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Research Article

Single and Multivalued Maps on Parametric Metric Spaces Endowed with an Equivalence Relation

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This article presents the E-parametric metric space, which is a generalized concept of parametric metric space. After that, the discussion is concerned with the existence of fixed points of single and multivalued maps on E-parametric metric spaces satisfying some contractive inequalities defined by an auxiliary function.

1. Introduction and Preliminaries

Metric fixed point theory is a very fruitful area of research belonging to nonlinear analysis and operator theory. It has much usage in other fields of mathematics. Numerous generalizations of Banach contraction principle have flourished this field in several aspects through using generalized forms of metric spaces or contraction conditions. It is not wrong to say that the contraction condition given by Salimi et al. in [1] is the most renowned condition of this decade, whereas on the account of metric spaces, we have many abstract forms like b-metric spaces [2–5], partial metric spaces [6, 7], and partial b-metric spaces [8, 9]. Recently, the notions of R-metric [10] and parametric metric spaces [11] have been initiated.

In the following, we recall the concept of parametric metric spaces [11] and the convergence of sequences.

A map $P_m : H \times H \times (0,\infty) \longrightarrow [0,\infty)$ (where $H \neq \varnothing$) is said to be a parametric metric on H if it satisfies the following axioms:

(i) $P_m(\ell, \ell', \curlyvee) = 0$ for all $\curlyvee>0 \Longleftrightarrow \ell = \ell'$

(ii) $P_m(\ell, \ell', \curlyvee) = P(\ell', \ell, \curlyvee)$ for all $\curlyvee>0$

(iii) $P_m(\ell, \ell', \curlyvee) \leq P_m(\ell, \ell'', \curlyvee) + P_m(\ell'', \ell', \curlyvee)$ for all $\ell, \ell', \ell'' \in H$ and for all $\curlyvee>0$

Then, (H, P_m) is a parametric metric space. If (H, P_m) be a parametric metric space and $\{\ell_n\} \subseteq H$, then

(1) $\ell_n \longrightarrow \ell \in H \Longleftrightarrow \lim_{n\longrightarrow\infty} P_m(\ell_n, \ell, \curlyvee) = 0$ for all $\curlyvee>0$

(2) $\{\ell_n\}$ is Cauchy if $\lim_{m,n\longrightarrow\infty} P_m(\ell_m, \ell_n, \curlyvee) = 0$ for all $\curlyvee>0$

Note that (H, P_m) is complete if each Cauchy sequence in H converges in H.

One of the ways to generalize the contraction condition is to define it via some auxiliary functions. For example, an auxiliary type function was given in [12], where Q_ξ represents the collection of functions $Q : (\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+ = [0,\infty)$ satisfying the below axioms:

(i) Q is nondecreasing and continuous with respect to its each coordinate

(ii) If $q_1, q_2 \in \mathbb{R}^+$ with $q_1 < q_2$ and $q_1 \leq Q(q_2, q_2, q_1, q_2)$, then $q_1 \leq \xi(q_2)$;

(iii) If $q_1, q_2 \in \mathbb{R}^+$ with $q_1 \geq q_2$ and $q_1 \leq Q(q_1, q_2, q_1, q_1)$, then $q_1 = 0$

(iv) If $q \in \mathbb{R}^+$ with $q \leq Q(0, 0, q, (1/2)q)$ or $q \leq Q(0, q, 0, (1/2)q)$ or $q \leq Q(q, 0, 0, q)$, then $q = 0$

and $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing map with $\sum_{n=1}^{\infty} \xi^n(t) < \infty$ for all $t \geq 0$ and $\xi(t) < t$ for all $t > 0$.

As examples of $Q \in Q_\xi$, we cite the following:

(i) Let $Q_1(q_1, q_2, q_3, q_4) = k \max \{q_1, q_2, q_3, q_4\}$ with $\xi(t) = kt$, where $k \in [0, 1)$.

(ii) Let $Q_2(q_1, q_2, q_3, q_4) = kq_4$ with $\xi(t) = kt$, where $k \in [0, 1)$.

(iii) Let $Q_3(q_1, q_2, q_3, q_4) = a^* q_1 + b^* q_2 + c^* q_4$ with $\xi(t) = (a^* + b^* + c^*)t$, where $a^*, b^*, c^* \geq 0$ and $a^* + b^* + c^* \in [0, 1)$.

Khalehoghli et al. [10] took a binary relation E on H and a simple metric d on H to define an R-metric space, denoted as (H, d, B_R). They also said that a self map T on H is a B_R-contraction, if

$$d\left(T\ell, T\ell'\right) \leq kd\left(\ell, \ell'\right), \tag{1}$$

for all $\ell, \ell' \in H$ with $\ell B_R \ell'$, where $0 < k < 1$. Along with this definition, the authors also defined the concepts of B_R-continuity and B_R-preserving property in order to extend the result of Banach on R-metric spaces.

2. Main Results

Let $H \neq \varnothing$ and E be an equivalence relation on H. We introduce the following: a function $P_m : H \times H \times (0,\infty) \longrightarrow [0, \infty)$ is an E-parametric-metric on H, provided that the following axioms hold:

(i) $P_m(\ell, \ell', \curlyvee) = 0 \Longleftrightarrow \ell = \ell'$ for all $\ell \sim \ell'$ and for all $\curlyvee > 0$

(ii) $(P_m(\ell, \ell', \curlyvee) = P_m(\ell', \ell, \curlyvee)$ for all $\curlyvee > 0$ and for all $\ell \sim \ell'$

(iii) $P_m(\ell, \ell', \curlyvee) \leq P_m(\ell, \ell'', \curlyvee) + P_m(\ell'', \ell', \curlyvee)$ for all $\ell, \ell', \ell'' \in H$, for all $\curlyvee > 0$ and for all $\ell \sim \ell' \sim \ell''$

Then, (H, E, P_m) is an E-parametric metric space.

Example 1. Take H as the collection of all functions $g : (0, \infty) \longrightarrow R$. Consider an equivalence relation E on H defined by gEf if $t | f(t) - g(t)$ for all $t \in (0,\infty)$ (note that | means that t divides $f(t) - g(t)$). Define $P_m : H \times H \times (0,\infty) \longrightarrow [0,\infty)$ by

$$P_m(g, f, \curlyvee) = |g(\curlyvee) - f(\curlyvee)|, \text{ for all } g, f \in H \text{ and for all } \curlyvee > 0. \tag{2}$$

Then, (H, E, P_m) is an E-parametric metric space.

Below, (H, E, P_m) stands for an E-parametric metric space.

Definition 1. Let $\{\kappa_m\}$ be an E-sequence in (H, E, P_m), that is, $\kappa_m \sim \kappa_{m+1}$ for each $m \in \mathbb{N}$. Then, we say that

(i) $\{\kappa_m\}$ is a convergent sequence in H and converges to $\kappa \in H$ if $\lim_{m\longrightarrow\infty} P_m(\kappa_m, \kappa, \curlyvee) = 0$ for all $\curlyvee > 0$ and $\kappa_m \sim \kappa$ for each $m \geq k$ (for some value of k)

(ii) $\{\kappa_m\}$ is Cauchy if $\lim_{m,n\longrightarrow\infty} P_m(\kappa_m, \kappa_n, \curlyvee) = 0$ for all $\curlyvee > 0$.

Definition 2. (H, E, P_m) is E-complete if each Cauchy E-sequence in H is convergent in H.

Definition 3. (H, E, P_m) is called regular whenever if $P_m(\ell, \ell', \curlyvee^0) = 0$ for some $\curlyvee^0 > 0$, then we have $P_m(\ell, \ell', \curlyvee) = 0$ for all $\curlyvee$.

We now present the first main theorem of the section.

Theorem 4. *Let (H, E, P_m) be an E-complete regular E-parametric metric space. Let $T : H \longrightarrow H$ and $\alpha : H \times H \longrightarrow [0,\infty)$ be two maps such that*

$$\alpha\left(\ell, \ell'\right) P_m\left(T\ell, T\ell', \curlyvee\right) \leq Q\left(P_m\left(\ell, \ell', \curlyvee\right), P_m(\ell, T\ell, \curlyvee), P_m\left(\ell', T\ell', \curlyvee\right), \frac{P_m\left(\ell', T\ell, \curlyvee\right) + P_m\left(\ell, T\ell', \curlyvee\right)}{2}\right), \tag{3}$$

for all $\curlyvee > 0$ and for all $\ell, \ell' \in H$ with $\ell \sim \ell'$, where $Q \in Q_\xi$. Also, consider the following assertions:

(a) T is E-preserving, that is, $\ell \sim \ell'$ implies $T\ell \sim T\ell'$

(b) There is $\ell_0 \in H$ such that $\ell_0 \sim T\ell_0$ and $\alpha(\ell_0, T\ell_0) \geq 1$

(c) For every $\ell, \ell' \in H$ such that $\ell \sim \ell'$ and $\alpha(\ell, \ell') \geq 1$, we have $\alpha(T\ell, T\ell') \geq 1$

(d) For each E-sequence $\{\ell_m\}$ in H with $\alpha(\ell_m, \ell_{m+1}) \geq 1$ for all $m \in \mathbb{N}$ and $\ell_m \longrightarrow \ell$, we have $\alpha(\ell_m, \ell) \geq 1$ for all $m \in \mathbb{N}$

Then, T has a fixed point.

Proof. By (b), there must be $\kappa_0 \in H$ of the form $\kappa_0 \sim T\kappa_0$ and $\alpha(\kappa_0, T\kappa_0) \geq 1$. Since T is E-preserving, we have $T\kappa_0 \sim T^2\kappa_0$, and by (c), $\alpha(T\kappa_0, T^2\kappa_0) \geq 1$. The repetition of these steps yields that $T^m\kappa_0 \sim T^{m+1}\kappa_0$ and $\alpha(T^m\kappa_0, T^{m+1}\kappa_0) \geq 1$ for all $m \in \mathbb{N}$. Take a sequence $\{\kappa_m\}$ with the terms as $\kappa_m = T^m\kappa_0$ for all $m \in \mathbb{N}$. Thus, $\kappa_m \sim \kappa_{m+1}$ and $\alpha(\kappa_m, \kappa_{m+1}) \geq 1$ for all $m \in \mathbb{W} = \mathbb{N} \cup \{0\}$. Before doing the next part of the proof, we take $\kappa_m \neq \kappa_{m+1}$ for all $m \in \mathbb{W}$. From (3), for each $m \in \mathbb{W}$, we get

$$\begin{aligned} P_m(T\kappa_m, T\kappa_{m+1},\curlyvee) &\leq \alpha(\kappa_m, \kappa_{m+1}) P_m(T\kappa_m, T\kappa_{m+1},\curlyvee) \\ &\leq Q\left(P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_m, T\kappa_m,\curlyvee), P_m(\kappa_{m+1}, T\kappa_{m+1},\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, T\kappa_m,\curlyvee) + P_m(\kappa_m, T\kappa_{m+1},\curlyvee))\right) \\ &= Q\left(P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_{m+1}, \kappa_{m+2},\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_{m+1},\curlyvee) + P_m(\kappa_m, \kappa_{m+2},\curlyvee))\right) \\ &\leq Q\left(P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_{m+1}, \kappa_{m+2},\curlyvee), \frac{1}{2}(P_m(\kappa_m, \kappa_{m+1},\curlyvee) + P_m(\kappa_{m+1}, \kappa_{m+2},\curlyvee))\right), \end{aligned} \tag{4}$$

for all $\curlyvee>0$.□ □

Here, our claim is, for each $\curlyvee>0$, $\{P_m(\kappa_m, \kappa_{m+1},\curlyvee)\}_{m\in\mathbb{W}}$ is a strictly decreasing sequence. Suppose this is not true in general, then we have at least one $m_0 \in \mathbb{W}$ such that $P_m(\kappa_{m_0}, \kappa_{m_0+1}, \curlyvee^0) \leq P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0)$ for some $\curlyvee^0 > 0$. Then, from (4) and property (i) of Q_ξ, we get

$$\begin{aligned} P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0) &= P_m(T\kappa_{m_0}, T\kappa_{m_0+1}, \curlyvee^0) \\ &\leq Q\left(P_m(\kappa_{m_0}, \kappa_{m_0+1}, \curlyvee^0), P_m(\kappa_{m_0}, \kappa_{m_0+1}, \curlyvee^0), P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0), \frac{1}{2}(P_m(\kappa_{m_0}, \kappa_{m_0+1}, \curlyvee^0) + P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0))\right) \\ &\leq Q(P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0), P_m(\kappa_{m_0}, \kappa_{m_0+1}, \curlyvee^0), P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0), P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0)). \end{aligned} \tag{5}$$

Property (iii) of Q_ξ together with the above inequality implies that

$$P_m(\kappa_{m_0+1}, \kappa_{m_0+2}, \curlyvee^0) = 0, \text{for} \curlyvee^0 > 0. \tag{6}$$

This fact and regularity condition give us

$$P_m(\kappa_{m_0+1}, \kappa_{m_0+2},\curlyvee) = 0, \text{for all} \curlyvee>0, \tag{7}$$

which is not possible under the assumption. Thus, $P_m(\kappa_m, \kappa_{m+1},\curlyvee) > P_m(\kappa_{m+1}, \kappa_{m+2},\curlyvee)$ for all $\curlyvee>0$ and $m \in \mathbb{W}$. Again, we use (4) and property (i) of Q_ξ to obtain

$$\begin{aligned} P_m(\kappa_{m+1}, \kappa_{m+2},\curlyvee) &= P_m(T\kappa_m, T\kappa_{m+1},\curlyvee) \\ &\leq Q(P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m \\ &\quad \cdot (\kappa_{m+1}, \kappa_{m+2},\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee)), \end{aligned} \tag{8}$$

for all $\curlyvee>0$ and $m \in \mathbb{W}$.

Property (ii) of Q_ξ together with (8) implies

$$P_m(\kappa_{m+1}, \kappa_{m+2},\curlyvee) \leq \xi(P_m(\kappa_m, \kappa_{m+1},\curlyvee)), \text{for all} \curlyvee>0 \text{ and } m \in \mathbb{W}. \tag{9}$$

After a few simplification, we get

$$P_m(\kappa_m, \kappa_{m+1},\curlyvee) \leq \xi^m(P_m(\kappa_0, \kappa_1,\curlyvee)), \text{for all} \curlyvee>0 \text{ and } m \in \mathbb{N}. \tag{10}$$

Hence, $\{\kappa_m\}$ is a sequence in H with $\kappa_m \sim \kappa_{m+1}$ and $P_m(\kappa_m, \kappa_{m+1},\curlyvee) \leq \xi^m(P_m(\kappa_0, \kappa_1,\curlyvee))$ for all $\curlyvee>0$ and $m \in \mathbb{N}$. Now, we will show that this E-sequence is also Cauchy. For each $n > m \in \mathbb{N}$, we can obtain

$$P_m(\kappa_m, \kappa_n,\curlyvee) \leq \sum_{j=m}^{n-1} P_m(\kappa_j, \kappa_{j+1},\curlyvee) \leq \sum_{j=m}^{\infty} \xi^j(P_m(\kappa_0, \kappa_1,\curlyvee)), \text{for all} \curlyvee>0. \tag{11}$$

The convergence of $\sum_{j=1}^{\infty} \xi^j(P_m(\kappa_0, \kappa_1,\curlyvee))$ for each $\curlyvee>0$ implies that $\{\kappa_m\}$ is a Cauchy E-sequence in (H, E, P_m).

The E-completeness of (H, E, P_m) leads to the existence of a point $\kappa_* \in H$ with $\lim_{m\longrightarrow\infty} P_m(\kappa_m, \kappa_*, \curlyvee) = 0$ for all $\curlyvee>0$ and $\kappa_m \sim \kappa_*$ for each $m \geq k$, for some k. By (d), we also conclude that $\alpha(\kappa_m, \kappa_*) \geq 1$ for each $m \geq k$. Now, from (3), we get

$$\begin{aligned} P_m(T\kappa_m, T\kappa_*,\curlyvee) &\leq \alpha(\kappa_m, \kappa_*) P_m(T\kappa_m, T\kappa_*,\curlyvee) \\ &\leq Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, T\kappa_m,\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_*, T\kappa_m,\curlyvee) + P_m(\kappa_m, T\kappa_*,\curlyvee))\right) \\ &= Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*,\curlyvee) + P_m(\kappa_m, T\kappa_*,\curlyvee))\right), \end{aligned} \tag{12}$$

for all $\curlyvee>0$.

By triangle property and (12), we get

$$\begin{aligned} P_m(\kappa_*, T\kappa_*,\curlyvee) &\leq P_m(\kappa_*, \kappa_{m+1},\curlyvee) + P_m(\kappa_{m+1}, T\kappa_*,\curlyvee) = P_m(\kappa_*, \kappa_{m+1},\curlyvee) + P_m(T\kappa_m, T\kappa_*,\curlyvee) \\ &\leq P_m(\kappa_*, \kappa_{m+1},\curlyvee) + Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*,\curlyvee) + P_m(\kappa_m, T\kappa_*,\curlyvee))\right) \\ &\leq P_m(\kappa_*, \kappa_{m+1},\curlyvee) + Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*,\curlyvee) + P_m(\kappa_m, \kappa_*,\curlyvee) + P_m(\kappa_*, T\kappa_*,\curlyvee))\right), \end{aligned} \tag{13}$$

for all $\curlyvee>0$.

Applying the limit when $m \longrightarrow \infty$ in (13), we get

$$P_m(\kappa_*, T\kappa_*,\curlyvee) \leq Q\left(0, 0, P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_*, T\kappa_*,\curlyvee))\right), \tag{14}$$

for all $\curlyvee>0$.

Thus, by property (iv) of Q_ξ, we get $P_m(\kappa_*, T\kappa_*,\curlyvee) = 0$ for all $\curlyvee>0$. Therefore, $\kappa_* = T\kappa_*$.

In the following result, (15) is a special case of (3), but with (15) we do not need the condition of the regularity of the E-parametric metric space.

Theorem 5. *Let (H, E, P_m) be an E-complete E-parametric metric space. Let $T : H \longrightarrow H$ and $\alpha : H \times H \longrightarrow [0,\infty)$ be two maps such that*

$$\alpha(\ell, \ell') P_m(T\ell, T\ell',\curlyvee) \leq \xi\left(\max\left\{P_m(\ell, \ell',\curlyvee), P_m(\ell, T\ell,\curlyvee), P_m(\ell', T\ell',\curlyvee), \frac{P_m(\ell', T\ell,\curlyvee) + P_m(\ell, T\ell',\curlyvee)}{2}\right\}\right), \tag{15}$$

for all $\curlyvee>0$ and for all $\ell, \ell' \in H$ with $\ell \sim \ell'$, where $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and nondecreasing map with $\sum_{n=1}^{\infty} \xi^n(t) < \infty$ for all $t \geq 0$ and $\xi(t) < t$ for all $t > 0$. Also, consider the following assertions:

(a) T is E-preserving, that is, $\ell \sim \ell'$ implies $T\ell \sim T\ell'$

(b) There is $\ell_0 \in H$ such that $\ell_0 \sim T\ell_0$ and $\alpha(\ell_0, T\ell_0) \geq 1$

(c) For every $\ell, \ell' \in H$ such that $\ell \sim \ell'$ and $\alpha(\ell, \ell') \geq 1$, we have $\alpha(T\ell, T\ell') \geq 1$

(d) For each E-sequence $\{\ell_m\}$ in H with $\alpha(\ell_m, \ell_{m+1}) \geq 1$ for all $m \in \mathbb{N}$ and $\ell_m \longrightarrow \ell$, we have $\alpha(\ell_m, \ell) \geq 1$ for all $m \in \mathbb{N}$

Then, T has a fixed point.

Proof. By (b), we know that there is $\kappa_0 \in H$ of the form $\kappa_0 \sim T\kappa_0$ and $\alpha(\kappa_0, T\kappa_0) \geq 1$. In view of the fact that T has E-preserving nature, thus $T\kappa_0 \sim T^2\kappa_0$ and by (c), $\alpha(T\kappa_0, T^2\kappa_0) \geq 1$. Continuing this process, we get $T^m\kappa_0 \sim$

$T^{m+1}\kappa_0$ and $\alpha(T^m\kappa_0, T^{m+1}\kappa_0) \geq 1$ for all $m \in \mathbb{N}$. Take a sequence $\{\kappa_m\}$ with the terms as $\kappa_m = T^m\kappa_0$ for all $m \in \mathbb{N}$. Thus, $\kappa_m \sim \kappa_{m+1}$ and $\alpha(\kappa_m, \kappa_{m+1}) \geq 1$ for all $m \in \mathbb{W}$. Now, we take $\kappa_m \neq \kappa_{m+1}$ for all $m \in \mathbb{W}$. From (15), one writes for each $m \in \mathbb{W}$,

$$\begin{aligned} P_m(\kappa_{m+1}, \kappa_{m+2}, Y) &= P_m(T\kappa_m, T\kappa_{m+1}, Y) \leq \alpha(\kappa_m, \kappa_{m+1}) P_m(T\kappa_m, T\kappa_{m+1}, Y) \\ &\leq \xi\left(\max\left\{P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_m, T\kappa_m, Y), P_m(\kappa_{m+1}, T\kappa_{m+1}, Y), \frac{1}{2}(P_m(\kappa_{m+1}, T\kappa_m, Y) + P_m(\kappa_m, T\kappa_{m+1}, Y))\right\}\right) \\ &= \xi\left(\max\left\{P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_{m+1}, \kappa_{m+2}, Y), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_{m+1}, Y) + P_m(\kappa_m, \kappa_{m+2}, Y))\right\}\right) \\ &\leq \xi(\max\{P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_{m+1}, \kappa_{m+2}, Y)\}), \end{aligned} \tag{16}$$

for all $Y>0$. □

Since $\kappa_m \neq \kappa_{m+1}$ for all $m \in \mathbb{W}$, (16) implies that

$$P_m(\kappa_{m+1}, \kappa_{m+2}, Y) \leq \xi(P_m(\kappa_m, \kappa_{m+1}, Y)), \text{ for all} Y>0 \text{ and } m \in \mathbb{W}. \tag{17}$$

Therefore,

$$P_m(\kappa_m, \kappa_{m+1}, Y) \leq \xi^m(P_m(\kappa_0, \kappa_1, Y)), \text{ for all} Y>0 \text{ and } m \in \mathbb{N}. \tag{18}$$

Now, by following the proof of the above result, we may ensure that $\{\kappa_m\}$ is a Cauchy E-sequence in (H, E, P_m), and there must exists $\kappa_* \in H$ with $\lim_{m\longrightarrow\infty} P_m(\kappa_m, \kappa_*, Y) = 0$ for all $Y>0$ and $\kappa_m \sim \kappa_*$ for each $m \geq k$, for some k. Also, using (d), we conclude that $\alpha(\kappa_m, \kappa_*) \geq 1$ for each $m \geq k$. Now, from (15), we get

$$\begin{aligned} P_m(T\kappa_m, T\kappa_*, Y) &\leq \alpha(\kappa_m, \kappa_*) P_m(T\kappa_m, T\kappa_*, Y) \\ &\leq \xi\left(\max\left\{P_m(\kappa_m, \kappa_*, Y), P_m(\kappa_m, T\kappa_m, Y), P_m(\kappa_*, T\kappa_*, Y), \frac{1}{2}(P_m(\kappa_*, T\kappa_m, Y) + P_m(\kappa_m, T\kappa_*, Y))\right\}\right) \\ &= \xi\left(\max\left\{P_m(\kappa_m, \kappa_*, Y), P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_*, T\kappa_*, Y), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*, Y) + P_m(\kappa_m, T\kappa_*, Y))\right\}\right), \end{aligned} \tag{19}$$

for all $Y>0$ and for all $m \geq k$.

By triangle property and (19), we get

$$\begin{aligned} P_m(\kappa_*, T\kappa_*, Y) &\leq P_m(\kappa_*, \kappa_{m+1}, Y) + P_m(\kappa_{m+1}, T\kappa_*, Y) = P_m(\kappa_*, \kappa_{m+1}, Y) + P_m(T\kappa_m, T\kappa_*, Y) \\ &\leq P_m(\kappa_*, \kappa_{m+1}, Y) + \xi\left(\max\left\{P_m(\kappa_m, \kappa_*, Y), P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_*, T\kappa_*, Y), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*, Y) + P_m(\kappa_m, T\kappa_*, Y))\right\}\right) \\ &\leq P_m(\kappa_*, \kappa_{m+1}, Y) + \xi\left(\max\left\{P_m(\kappa_m, \kappa_*, Y), P_m(\kappa_m, \kappa_{m+1}, Y), P_m(\kappa_*, T\kappa_*, Y), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*, Y) + P_m(\kappa_m, \kappa_*, Y) + P_m(\kappa_*, T\kappa_*, Y))\right\}\right), \end{aligned} \tag{20}$$

for all $\curlyvee>0$ and for all $m \geq k$.

Applying the limit when $m \longrightarrow \infty$ in (20) and using the continuity of ξ, we get

$$P_m(\kappa_*, T\kappa_*,\curlyvee) \leq \xi\left(\max\left\{P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_*, T\kappa_*,\curlyvee))\right\}\right), \quad (21)$$

for all $\curlyvee>0$.

The above inequality is only possible when $P_m(\kappa_*, T\kappa_*, \curlyvee) = 0$ for all $\curlyvee>0$. Therefore, $\kappa_* = T\kappa_*$.

Example 6. Take $H = C[1,\infty)$ with equivalence relation E on H defined as $h_g \sim h_f$ if $t \mid h_g(t) - h_f(t)$. Define $P_m : H \times H \times (0,\infty) \longrightarrow [0,\infty)$ by

$$P_m(h_g, h_f,\curlyvee) = \begin{cases} |h_g(\curlyvee) - h_f(\curlyvee)|, \text{for all } h_g, h_f \in H \text{ and for all} \curlyvee\geq 1, \\ 0, \text{for all } h_g, h_f \in H \text{ and for all} \curlyvee\in(0,1). \end{cases} \quad (22)$$

Define the maps $T : H \longrightarrow H$ and $\alpha : H \times H \longrightarrow [0,\infty)$ by

$$T(h_g(t)) = \begin{cases} \dfrac{h_g(t)+1}{2}, & \text{if } h_g(t) \geq 0, \\ -(h_g(t))^2, & \text{otherwise}, \end{cases} \quad (23)$$

$$\alpha(h_g, h_f) = \begin{cases} 1, & \text{if } h_g, h_f \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Here, one can verify that all the axioms of Theorem 5 are valid, and there is a fixed point of T in H.

Let (H, E, P_m) be an E-parametric metric space. Let $\mathcal{N}(H, E)$ denotes the collection of all nonempty subsets of H having the following properties:

(i) For any $A_H \in \mathcal{N}(H, E)$, $\sup\{P_m(\kappa, \omega,\curlyvee): \kappa, \omega \in A_H\}$ exists for each $\curlyvee>0$

(ii) If $\{\kappa_n\} \subset A_H$ and $\kappa_n \longrightarrow \kappa$, then $\kappa \in A_H$

The Hausdroff E-parametric distance H_{P_m} is a mapping from $\mathcal{N}(H, E) \times \mathcal{N}(H, E) \times (0,\infty)$ into $[0,\infty)$ defined by

$$H_{P_m}(A_H, B_H,\curlyvee) = \max\{\lambda(A_H, B_H,\curlyvee), \lambda(B_H, A_H,\curlyvee)\}, \quad (24)$$

for all $\curlyvee>0$ where

$$\begin{aligned} \lambda(A_H, B_H,\curlyvee) &= \sup\{\inf\{P_m(\kappa, \omega,\curlyvee): \omega \in B_H\}: \kappa \in A_H\}, \\ \lambda(B_H, A_H,\curlyvee) &= \sup\{\inf\{P_m(\omega, \kappa,\curlyvee): \kappa \in A_H\}: \omega \in B_H\}. \end{aligned} \quad (25)$$

Here, everyone should know that for each $A_H \in \mathcal{N}(H, E)$, $\omega \in (H, E, P_m)$ and for each value of $\curlyvee$ there must exists $\kappa_q \in A_H$ with $P_m(\kappa_q, \omega,\curlyvee) \leq q \inf\{P_m(\kappa, \omega,\curlyvee): \kappa \in A_H\}$, where $q > 1$. Subsequently, we take $\inf\{P_m(\omega, \kappa,\curlyvee): \kappa \in A_H\} = P_m(\omega, A_H,\curlyvee)$.

In the following theorem, we will discuss the case of the existence of fixed points for multivalued maps.

Theorem 7. *Let (H, E, P_m) be an E-complete regular E-parametric metric space. Let $T : \mathcal{N}(H, E) \longrightarrow \mathcal{N}(H, E)$ and $\alpha : H \times H \longrightarrow [0,\infty)$ be two maps such that*

$$\alpha_c(T\ell, T\ell')H_{P_m}(T\ell, T\ell',\curlyvee) \leq Q\left(P_m(\ell, \ell',\curlyvee), P_m(\ell, T\ell,\curlyvee), P_m(\ell', T\ell',\curlyvee), \frac{P_m(\ell', T\ell,\curlyvee) + P_m(\ell, T\ell',\curlyvee)}{2}\right), \quad (26)$$

for all $\curlyvee>0$ and for all $\ell, \ell' \in H$ with $\ell \sim \ell'$, where $Q \in Q_\xi$ and $\alpha_c(T\ell, T\ell') = \inf\{\alpha(j, j'): j \in T\ell, j' \in T\ell'\}$. Also, consider the following assertions:

(a) T is E-preserving, that is, $\ell \sim \ell'$ implies $j \sim j'$ for each $j \in T\ell$ and $j' \in T\ell'$

(b) There is $\kappa_0 \in H$ and $\kappa_1 \in T\kappa_0$ such that $\kappa_0 \sim \kappa_1$ and $\alpha(\kappa_0, \kappa_1) > 1$

(c) For every $\ell, \ell' \in H$ such that $\ell \sim \ell'$ and $\alpha(\ell, \ell') > 1$, we have $\alpha_c(T\ell, T\ell') > 1$

(d) For each E-sequence $\{\ell_m\}$ in H with $\alpha(\ell_m, \ell_{m+1}) > 1$ for all $m \in \mathbb{N}$ and $\ell_m \longrightarrow \ell$, we have $\alpha(\ell_m, \ell) > 1$ for all $m \in \mathbb{N}$

(e) For each $q > 1$, $T\kappa \in \mathcal{N}(H, E)$ and $\omega \in H$ there is $\kappa_q \in T\kappa$ with $P_m(\kappa_q, \omega,\curlyvee) \leq qP_m(\omega, T\kappa,\curlyvee)$ for all $\curlyvee>0$, where $\inf\{P_m(\omega, \kappa_*,\curlyvee): \kappa_* \in T\kappa\} = P_m(\omega, T\kappa,\curlyvee)$.

Then, T has a fixed point.

Proof. By (b) and (c), we observe that $\alpha_c(T\kappa_0, T\kappa_1) > 1$, since $\kappa_0 \in H$ and $\kappa_1 \in T\kappa_0$ are of the form $\kappa_0 \sim \kappa_1$ and $\alpha(\kappa_0, \kappa_1) > 1$.

Here, we take $\kappa_0 \neq \kappa_1$. Otherwise, $\kappa_0 \in T\kappa_0$. By (26), we get

$$\begin{aligned}\alpha_c(T\kappa_0, T\kappa_1)P_m(\kappa_1, T\kappa_1,\curlyvee) &\leq \alpha_c(T\kappa_0, T\kappa_1)H_{P_m}(T\kappa_0, T\kappa_1,\curlyvee)\\ &\leq Q\left(P_m(\kappa_0, \kappa_1,\curlyvee), P_m(\kappa_0, T\kappa_0,\curlyvee), P_m(\kappa_1, T\kappa_1,\curlyvee), \frac{1}{2}(P_m(\kappa_1, T\kappa_0,\curlyvee) + P_m(\kappa_0, T\kappa_1,\curlyvee))\right),\end{aligned} \tag{27}$$

for all $\curlyvee>0$. □

Since $\alpha_c(T\kappa_0, T\kappa_1) > 1$, by (e), there is $\kappa_2 \in T\kappa_1$ such that

$$P_m(\kappa_1, \kappa_2,\curlyvee) \leq \alpha_c(T\kappa_0, T\kappa_1)P_m(\kappa_1, T\kappa_1,\curlyvee), \tag{28}$$

for all $\curlyvee>0$. Here, again, we take $\kappa_1 \neq \kappa_2$. Otherwise, $\kappa_1 \in T\kappa_1$. Thus, by using it in (27), we get

$$\begin{aligned}P_m(\kappa_1, \kappa_2,\curlyvee) &\leq \alpha_c(T\kappa_0, T\kappa_1)P_m(\kappa_1, T\kappa_1,\curlyvee) \leq Q\left(P_m(\kappa_0, \kappa_1,\curlyvee), P_m(\kappa_0, T\kappa_0,\curlyvee), P_m(\kappa_1, T\kappa_1,\curlyvee), \frac{1}{2}(P_m(\kappa_1, T\kappa_0,\curlyvee) + P_m(\kappa_0, T\kappa_1,\curlyvee))\right)\\ &\leq Q\left(P_m(\kappa_0, \kappa_1,\curlyvee), P_m(\kappa_0, \kappa_1,\curlyvee), P_m(\kappa_1, \kappa_2,\curlyvee), \frac{1}{2}(P_m(\kappa_0, \kappa_1,\curlyvee) + P_m(\kappa_1, \kappa_2,\curlyvee))\right),\end{aligned} \tag{29}$$

for all $\curlyvee>0$.

Here, our claim is $P_m(\kappa_0, \kappa_1,\curlyvee) > P_m(\kappa_1, \kappa_2,\curlyvee)\forall\curlyvee > 0$. Suppose this is not true. Then, $P_m(\kappa_0, \kappa_1, \curlyvee^0) \leq P_m(\kappa_1, \kappa_2, \curlyvee^0)$ for some $\curlyvee^0 > 0$. Then, from (29) and by property (i) of Q_ξ, for $\curlyvee^0 > 0$, we get

$$P_m(\kappa_1, \kappa_2, \curlyvee^0) \leq Q(P_m(\kappa_1, \kappa_2, \curlyvee^0), P_m(\kappa_0, \kappa_1, \curlyvee^0), P_m \cdot (\kappa_1, \kappa_2, \curlyvee^0), P_m(\kappa_1, \kappa_2, \curlyvee^0)). \tag{30}$$

From property (iii) of Q_ξ and the above inequality, we get

$$P_m(\kappa_1, \kappa_2, \curlyvee^0) = 0, \text{for some}\curlyvee^0 > 0. \tag{31}$$

This fact together with regularity condition implies

$$P_m(\kappa_1, \kappa_2,\curlyvee) = 0, \text{for all}\curlyvee>0, \tag{32}$$

and it is not possible under the assumption. Thus, $P_m(\kappa_0, \kappa_1,\curlyvee) > P_m(\kappa_1, \kappa_2,\curlyvee)$ for all $\curlyvee>0$. Again, we use (29) and property (i) of Q_ξ to obtain

$$P_m(\kappa_1, \kappa_2,\curlyvee) \leq Q(P_m(\kappa_0, \kappa_1,\curlyvee), P_m(\kappa_0, \kappa_1,\curlyvee), P_m \cdot (\kappa_1, \kappa_2,\curlyvee), P_m(\kappa_0, \kappa_1,\curlyvee)), \text{for all}\curlyvee>0. \tag{33}$$

The property (ii) of Q_ξ and (33) imply

$$P_m(\kappa_1, \kappa_2,\curlyvee) \leq \xi(P_m(\kappa_0, \kappa_1,\curlyvee)), \text{for all}\curlyvee>0. \tag{34}$$

Clearly, $\alpha(\kappa_1, \kappa_2) > 1$ due to the fact that $\kappa_1 \in T\kappa_0$, $\kappa_2 \in T\kappa_1$ and $\alpha_c(T\kappa_0, T\kappa_1) > 1$. Also, the E-preserving characteristic of T yields $\kappa_1 \sim \kappa_2$. Thus, by (c), we get $\alpha_c(T\kappa_1, T\kappa_2) > 1$. The repeated application of the above steps yields the sequence $\{\kappa_m\}$ with $\kappa_m \in T\kappa_{m-1}$, $\kappa_{m-1} \sim \kappa_m$, $\alpha(\kappa_{m-1}, \kappa_m) > 1$ and $P_m(\kappa_{m-1}, \kappa_m,\curlyvee) > P_m(\kappa_m, \kappa_{m+1},\curlyvee)$ for all $\curlyvee>0$ and

$$P_m(\kappa_m, \kappa_{m+1},\curlyvee) \leq \xi^m(P_m(\kappa_0, \kappa_1,\curlyvee)), \tag{35}$$

for all $\curlyvee>0$ and for all $m \in \mathbb{N}$.

The steps given below will show that $\{\kappa_m\}$ is a Cauchy E-sequence. For each $n > m \in \mathbb{N}$, the triangle inequality and (35) imply that

$$P_m(\kappa_m, \kappa_n,\curlyvee) \leq \sum_{j=m}^{n-1} P_m(\kappa_j, \kappa_{j+1},\curlyvee) \leq \sum_{j=m}^{\infty} \xi^j(P_m(\kappa_0, \kappa_1,\curlyvee)), \tag{36}$$

for all $\curlyvee>0$.

The convergence of $\sum_{j=1}^{\infty} \xi^j(P_m(\kappa_0, \kappa_1,\curlyvee))$ together with the above inequality confirm that $\{\kappa_m\}$ is a Cauchy E-sequence in (H, E, P_m). The E-completeness of (H, E, P_m) ensures the existence of a point $\kappa_* \in H$ with $\lim_{m\longrightarrow\infty}P_m(\kappa_m, \kappa_*,\curlyvee) = 0$ for all $\curlyvee>0$ and $\kappa_m \sim \kappa_*$ for each $m \geq k$, for some k. We now able to conclude that, by (d), $\alpha(\kappa_m, \kappa_*) > 1$ for each $m \in \mathbb{N}$. Since $\kappa_m \sim \kappa_*$ and $\alpha(\kappa_m, \kappa_*) > 1$ for each $m \geq k$, using (c), we have $\alpha(T\kappa_m, T\kappa_*) > 1$. Thus, by (26), we get

$$\begin{aligned}H_{P_m}(T\kappa_m, T\kappa_*,\curlyvee) &< \alpha(T\kappa_m, T\kappa_*)H_{P_m}(T\kappa_m, T\kappa_*,\curlyvee)\\ &\le Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, T\kappa_m,\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_*, T\kappa_m,\curlyvee) + P_m(\kappa_m, T\kappa_*,\curlyvee))\right)\\ &= Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*,\curlyvee) + P_m(\kappa_m, T\kappa_*,\curlyvee))\right),\end{aligned} \tag{37}$$

for all $\curlyvee>0$.

By triangle property and (37), we get

$$\begin{aligned}P_m(\kappa_*, T\kappa_*,\curlyvee) &\le P_m(\kappa_*, \kappa_{m+1},\curlyvee) + P_m(\kappa_{m+1}, T\kappa_*,\curlyvee) \le P_m(\kappa_*, \kappa_{m+1},\curlyvee) + H_{P_m}(T\kappa_m, T\kappa_*,\curlyvee) < P_m(\kappa_*, \kappa_{m+1},\curlyvee)\\ &\quad + Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*,\curlyvee) + P_m(\kappa_m, T\kappa_*,\curlyvee))\right)\\ &\le P_m(\kappa_*, \kappa_{m+1},\curlyvee) + Q\left(P_m(\kappa_m, \kappa_*,\curlyvee), P_m(\kappa_m, \kappa_{m+1},\curlyvee), P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_{m+1}, \kappa_*,\curlyvee) + P_m(\kappa_m, \kappa_*,\curlyvee) + P_m(\kappa_*, T\kappa_*,\curlyvee))\right),\end{aligned} \tag{38}$$

for all $\curlyvee>0$.

Applying the limit when $m \longrightarrow \infty$ in (38), we get

$$P_m(\kappa_*, T\kappa_*,\curlyvee) \le Q\left(0, 0, P_m(\kappa_*, T\kappa_*,\curlyvee), \frac{1}{2}(P_m(\kappa_*, T\kappa_*,\curlyvee))\right), \text{for all} \curlyvee>0. \tag{39}$$

Thus, by property (iv) of Q_ξ, we get $P_m(\kappa_*, T\kappa_*,\curlyvee) = 0$ for all $\curlyvee>0$. Therefore, $\kappa_* \in T\kappa_*$.

In the following theorem, (40) is a special case of (26), but with (40), we do not need the condition of regularity of the E-parametric metric space.

Theorem 8. *Let (H, E, P_m) be an E-complete E-parametric metric space. Let $T : \mathcal{N}(H, E) \longrightarrow \mathcal{N}(H, E)$ and $\alpha : H \times H \longrightarrow [0,\infty)$ be two maps such that*

$$\alpha_c\left(T\ell, T\ell'\right)H_{P_m}\left(T\ell, T\ell',\curlyvee\right) \le \xi\left(\max\left\{P_m\left(\ell, \ell',\curlyvee\right), P_m(\ell, T\ell,\curlyvee), P_m\left(\ell', T\ell',\curlyvee\right), \frac{P_m\left(\ell', T\ell,\curlyvee\right) + P_m\left(\ell, T\ell',\curlyvee\right)}{2}\right\}\right), \tag{40}$$

for all $\curlyvee>0$ and for all $\ell, \ell' \in H$ with $\ell \sim \ell'$, where $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and nondecreasing mapping with $\sum_{n=1}^{\infty} \xi^n(t) < \infty$ for all $t \ge 0$ and $\xi(t) < t$ for all $t > 0$, and $\alpha_c(T\ell, T\ell') = \inf\{\alpha(j, j'): j \in T\ell, j' \in T\ell'\}$. Also, consider the following assertions:

(a) T is E-preserving, that is, $\ell \sim \ell'$ implies $j \sim j'$ for each $j \in T\ell$ and $j' \in T\ell'$

(b) There is $\kappa_0 \in H$ and $\kappa_1 \in T\kappa_0$ such that $\kappa_0 \sim \kappa_1$ and $\alpha(\kappa_0, \kappa_1) > 1$

(c) For every $\ell, \ell' \in H$ such that $\ell \sim \ell'$ and $\alpha(\ell, \ell') > 1$, we have $\alpha_c(T\ell, T\ell') > 1$

(d) For each E-sequence $\{\ell_m\}$ in H with $\alpha(\ell_m, \ell_{m+1}) > 1$ for all $m \in \mathbb{N}$ and $\ell_m \longrightarrow \ell$, we have $\alpha(\ell_m, \ell) > 1$

(e) For each $q > 1$, $T\kappa \in \mathcal{N}(H, E)$ and $\omega \in H$ there exists $\kappa_q \in T\kappa$ with $P_m(\kappa_q, \omega,\curlyvee) \le qP_m(\omega, T\kappa,\curlyvee)$ for all $\curlyvee>0$, where $\inf\{P_m(\omega, \kappa_*,\curlyvee): \kappa_* \in T\kappa\} = P_m(\omega, T\kappa,\curlyvee)$.

Then, T has a fixed point.

The proof of this theorem will be done by following the techniques of Theorems 5 and 6.

3. Application to Existence of Solutions of Integral Equations

Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$ and $P_m : X \times X \times (0,\infty) \longrightarrow [0,+\infty)$ be defined by

$$P_m(x, y, \alpha) = \sup_{t\in[0,T]} \frac{1}{1 + (\alpha t^2)} |x(t) - y(t)|, \tag{41}$$

for all $x, y \in X$ and all $t \in [0, T]$. Then, (X, P_m) is a complete parametric metric space. Let $\sim$ be the equivalence relation on X defined by $x \sim y$ if and only if $t \mid x(t) - y(t)$. Then, $(X, P_m, \sim)$ is a complete E-parametric metric space. Consider the following integral equation:

$$x(t) = p(t) + \int_0^T S(t, s)f(s, x(s))ds, \tag{42}$$

where

(A) $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous

(B) $p : [0, T] \longrightarrow \mathbb{R}$ is continuous

(C) $S : [0, T] \times [0, T] \longrightarrow [0,+\infty)$ is continuous and

$$\sup_{t\in[0,T]} \left(\int_0^T S(t, s)ds\right) \le 1, \tag{43}$$

(D) There exists $k \in (0, 1)$ such that

$$0 \le [f(s, y(s)) - f(s, x(s))] \le \frac{k}{1 + (\alpha s^2)} \max\left\{|x(s) - y(s)|, |x(s) - Hx(s)|, |y(s) - Hy(s)|, \frac{|x(s) - Hy(s)| + |y(s) - Hx(s)|}{2}\right\}, \tag{44}$$

for all $x, y \in X$ with $x \sim y$ and $s \in [0, T]$ where

$$Hx(t) = p(t) + \int_0^T S(t, s)f(s, x(s))ds, t \in [0, T], \text{for all } x \in X. \tag{45}$$

(E) There exists $x_0 \in X$ such that

$$\begin{aligned} &x_0(t) \le p(t) + \int_0^T S(t, s)f(s, x_0(s))ds, \\ &t\left|x_0(t) - \left[p(t) + \int_0^T S(t, s)f(s, x_0(s))ds\right].\right. \end{aligned} \tag{46}$$

(F) $s \mid \int_0^T S(t, s)[f(s, y(s)) - f(s, x(s))]ds$ for all $x, y \in X$ with $x \sim y$ and $s \in [0, T]$

We have the following result on the existence of solutions for integral equations.

Theorem 9. *Under assumptions $(A) - (F)$, the integral Equation (42) has a unique solution in $X = C([0, T], R)$.*

Proof. Let $H : X \longrightarrow X$ be defined by

$$Hx(t) = p(t) + \int_0^T S(t, s)f(s, x(s))ds, t \in [0, T], \text{for all } x \in X. \tag{47}$$

□

First, we will prove that H is E-preserving. Let $x \sim y$. Then, we have $s \mid \int_0^T S(t, s)f(s, y(s))ds - \int_0^T S(t, s)f(s, x(s))ds$ for all $s \in [0, T]$. On the other hand, by the definition of H, we have

$$Hy - Hx = \int_0^T S(t, s)[f(s, y(s)) - f(s, x(s))]ds, \text{for all } t \in [0, T]. \tag{48}$$

So, $Hx \sim Hy$.

Now, suppose that $x, y \in X$ with $x \sim y$. Then, by (C), (D) and the definition of H, we get

$$\begin{aligned}
P_m(Hx, Hy, \alpha) &= \sup_{t\in[0,T]} \frac{1}{1+(\alpha t^2)} |Hx(t) - Hy(t)| \\
&\le \sup_{t\in[0,T]} \frac{1}{1+(\alpha t^2)} \int_0^T S(t,s)|f(s,x(s)) - f(s,y(s))|ds \\
&\le k\left(\frac{1}{1+(\alpha t^2)} \max\left\{|x(s)-y(s)|, |x(s)-Hx(s)|, |y(s)-Hy(s)|, \frac{|x(s)-Hy(s)| + |y(s)-Hx(s)|}{2}\right\}\right) \\
&\le k\left(\frac{1}{1+(\alpha t^2)} \max\left\{\sup_{s\in[0,T]}|x(s)-y(s)|, \sup_{s\in[0,T]}|x(s)-Hx(s)|, \sup_{s\in[0,T]}|y(s)-Hy(s)|, \frac{\sup_{s\in[0,T]}|x(s)-Hy(s)| + \sup_{s\in[0,T]}|y(s)-Hx(s)|}{2}\right\}\right) \\
&\le k\left[\max\left\{P_m(x,y,\alpha), P_m(x,Hx,\alpha), P_m(y,Hy,\alpha), \frac{P_m(x,Hy,\alpha)+P_m(y,Hx,\alpha)}{2}\right\}\right].
\end{aligned} \tag{49}$$

Now, by (E), there exists $x_0 \in X$ such that $x_0 \sim Hx_0$. Then, the conditions of Theorem 4 are satisfied and hence the integral Equation (42) has a unique solution in $X = C([0,T], \mathbb{R})$. Note that $\alpha \equiv 1$ when $\ell \sim \ell'$ and $\alpha \equiv 0$ when $\ell \nsim \ell'$.

4. Conclusion

This article presents fixed point theorems for mappings satisfying the contraction type conditions defined with auxiliary functions over E-parametric metric spaces. The readers will get many fixed point results as the outcome of this work. These results have been obtained in the following ways:

(1) Defining an auxiliary function through a particular form

(2) Equipping the E-parametric metric space (H, E, P_m) with a directed graph and defining $\alpha(x, y) = 1$, whenever (x, y) is an edge in a directed graph defined on H and $\alpha(x, y) = 0$ otherwise. Similarly, we may consider a partial order on H instead of a directed graph

(3) Defining an equivalence relation on H by edge/path of a graph, a partial order, a preorder, etc.

(4) Taking $P_m(\ell, \ell', \curlyvee) = d(\ell, \ell')$ for all $\curlyvee$, where d is used for a metric on H

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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