.26021	0.24325	0.73005	1.20018	2.51057
0.2	0.1	0.50000	0.00045	0.24683	0.48487	0.71458	1.35370
	0.2	0.66667	-0.01960	0.30944	0.62366	0.92307	1.73240
	0.3	1.00000	-0.10992	0.38784	0.85226	1.28336	2.37663
	0.4	2.00000	-0.54568	0.48203	1.37640	2.13744	3.62055
0.3	0.1	0.75000	0.00527	0.36804	0.70268	1.00920	1.75999
	0.2	1.00000	-0.02614	0.45936	0.89486	1.28036	2.13685
	0.3	1.50000	-0.16847	0.57396	1.20389	1.72132	2.59862
	0.4	3.00000	-0.88166	0.71184	1.85534	2.54885	1.92936
0.4	0.1	1.00000	0.01561	0.48661	0.89095	1.22861	1.84161
	0.2	1.33333	-0.02877	0.60352	1.11729	1.51255	1.98720
	0.3	2.00000	-0.23134	0.75072	1.46612	1.91484	1.66103
	0.4	4.00000	-1.29343	0.92821	2.08319	2.17149	-3.96359

where

$$u_{11}(x,t)=\frac{1}{2}\left(f_{00}(x)+g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x+f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+g_{10}(t)\frac{x^{\beta}}{\Gamma(1+\beta)}+\lambda(g_{11}(t)-g_{10}(t))\frac{x^{\beta+1}}{\Gamma(2+\beta)}\right). \quad (23)$$

Using Equation (15) when $k=1$ and $l=1$, we get

$$g_{11}(t)=g_{10}(t)+D_t^{\alpha}g_{01}-D_t^{\alpha}g_{00}+\frac{1}{\lambda}\left(D_xf_{10}(0)-D_xD_x^{\beta}f_{00}(0)-D_xD_x^{\beta}f_{10}(0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}-D_xF(u_{11}(0,t))\right). \quad (24)$$

To determine $g_{21}(t)$, substituting $k=2$ and $l=1$ into Equation (14), then

$$\operatorname{Re}s_{21}(x,t)=D_t^{\alpha}u_{21}-D_x^{\beta}u_{21}-F(u_{21}), \quad (25)$$

where

$$u_{21}(x,t)=\frac{1}{2}\left(f_{00}(x)+g_{00}(t)+\lambda(g_{01}(t)-g_{00}(t))x+f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_{20}(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+g_{10}(t)\frac{x^{\beta}}{\Gamma(1+\beta)}+g_{20}(t)\frac{x^{2\beta}}{\Gamma(1+2\beta)}+\lambda(g_{11}(t)-g_{10}(t))\frac{x^{\beta+1}}{\Gamma(2+\beta)}+\lambda(g_{21}(t)-g_{20}(t))\frac{x^{2\beta+1}}{\Gamma(2+2\beta)}\right). \quad (26)$$

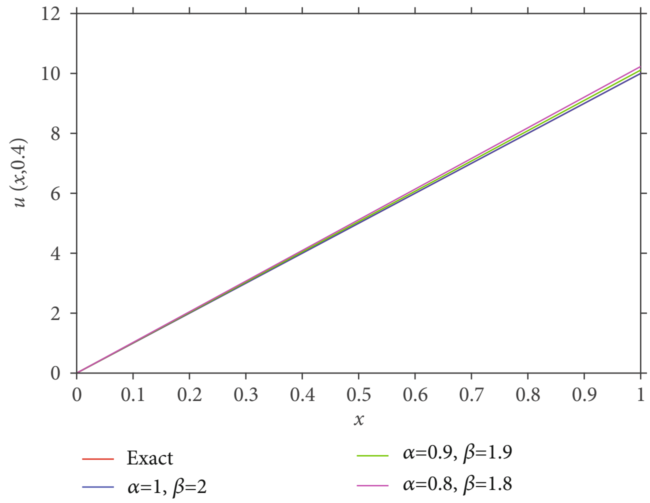

Figure 3: The graphics of approximate solution $u_{21}(x,t)$ at $t=0.8$ and $\lambda=1.16$ for various orders α and β.

Using Equation (15) when $k=2$ and $l=1$, we get

$$g_{21}(t)=g_{20}(t)+D_t^{\alpha}g_{11}-D_t^{\alpha}g_{10}+\frac{1}{\lambda}\left(D_x^{\beta}D_xf_{10}(0)-D_x^{\beta}D_xf_{20}(0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}-D_x^{\beta}D_xD_x^{\beta}f_{00}(0)-D_x^{\beta}D_xD_x^{\beta}f_{10}(0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}-D_x^{\beta}D_xD_x^{\beta}f_{20}(0)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}-D_x^{\beta}D_xF(u_{21}(0,t))\right), \quad (27)$$

and so on.

Now, the next step is to obtain the coefficients $f_{i0}(x), i=1,2,3,\cdots,k$, by Equation (9).

The equation

$$D_t^{(k-1)\alpha}\operatorname{Re}s_{k0}(x,0)=0 \quad (28)$$

allows us to establish the required coefficients $f_{ij}(x), i,j=1,2,3,\cdots,k$. Taking into $k=1$ in Equation (14) leads to

$$\operatorname{Re}s_{10}(x,t)=D_t^{\alpha}u_{10}-D_x^{\beta}u_{10}-F(u_{10}), \quad (29)$$

where

$$u_{10}(x,t)=f_{00}(x)+f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad (30)$$

which allows us to determine the coefficients $f_{10}(x)$ as follows:

$$f_{10}(x)=D_x^{\beta}f_{00}(x)+F(u_{10}(x,0)). \quad (31)$$

In a similar form, $f_{20}(x)$ is acquired by substituting $k=2$ into Equation (14) then

$$\operatorname{Re}s_{20}(x,t)=D_t^{\alpha}u_{20}-D_x^{\beta}u_{20}-F(u_{20}), \quad (32)$$

Table 6: The approximate values of $u_{21}(x,t)$ with the parameter $\lambda=1.16$ for various orders α and β.

x	t	Exact	$\alpha=1$ $\beta=2$	$\alpha=1$ $\beta=1.9$	$\alpha=1$ $\beta=1.8$	$\alpha=0.9$ $\beta=2$	$\alpha=0.9$ $\beta=1.9$	$\alpha=0.9$ $\beta=1.8$
0.1	0.1	0.25000	0.26702	0.26618	0.26499	0.27667	0.27579	0.27454
	0.2	0.33333	0.34607	0.34468	0.34272	0.36510	0.36365	0.36160
	0.3	0.50000	0.47993	0.47735	0.47371	0.50879	0.50614	0.50241
	0.4	1.00000	0.80639	0.79890	0.78833	0.84529	0.83781	0.82726
0.2	0.1	0.50000	0.52218	0.51696	0.51010	0.54089	0.53541	0.52822
	0.2	0.66667	0.67256	0.66395	0.65263	0.70971	0.70069	0.68884
	0.3	1.00000	0.92348	0.90747	0.88643	0.98027	0.96386	0.94229
	0.4	2.00000	1.50712	1.46065	1.39956	1.58512	1.53873	1.47776
0.3	0.1	0.75000	0.75361	0.73882	0.72018	0.78022	0.76470	0.74514
	0.2	1.00000	0.95990	0.93549	0.90473	1.01331	0.98774	0.95554
	0.3	1.50000	1.29424	1.24887	1.19170	1.37713	1.33062	1.27201
	0.4	3.00000	1.99655	1.86481	1.69883	2.11404	1.98255	1.81688
0.4	0.1	1.00000	0.94945	0.91897	0.88172	0.98220	0.95022	0.91114
	0.2	1.33333	1.18850	1.13820	1.07672	1.25540	1.20273	1.13835
	0.3	2.00000	1.55583	1.46234	1.34806	1.66207	1.56622	1.44907
	0.4	4.00000	2.16900	1.89754	1.56576	2.32658	2.05563	1.72447

where

$$u_{20}(x,t)=f_{00}(x)+f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_{20}(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{33}$$

Hence, we have the coefficient $f_{20}(x)$ in the following form:

$$f_{20}(x)=D_x^{\beta}f_{10}(x)+D_t^{\alpha}F(u_{20}(x,0)), \tag{34}$$

and so on. By substituting the coefficients $f_{ij}(x), i,j=1,2,3,\cdots,k$, into Equations (18), (21), (24), and (27) lead to the coefficients $g_{ij}(t), i,j=1,2,3,\cdots,k$.

As a verification of the new RPS technique, we observe that we obtain as a classical exact solution of the problems (6)–(8) with integer order as $\beta\longrightarrow 2$ and $\alpha\longrightarrow 1$.

4. Convergence Analysis

Theorem 6. *Let $u(x,t), D_t^{k\alpha}u(x,t)\in C[R_1,t_0]\times[R_1,t_0+R_2]$, and $D_x^{k\beta}u(x,t)\in C[x_0,R_2]\times[x_0+R_2,R_2]$, where $k=0,1,2,\cdots,p_1+1, j=0,1,2,\cdots,n-1$, and $l=0,1,2,\cdots,p_2+1, i=0,1,2,\cdots,m-1$. Moreover, $D_t^{k\alpha}u(x,t)$ can be differentiated $n-1$ times with respect to $'t'$ on (t_0,t_0+R_1) and $D_x^{l\beta}u(x,t)$ can be differentiated $m-1$ times with respect to $'x'$ on (x_0, x_0+R_2). Then,*

$$u(x,t)\cong\frac{1}{2}\sum_{j=0}^{n-1}\sum_{i=0}^{p_1}F_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}+\frac{\lambda}{2}\sum_{j=0}^{m-1}\sum_{i=0}^{p_2}G_{j+i\beta}(t)(x-x_0)^{j+i\beta}, \tag{35}$$

where $F_{j+i\alpha}(x)=(D_t^{j+i\alpha}/(\Gamma(j+i\alpha+1))\Gamma(j+i\alpha+1))u(x,t_0)$ and $G_{j+i\beta}(t)=(D_x^{j+i\beta}/(\Gamma(j+i\beta+1))\Gamma(j+i\beta+1))u(x_0,t)$.

Moreover, for $\exists$ values ε_1 and ε_2 such that $0\le\varepsilon_1\le t$ and $0\le\varepsilon_2\le x$, the error term has the term as follows:

$$\|E_p(x,t)\|\le\frac{1}{2}\left[\sup_{t\in[0,T]}\left|\sum_{j=0}^{n-1}\left(\frac{D_t^{j+(p_1+1)\alpha}}{\Gamma(j+(p_1+1)\alpha+1)}u(x,\varepsilon_1)t^{j+(p_1+1)\alpha}\right)\right|+\inf_{\lambda\in[1,2]}\sup_{x\in[0,L]}\left|\lambda\sum_{j=0}^{m-1}\left(\frac{D_x^{j+(p_2+1)\beta}}{\Gamma(j+(p_2+1)\beta+1)}u(\varepsilon_2,t)x^{j+(p_2+1)\beta}\right)\right|\right]. \tag{36}$$

Proof. The proof of this theorem follows from the proof 4.3 [27]. □

5. Numerical Examples

In this section, we provide illustrative examples to prove how effective and reliable the RPS technique proposed in this study is.

Example 1. Consider the STFDE with initial-boundary conditions

$$\begin{gathered}D_t^{\alpha}u=D_x^{\beta}u,\\ u(x,0)=\sin(x),\\ u(0,t)=0,\\ u(1,t)=\exp(-t)\sin(1),\end{gathered} \tag{37}$$

for which

$$u(x,t) = \exp(-t)\sin(x) \tag{38}$$

represents the exact solution for $\alpha = 1$ and $\beta = 2$. Based on the RPS technique and above problem, we have the following coefficients:

$$\begin{gathered}
f_{00}(x) = \sin(x),\\
g_{00}(t) = 0,\\
g_{01}(t) = \exp(-t)\sin(1),\\
f_{10}(x) = \frac{1}{2i}\left(\sum_{k=0}^{\infty} i^{k+2}\frac{x^{k+2-\beta}}{\Gamma(k+3-\beta)} - \sum_{k=0}^{\infty}(i)^{k+2}\frac{x^{k+2-\beta}}{\Gamma(k+3-\beta)}\right),\\
g_{10}(t) = 0,\\
g_{11}(t) = \sin(1)\left(\sum_{k=0}^{\infty}(-1)^{k+1}\frac{t^{k+1-\alpha}}{\Gamma(k+2-\alpha)}\right),\\
f_{20}(x) = \frac{1}{2i}\left(\sum_{k=0}^{\infty} i^{k+4}\frac{x^{k+4-2\beta}}{\Gamma(k+5-2\beta)} - \sum_{k=0}^{\infty}(-i)^{k+4}\frac{x^{k+4-2\beta}}{\Gamma(k+5-2\beta)}\right),\\
g_{20}(t) = 0,\\
g_{21}(t) = \sin(1)\left(\sum_{k=0}^{\infty}(-1)^{k+2}\frac{t^{k+2-2\alpha}}{\Gamma(k+3-2\alpha)}\right),\\
f_{30}(x) = \frac{1}{2i}\left(\sum_{k=0}^{\infty} i^{k+6}\frac{x^{k+6-3\beta}}{\Gamma(k+7-3\beta)} - \sum_{k=0}^{\infty}(-i)^{k+6}\frac{x^{k+6-3\beta}}{\Gamma(k+7-3\beta)}\right),\\
g_{30}(t) = 0,\\
g_{31}(t) = \sin(1)\left(\sum_{k=0}^{\infty}(-1)^{k+3}\frac{t^{k+3-3\alpha}}{\Gamma(k+4-3\alpha)}\right).
\end{gathered} \tag{39}$$

In Table 1, the approximate solutions are given for various values of λ with $\alpha = 1$ and $\beta = 2$. Table 2 provides the values of exact solution and an approximate solution $u_{31}(x, t)$ for the parameter $\lambda = 1.22$ and various orders of α and β. Based on Table 1, we conclude that approximate solutions have a higher accuracy as adding more components. Moreover, Figure 1 supports our conclusion.

Example 2. Take the initial-boundary value problem including STFDE with source function into consideration

$$\begin{gathered}
D_t^{\alpha}u = D_x^{\beta}u + 3u,\\
u(x,0) = \cos\left(x + \frac{\pi}{2}\right),\\
u(0,t) = 0,\ u(1,t) = \exp(2t)\cos\left(1 + \frac{\pi}{2}\right),
\end{gathered} \tag{40}$$

for which

$$u(x,t) = \exp(2t)\cos\left(x + \frac{\pi}{2}\right) \tag{41}$$

denotes the exact solution for $\alpha = 1$ and $\beta = 2$. Based on the RPS technique and above problem, the following coefficients are obtained:

$$\begin{gathered}
f_{00}(x) = \cos\left(x + \frac{\pi}{2}\right),\\
g_{00}(t) = 0,\\
g_{01}(t) = \exp(2t)\cos\left(1 + \frac{\pi}{2}\right),\\
f_{10}(x) = \frac{1}{2i}\left(\sum_{k=0}^{\infty}(-i)^{k+2}\frac{x^{k+2-\beta}}{\Gamma(k+3-\beta)} - \sum_{k=0}^{\infty} i^{k+2}\frac{x^{k+2-\beta}}{\Gamma(k+3-\beta)}\right) + 3\cos\left(x + \frac{\pi}{2}\right),\\
g_{10}(t) = 0,\\
g_{11}(t) = -3\cos\left(1 + \frac{\pi}{2}\right)\exp(2t) + \cos\left(1 + \frac{\pi}{2}\right)\sum_{k=0}^{\infty} 2^{k+1}\frac{t^{k+1-\alpha}}{\Gamma(k+2-\alpha)},\\
f_{20}(x) = \frac{1}{2i}\left(\sum_{k=0}^{\infty} i^{k+2}\frac{x^{k+4-2\beta}}{\Gamma(k+5-2\beta)} - \sum_{k=0}^{\infty}(-i)^{k+2}\frac{x^{k+4-2\beta}}{\Gamma(k+5-2\beta)}\right) + \frac{6}{2i}\left(\sum_{k=0}^{\infty}(-i)^{k+2}\frac{x^{k+2-\beta}}{\Gamma(k+3-\beta)} - \sum_{k=0}^{\infty} i^{k+2}\frac{x^{k+2-\beta}}{\Gamma(k+3-\beta)}\right) + 3\cos\left(x + \frac{\pi}{2}\right) + 9\cos\left(x + \frac{\pi}{2}\right),\\
g_{20}(t) = 0,\\
g_{21}(t) = 9\cos\left(1 + \frac{\pi}{2}\right)\exp(2t) - 6\cos\left(1 + \frac{\pi}{2}\right)\sum_{k=0}^{\infty} 2^{k+1}\frac{t^{k+1-\alpha}}{\Gamma(k+2-\alpha)} + \cos\left(1 + \frac{\pi}{2}\right)\sum_{k=0}^{\infty} 2^{k+2}\frac{t^{k+2-2\alpha}}{\Gamma(k+3-2\alpha)}.\\
\vdots
\end{gathered} \tag{42}$$

In Table 3, the approximate solution is given for various values of λ with $\alpha = 1$ and $\beta = 2$. Table 4 provides the values of exact solution and the approximation $u_{21}(x, t)$ with the parameter $\lambda = 1.47$ and various orders α and β. Based on Table 3, we conclude that the approximate solution has a higher accuracy as adding more components. Moreover, Figure 2 confirms our conclusion.

Example 3. Consider the initial-boundary value problem including fractional Burger equation

$$\begin{gathered}
D_t^{\alpha}u = D_x^{\beta}u + uu_x,\\
u(x,0) = 2x,\\
u(0,t) = 0,\ u(1,t) = \frac{2}{1-2t},
\end{gathered} \tag{43}$$

for which

$$u(x,t)=\frac{2x}{1-2t} \tag{44}$$

represents the exact solution for integer order derivatives. Based on the RPS technique and above problem, the following coefficients are established:

$$\begin{aligned} f_{00}(x)&=2x,\\ g_{00}(t)&=0,\\ g_{01}(t)&=\frac{2}{1-2t},\\ f_{10}(x)&=4x,\\ g_{10}(t)&=0,\\ f_{20}(x)&=16x\\ g_{20}(t)&=0. \end{aligned} \tag{45}$$

In Table 5, the approximate solution is given for various values of λ with $\alpha=1$ and $\beta=2$. Table 6 provides the values of exact solution and the approximation $u_{21}(x,t)$ with the parameter $\lambda=1.16$ and various orders α and β. Based on Table 5, we conclude that the approximate solution has a higher accuracy as adding more components. Moreover, our conclusion is supported by Figure 3.

6. Conclusion

In this study, the enhanced version of RPS technique is proposed to establish better approximate solutions of linear and nonlinear space-time fractional problems with Dirichlet boundary conditions by introducing new parameter λ. The illustrated examples present that the best approximate solutions are achieved for specific values of the parameter λ. Moreover, the value of parameter λ which leads to the best approximate solution depends on the fractional mathematical problem. The numerical examples also prove that this new effective and reliable technique is easy to implement.

Data Availability

There is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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