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Research Article

Two New Modifications of the Exp-Function Method for Solving the Fractional-Order Hirota-Satsuma Coupled KdV

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Two novel modifications of the exp-function method have been suggested to solve the nonlinear system of the partial differential equation of the fractional order for the first time. The fractional-order Hirota-Satsuma coupled KdV system has been solved precisely; as a result, some exact solutions, which include soliton-type and rational solutions, will be derived. Eventually, the graphs of the obtained results have been illustrated at the end of the article. The newly used methods are highly accurate, flexible, effective, and programmable to solve fractional-order devices.

1. Introduction

One of the best ways to model physical processes is to use fractional partial differential equations (FPDE). FPDE can be a great instrument for elaborating the properties of different materials and processes; therefore, this sort of equations plays an important role in various scientific fields.

Plenty of definitions had been presented about the derivative of the fractional order. In the current paper, we consider the conformable derivative definition [1, 2]. Given a function $h : [0,\infty) \longrightarrow R$, then, the conformable fractional derivative of f of order α is defined by

$$D_t^{\lambda} h = \lim_{\varepsilon \longrightarrow 0} \frac{h\left(t + \varepsilon t^{1-\lambda}\right) - h(\mathrm{t})}{\varepsilon}, \quad \text{for all } t > 0, 0 < \lambda < 1. \tag{1}$$

One can easily show that the abovementioned definition satisfies all the properties in the following properties [1]:

(1) $D_t^{\lambda} c = 0$, c is constant

(2) $D_t^{\lambda} t^k = k t^{k-\lambda}$, $k \in R$

(3) $D_t^{\lambda}(c_1 h_1 + c_2 h_2) = c_1 D_t^{\lambda}(h_1) + c_2 D_t^{\lambda}(h_2)$

(4) $D_t^{\lambda}(h_1 h_2) = h_1 D_t^{\lambda}(h_2) + h_2 D_t^{\lambda}(h_1)$

(5) $D_t^{\lambda}(h_1/h_2) = (h_2 D_t^{\lambda}(h_1) - h_1 D_t^{\lambda}(h_2))/h_2^2$

(6) $D_t^{\lambda} h(t) = t^{1-\lambda} D_t h$

Recently, the fractional differential equations have become the focus of many scientists in the field of physics and mathematics and also many researchers to focus on this topic [3–6]. It can provide many methods for obtaining their exact solutions, such as the G'/G-expansion method [7–9], the exp-function method [10–28], the homotopy analysis method [29, 30], and spectral methods [31–33]. And in the meantime, obtaining the exact soliton solutions to these equations is more important because this study gives us a great deal of information on various sciences such as fluid mechanics, physics, and mathematics. So, a lot of research has begun in recent years to get this style of solutions [34–44]. This concept was first introduced by John Scott Russell (1818–1388). The "translation wave" was the name he gave to the phenomenon that was reproduced in a wave reservoir. Although solitons can be said to be solitary waves that propagate like a particle at constant velocity and shape, over the years, there is still no single definition for them. In short, solitons have three distinct characteristics.

(1) They are in constant shape

(2) They are localized within a region

(3) In interaction with other solitons, they come out unchanged (except phase shift)

In the current article, two interesting forms of the modification of the exp-function method has been applied to attain the soliton solution of fractional-order Hirota-Satsuma coupled KdV equations as the following.

$$\begin{aligned} D_t^{\alpha} u-\frac{1}{4} u_{xxx}-3 u u_x-3\left(-v^2+w\right)_x=0, \\ D_t^{\alpha} v+\frac{1}{2} v_{xxx}+3 u v_x=0, \\ D_t^{\alpha} w+\frac{1}{2} w_{xxx}+3 u w_x=0. \end{aligned} \tag{2}$$

KdV equations are of great importance due to their various applications in the different fields of study [36–38]. For instance, in plasma physics, the abovementioned system can cause ion-acoustic solutions [39, 40]. Some types of solutions for the coupled-KdV equation have been investigated by many researchers. For example, this equation has been studied by a variation iteration method in [41, 42].

2. Modification of the Exp-Function Method for a System of FPDE

To explain the first modification, the following nonlinear system of FPDE should be noted:

$$\begin{aligned} F\left(u, D_t^{\alpha} u, D_x^{\beta} u \cdots\right)=0, \quad 0<\alpha, \beta \leq 1, \\ G\left(u, D_t^{\alpha} u, D_x^{\beta} u \cdots\right)=0, \quad 0<\alpha, \beta \leq 1, \end{aligned} \tag{3}$$

where $D_t^{\alpha} u, D_x^{\beta} u \cdots$ are the fractional derivatives and F, G are polynomial in $u(x, t)$ and its fractional partial derivatives.

By using the nonlinear fractional complex transformation,

$$\xi=\frac{\tau x^{\beta}}{\beta}+\frac{v t^{\alpha}}{\alpha}, \tag{4}$$

where τ and v are nonzero parameters and equation (3) turns to a system of ODE:

$$\begin{aligned} Q\left(u, u', u'', u''' \cdots\right)=0, \\ P\left(u, u', u'', u''' \cdots\right)=0. \end{aligned} \tag{5}$$

Now, assume that

$$\begin{aligned} u(x, t)=u(\xi)=\frac{a_1+b_1 \exp (\xi)}{\left(1+c_1 \exp (\xi)\right)^2}, \\ v(x, t)=v(\xi)=\frac{a_2+b_2 \exp (\xi)}{\left(1+c_2 \exp (\xi)\right)^2}. \end{aligned} \tag{6}$$

Placing equation (6) into (5) precedes an algebraic system including powers of exp (ξ). By setting the coefficient of these powers to zero, a_i, b_i, c_i, τ, and v are acquired. By substituting the overdetermined value in eq. (6) the exact solutions of (3) will be derived.

To apply the second modification of the exp-function method, the solution of system (5) has been imaged as follows:

$$\begin{aligned} u(\xi)=\sum_{i=-m}^{m} a_i\left(\exp (\varphi(\xi))^i,\right. \\ v(\xi)=\sum_{i=-n}^{n} b_i\left(\exp (\varphi(\xi))^i,\right. \end{aligned} \tag{7}$$

where $\varphi(\xi)$ satisfies the nonlinear ODE in the form as follows:

$$\varphi'=\exp (-\varphi)+\alpha \exp (\varphi)+\beta. \tag{8}$$

a_i, b_i, α, and β are parameters to be handled further. The solution of equation (8) is as follows:

(1) If $\alpha \neq 0, \beta^2-4 \alpha>0$,

$$\varphi(\xi)=\ln \left(-\frac{\sqrt{\beta^2-4 \alpha}}{2 \alpha} \tan h\left(\frac{\sqrt{\beta^2-4 \alpha}}{2}(\xi+c)\right)-\frac{\beta}{2 \alpha}\right) \tag{9}$$

(2) If $\alpha \neq 0, \beta^2-4 \alpha<0$,

$$\varphi(\xi)=\ln \left(\frac{\sqrt{4 \alpha-\beta^2}}{2 \alpha} \tan \left(\frac{\sqrt{4 \alpha-\beta^2}}{2}(\xi+c)\right)-\frac{\beta}{2 \alpha}\right) \tag{10}$$

(3) If $\alpha \neq 0, \beta^2-4 \alpha=0$,

$$\varphi(\xi)=\ln \left(-\frac{2 \beta(\xi+c)+4}{\beta(\xi+c)}\right) \tag{11}$$

(4) If $\alpha=0, \beta \neq 0$,

$$\varphi(\xi)=-\ln \left(-\frac{\beta}{\exp (\beta(\xi+c))-1}\right) \tag{12}$$

(5) If $\alpha = 0, \beta = 0$,

$$\varphi(\xi) = \ln(\xi + c) \tag{13}$$

To obtain the numbers m and n, we strike a balance between the sentences with the topmost derivative and the topmost nonlinear order in equation (5). Placing equation (7) into (4) and considering equation (8) precede an algebraic system including powers of exp $(\varphi(\xi))$. By setting the coefficient of these powers to zero, a_i, b_i, τ, υ, α, and β are acquired. Finally, By substituting the overdetermined value in eq. (7) and the general solutions of (8) the exact solution of eq. (3) will be derived.

3. Application First Modified Exp-Function on the Fractional-Order Hirota-Satsuma Coupled KdV Equations

By using the nonlinear fractional complex transformation,

$$\xi = \tau x - \frac{\upsilon t^{\alpha}}{\alpha}, \tag{14}$$

where τ and υ are nonzero arbitrary constants and system (2) turns to a system of ODE as follows:

$$\begin{aligned} -\upsilon u' - \frac{1}{4}\tau^3 u''' - 3\tau u u' - 3\tau\left(-v^2 + w\right)' &= 0, \\ -\upsilon v' + \frac{1}{2}\tau^3 v''' + 3\tau u v' &= 0, \\ D_t^{\alpha} w + \frac{1}{2}\tau^3 w''' + 3\tau u w' &= 0. \end{aligned} \tag{15}$$

We define the solution of the abovementioned system as follows:

$$\begin{aligned} u(x,t) &= \frac{a_0 + a_1 e^{\xi}}{\left(1 + a_2 e^{\xi}\right)^2}, \\ v(x,t) &= \frac{b_0 + b_1 e^{\xi}}{\left(1 + b_2 e^{\xi}\right)^2}, \\ w(x,t) &= \frac{c_0 + c_1 e^{\xi}}{\left(1 + c_2 e^{\xi}\right)^2}. \end{aligned} \tag{16}$$

Substituting (16) into (15) and setting the coefficient of exp-function in the numerator equal to zero yield a system of algebraic equations, which solving by maple leads to the following:

Case 1.

$$\begin{aligned} \upsilon &= \frac{1}{2}\tau^3, \\ a_0 &= 0, \\ a_1 &= \tau^2 c_2, \\ a_2 &= c_2, \\ b_0 &= \frac{b_1}{c_2}, \\ b_2 &= c_2, \\ c_0 &= \frac{b_1^2}{c_2^2}, \\ c_1 &= -\frac{1}{4}\tau^4 c_2. \end{aligned} \tag{17}$$

Case 2.

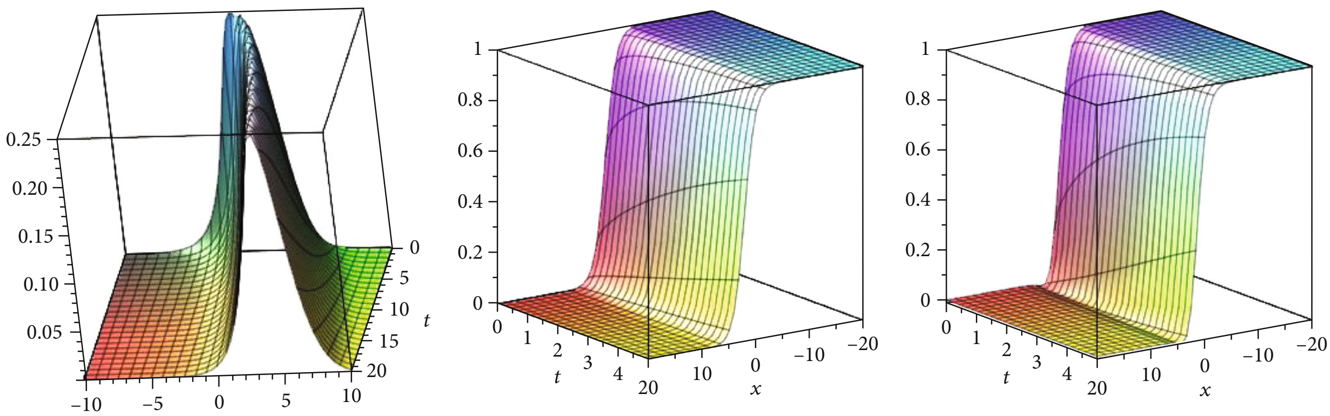
$$\begin{aligned} \upsilon &= \frac{1}{2}\tau^3, \\ a_0 &= 0, \\ a_1 &= 2\tau^2 c_2, \\ a_2 &= c_2, \\ b_0 &= 0, \\ b_1 &= \pm\tau^2 c_2, \\ b_2 &= c_2, \\ c_0 &= 0, \\ c_1 &= -\frac{1}{2}\tau^4 c_2. \end{aligned} \tag{18}$$

So, the corresponding soliton solutions will be achieved.

$$\begin{aligned} u_1(x,t) &= \frac{\tau^2 c_2 e^{\xi}}{(1 + c_2 e)^{\xi 2}}, \\ v_1(x,t) &= \frac{b_1}{c_2\left(1 + c_2 e^{\xi}\right)}, \\ w_1(x,t) &= \frac{b_1^2 - (1/4)\tau^4 {c_2}^3 e^{\xi}}{c_2^2\left(1 + c_2 e^{\xi}\right)^2}, \end{aligned} \tag{19}$$

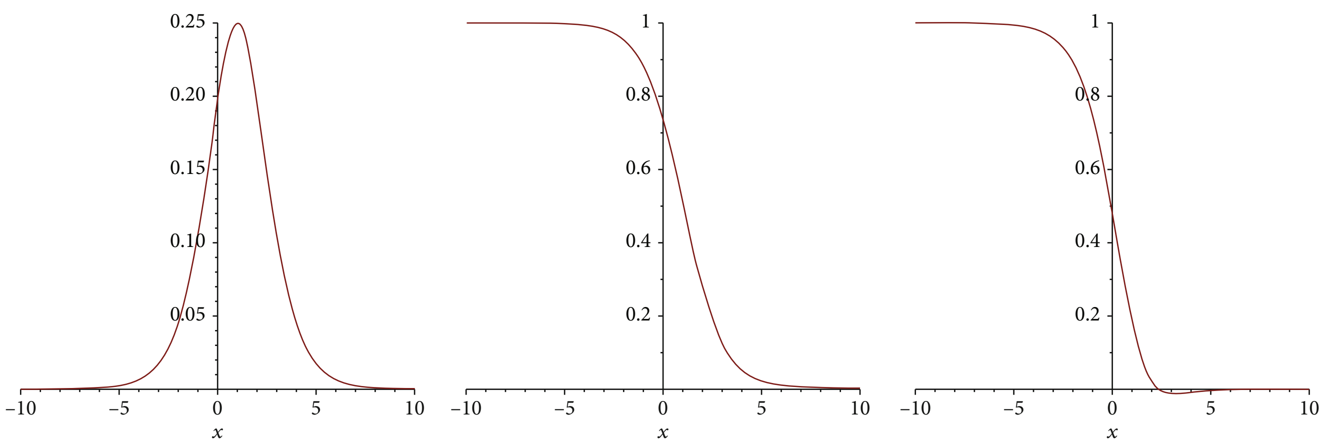
$$\begin{aligned} u_2(x,t) &= \frac{2\tau^2 c_2 e^{\xi}}{\left(1 + c_2 e^{\xi}\right)^2}, \\ v_2(x,t) &= \frac{\pm\tau^2 c_2 e^{\xi}}{\left(1 + c_2 e^{\xi}\right)^2}, \\ w_2(x,t) &= \frac{-\tau^4 c_2 e^{\xi}}{2\left(1 + c_2 e^{\xi}\right)^2}, \end{aligned} \tag{20}$$

where $\xi = \tau x - (\tau^3 t^{\alpha}/2\alpha)$.



(a) $u(x,t)$ (b) $v(x,t)$ (c) $w(x,t)$

Figure 1: Plots of solution (19) for $\alpha = 0.5$, $b_1 = \tau = c_2 = 1$.



(a) (b) (c)

Figure 2: Plots of solution (19) for $\alpha = 0.5$, $b_1 = \tau = c_2 = 1$, and $t = 1$.

In Figures 1–4, the plots of solutions (19) and (20), for some values of parameters, have been illustrated.

The plots of solutions (19) and (20) for different values of $\alpha = 0.1, 0.3, 0.5, 0.9, 1$ are illustrated in Figures 5 and 6. These figures show the effect of the order of the fractional derivative on the solutions. In fact, one can compare the solutions for different values of α.

4. Application of the Second Modified Exp-Function on the Fractional-Order Hirota-Satsuma Coupled KdV Equations

By using the nonlinear fractional complex transformation,

$$\xi = \tau x - \frac{\upsilon t^{\alpha}}{\alpha}, \tag{21}$$

where τ and υ are nonzero arbitrary parameters and system (2) changes to a system of ODE differential equations.

$$\begin{aligned} -\upsilon u' - \frac{1}{4}\tau^3 u''' - 3\tau u u' - 3\tau\left(-v^2 + w\right)' &= 0, \\ -\upsilon v' + \frac{1}{2}\tau^3 v''' + 3\tau u v' &= 0, \\ D_t^{\alpha} w + \frac{1}{2}\tau^3 w''' + 3\tau u w' &= 0. \end{aligned} \tag{22}$$

The solution of (22) will be imaged as follows:

$$\begin{aligned} u(\xi) &= \sum_{i=-m}^{m} a_i \left(\exp\left(\varphi(\xi)\right)\right)^i, \\ v(\xi) &= \sum_{i=-n}^{n} b_i \left(\exp\left(\varphi(\xi)\right)\right)^i, \\ w(\xi) &= \sum_{i=-k}^{k} c_i \left(\exp\left(\varphi(\xi)\right)\right)^i, \end{aligned} \tag{23}$$

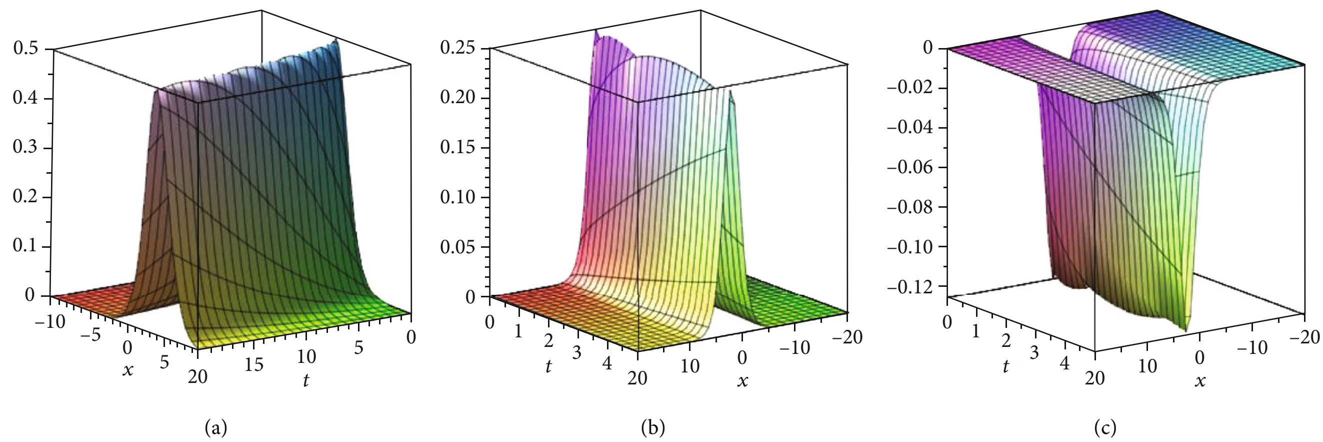


Figure 3: Plots of solution (20) for $\alpha = 0.5$, $b_1 = \tau = c_2 = 1$.

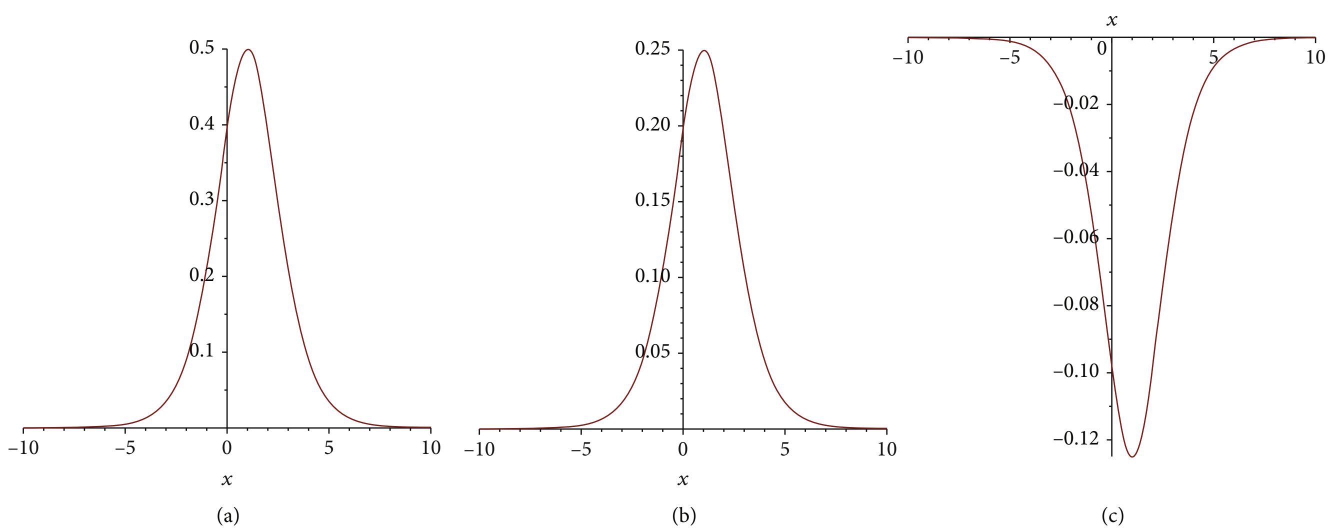


Figure 4: Plots of solution (20) for $\alpha = 0.5$, $b_1 = \tau = c_2 = 1$, and $t = 1$.

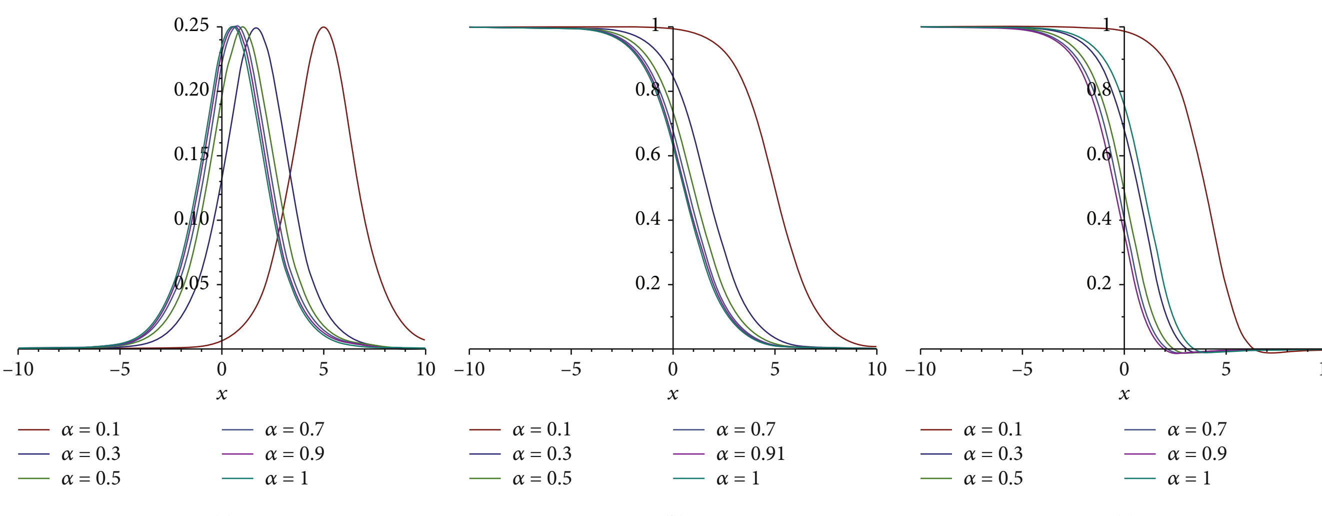
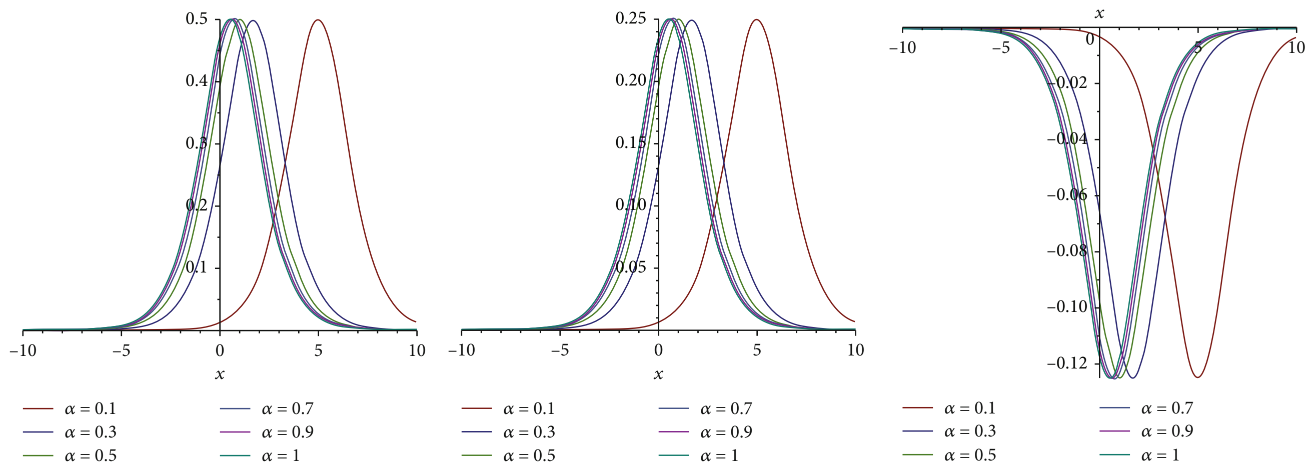


Figure 5: Plots of solution (19) for $b_1 = \tau = c_2 = 1$ and $t = 1$ for different values of α.



(a) (b) (c)

FIGURE 6: Plots of solution (20) for $b_1 = \tau = c_2 = 1$ and $t = 1$ for different values of α.

where $\varphi(\xi)$ is satisfied in equation (8). The homogeneous balance between linear and nonlinear terms in each equation of (22) leads to $m = 2$, $n = 1$ or 2, and k is arbitrary. So, solution (23) will be written as follows:

$$\begin{aligned} u(\xi) &= a_2 e^{2\varphi(\xi)} + a_1 e^{\varphi(\xi)} + a_0 + a_{-1} e^{-\varphi(\xi)} + a_{-2} e^{-2\varphi(\xi)}, \\ v(\xi) &= b_1 e^{\varphi(\xi)} + b_0 + b_{-1} e^{-\varphi(\xi)}, \\ w(\xi) &= c_1 e^{\varphi(\xi)} + c_0 + c_{-1} e^{-\varphi(\xi)}. \end{aligned} \tag{24}$$

Putting (24) in (22) and putting the coefficient of $e^{\varphi(x)}$ equal to zero yield a system of algebraic equations, which solving by maple leads to the following:

Case 1.

$$\begin{gathered} \alpha = 0, \\ \beta = 0, \\ \nu = -\frac{3\tau\left(-b_{-1}^2 + a_{-2}\right)}{a_{-2}}, \\ a_{-1} = 0, \\ a_0 = -\frac{-b_{-1}^2 + a_{-2}}{a_{-2}}, \\ a_1 = 0, \\ a_2 = 0, \\ b_1 = 0, \\ c_1 = 0, \\ c_{-1} = 2b_{-1}b_0, \\ c_0 = \frac{a_{-2}^2 b_0^2 - b_{-1}^4 + a_{-2} b_{-1}^2}{a_{-2}^2}. \end{gathered} \tag{25}$$

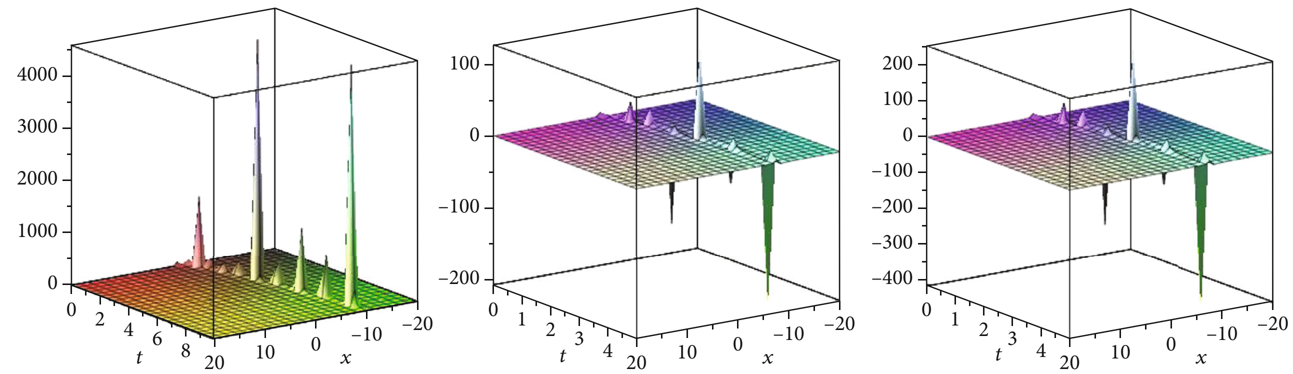
By placing the abovementioned solution into (24), the following exact solution will be derived:

$$\begin{aligned} u(\xi) &= -\frac{-b_{-1}^2 + a_{-2}}{a_{-2}} + a_{-2}(\xi + c)^{-2}, \\ v(\xi) &= b_0 + \frac{b_{-1}}{(\xi + c)}, \\ w(\xi) &= \frac{a_{-2}^2 b_0^2 - b_{-1}^4 + a_{-2} b_{-1}^2}{a_{-2}^2} + \frac{2b_{-1}b_0}{(\xi + c)}, \end{aligned} \tag{26}$$

where $\xi = \tau x + (3\tau(-b_{-1}^2 + a_{-2})t^{\alpha}/a_{-2}\alpha)$.

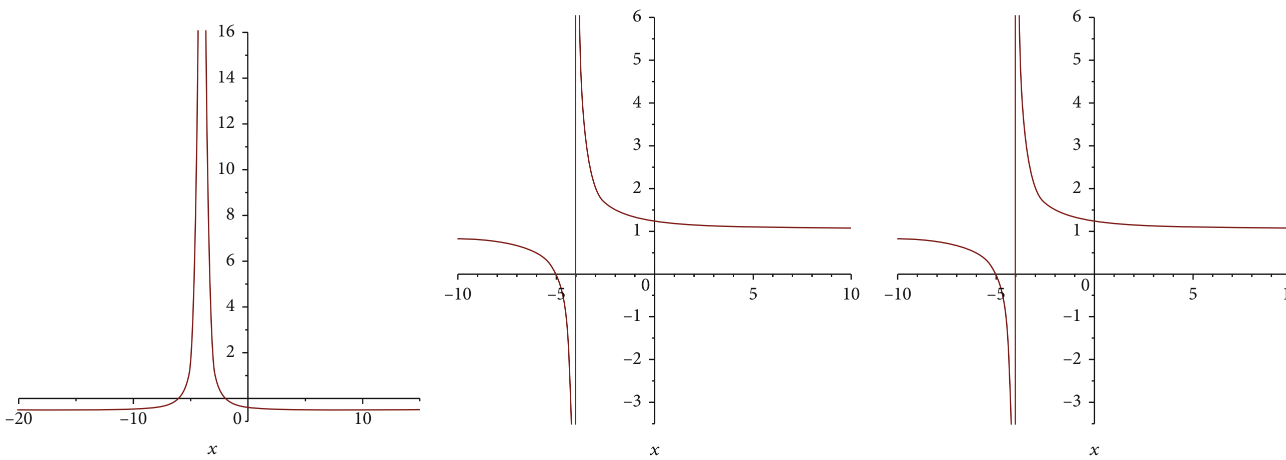
Case 2.

$$\begin{gathered} \alpha = -\frac{1}{2}\frac{3\tau^2 + \tau\upsilon + 3b_{-1}^2}{\tau^4}, \\ \beta = 0, \\ a_{-2} = -\tau^2, \\ a_{-1} = 0, \\ a_0 = \frac{1}{2}\frac{\tau^2 + \tau\upsilon + b_{-1}^2}{\tau^2}, \\ a_1 = 0, \\ a_2 = 0, \\ b_0 = \frac{1}{2}, \\ b_1 = 0, \\ c_1 = 0, \\ c_{-1} = b_{-1}, \\ c_0 = \frac{1}{24}\frac{3\tau^2 - 10\tau^3\upsilon - 3\tau^2\upsilon^2 + 6\tau^2 b_{-1}^2 + 2\tau\upsilon b_{-1}^2 + 9b_{-1}^4}{\tau^4}. \end{gathered} \tag{27}$$



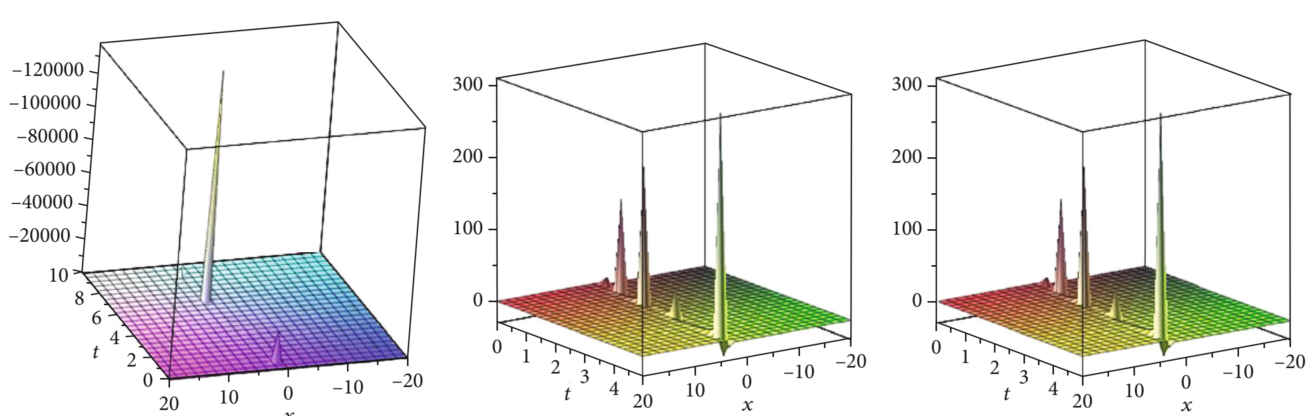
(a) (b) (c)

Figure 7: Plots of solution (26) for $\alpha = 0.5$, $b_{-1} = b_0 = \tau = c = 1$, and $a_{-2} = 2$.



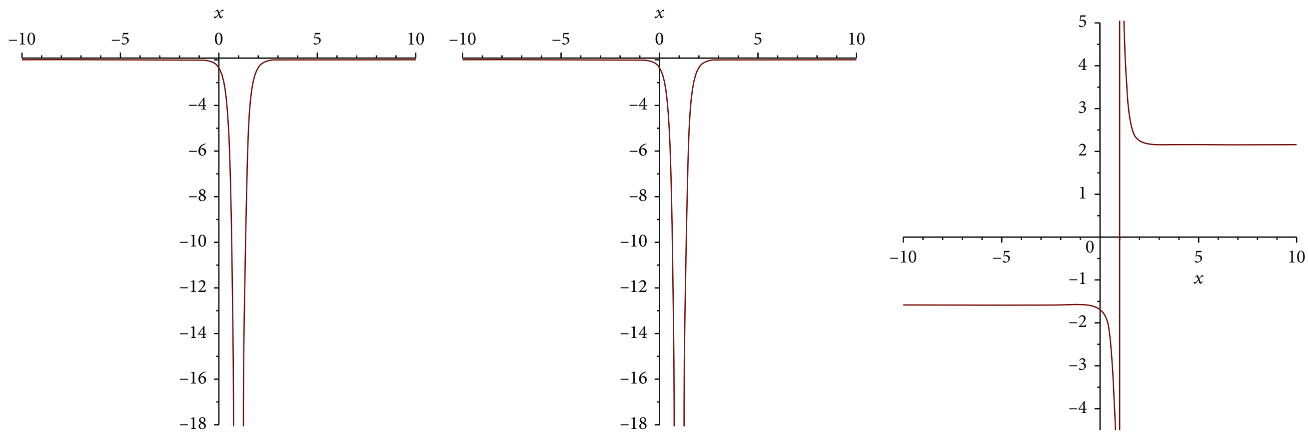
(a) (b) (c)

Figure 8: Plots of solution (26) for $\alpha = 0.5$, $b_{-1} = b_0 = \tau = c = t = 1$, and $a_{-2} = 2$.



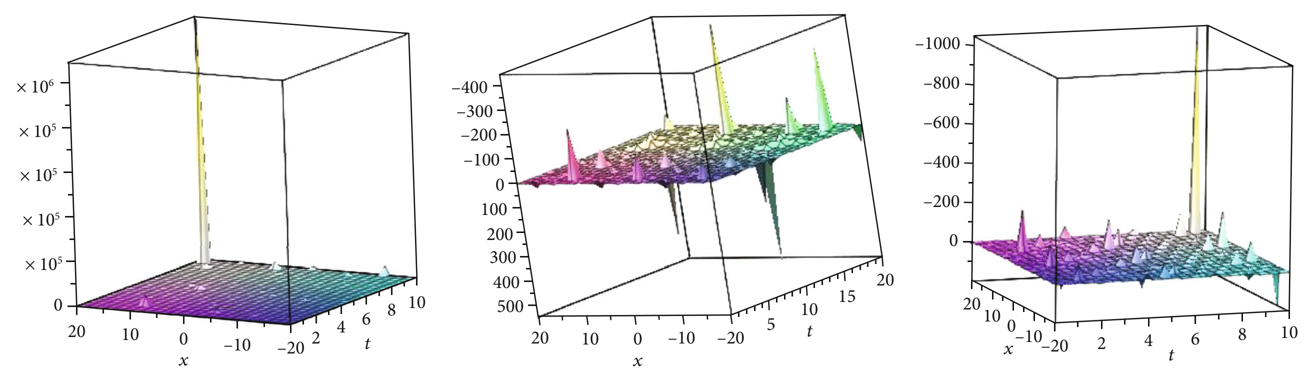
(a) (b) (c)

Figure 9: Plots of solution (28) for $\alpha = 0.5$, $b_{-1} = \nu = \tau = c = 1$.



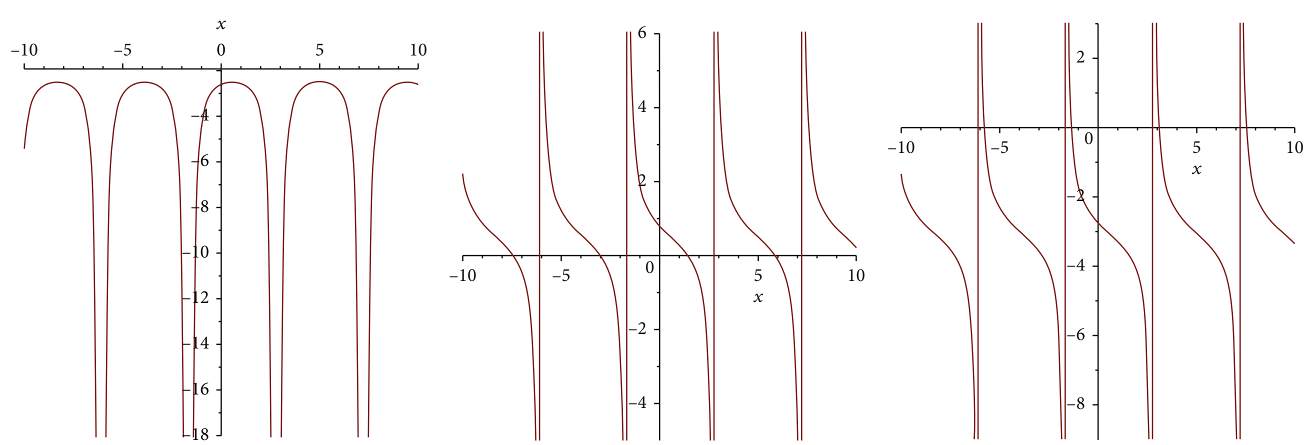
(a) (b) (c)

Figure 10: Plots of solution (28) for $\alpha = 0.5$, $b_{-1} = \nu = \tau = c = t = 1$.



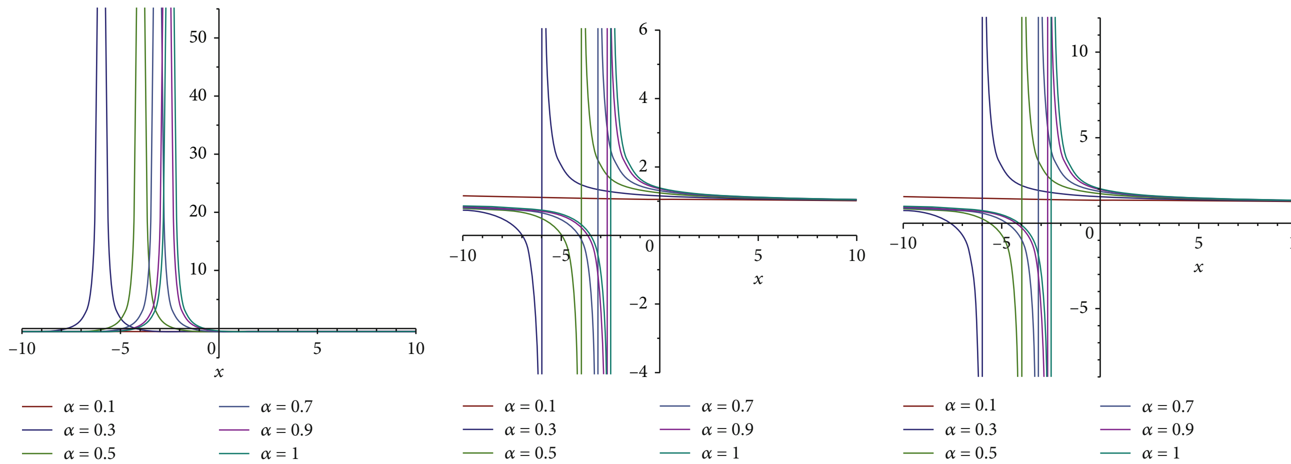
(a) (b) (c)

Figure 11: Plots of solution (29) for $\alpha = 0.5$, $b_{-1} = \tau = c = 1$, $\nu = -7$.



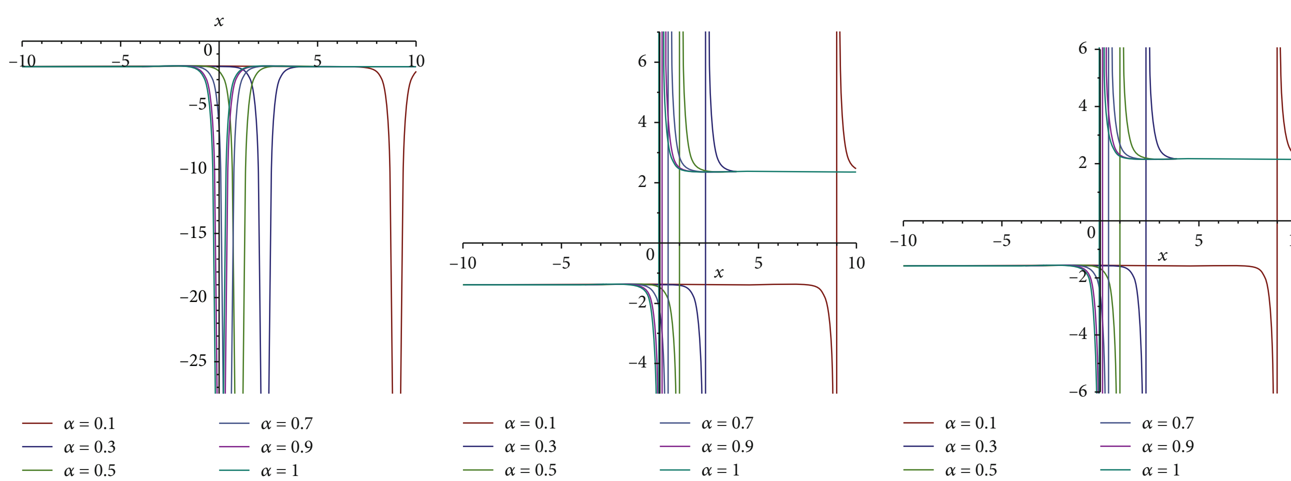
(a) (b) (c)

Figure 12: Plots of solution (29) for $\alpha = 0.5$, $b_{-1} = \tau = c = t = 1$, $\nu = -7$..



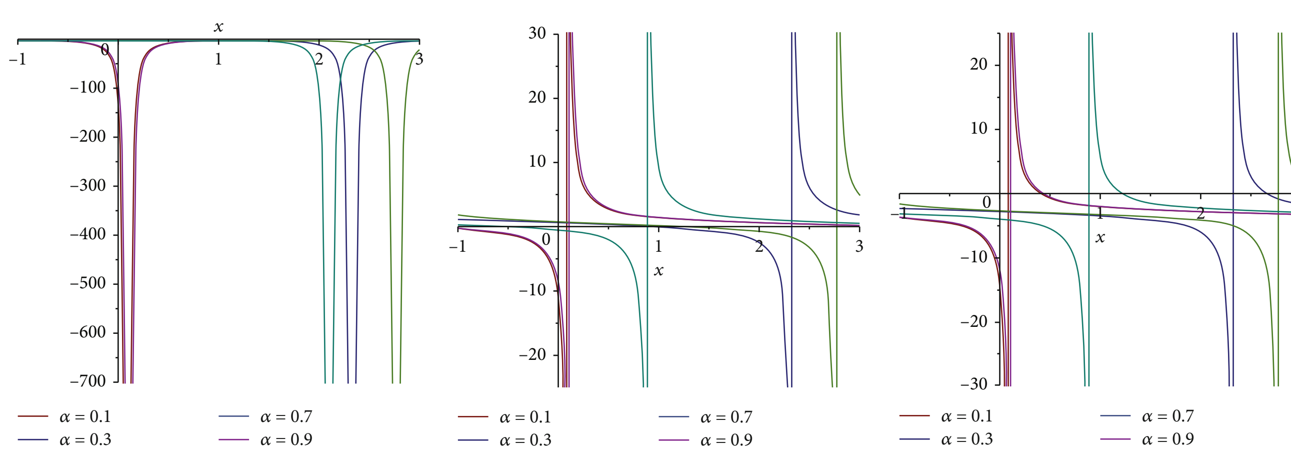
(a) (b) (c)

Figure 13: Plots of solution (26) for $b_{-1} = b_0 = \tau = c = 1$, $t = 1$, and $a_{-2} = 2$ for differnt values of α.



(a) (b) (c)

Figure 14: Plots of solution (28) for $\alpha = 0.5$, $b_{-1} = \nu = \tau = c = 1$ and $t = 1$ for differnt values of α.



(a) (b) (c)

Figure 15: Plots of solution (29) for $b_{-1} = \tau = c = 1$, $t = 1$, $\nu = -7$ for different value of α.

By placing the abovementioned solution into (24), the following exact solution will be derived:

If $3\tau^2+\tau\upsilon+3b_{-1}^2>0$,

$$\begin{aligned}
u(\xi)&=\frac{1}{2}\frac{\tau^2+\tau\upsilon+b_{-1}^2}{\tau^2}-\frac{3\tau^2+\tau\upsilon+3b_{-1}^2}{2\tau^2}\tan h^{-2}\\
&\quad\cdot\left(\frac{\sqrt{2\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{2\tau^2}(\xi+c)\right),\\
v(\xi)&=\frac{1}{2}+b_{-1}\frac{\sqrt{3\tau^2+\tau\upsilon+3b_{-1}^2}}{\tau^2\sqrt{2}}\tan h^{-1}\\
&\quad\cdot\left(\frac{\sqrt{2\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{2\tau^2}(\xi+c)\right),\\
w(\xi)&=\frac{1}{24}\frac{3\tau^2-10\tau^3\upsilon-3\tau^2\upsilon^2+6\tau^2b_{-1}^2+2\tau\upsilon b_{-1}^2+9b_{-1}^4}{\tau^4}\\
&\quad+b_{-1}\frac{\sqrt{3\tau^2+\tau\upsilon+3b_{-1}^2}}{\tau^2\sqrt{2}}\tan h^{-1}\\
&\quad\cdot\left(\frac{\sqrt{2\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{2\tau^2}(\xi+c)\right).
\end{aligned}\tag{28}$$

And, if $3\tau^2+\tau\upsilon+3b_{-1}^2<0$, so, we get

$$\begin{aligned}
u(\xi)&=\frac{1}{2}\frac{\tau^2+\tau\upsilon+b_{-1}^2}{\tau^2}+\frac{\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}{2\tau^2}\tan^{-2}\\
&\quad\cdot\left(\frac{\sqrt{-2\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{2\tau^2}(\xi+c)\right),\\
v(\xi)&=\frac{1}{2}+b_{-1}\frac{\sqrt{-\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{\sqrt{2}\tau^2}\tan^{-1}\\
&\quad\cdot\left(\frac{\sqrt{-2\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{2\tau^2}(\xi+c)\right),\\
w(\xi)&=\frac{1}{24}\frac{3\tau^2-10\tau^3\upsilon-3\tau^2\upsilon^2+6\tau^2b_{-1}^2+2\tau\upsilon b_{-1}^2+9b_{-1}^4}{\tau^4}\\
&\quad+b_{-1}\frac{\sqrt{-\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{\sqrt{2}\tau^2}\tan^{-1}\\
&\quad\cdot\left(\frac{\sqrt{-2\left(3\tau^2+\tau\upsilon+3b_{-1}^2\right)}}{2\tau^2}(\xi+c)\right),
\end{aligned}\tag{29}$$

where $\xi=\tau x-(\upsilon t^\alpha/\alpha)$.

In Figures 7–12, the plots of solutions (26)–(29), for some values of parameters, have been illustrated.

The plots of solutions (19) and (20) for different values of $\alpha=0.1,0.3,0.5,0.9,1$ are illustrated in Figures 13–15. These figures show the effect of the order of the fractional derivative on the solutions. In fact, one can compare the solutions for different values of α.

5. Conclusions

In the current paper, the soliton solutions of a system of fractional-order Hirota-Satsuma coupled KdV equations have been derived by two modifications of the Exp-function method. The outcomes indicate that the proposed methods are powerful, effective, and simple techniques to obtain the solution of partial differential equations of the fractional order in applied sciences. Most of the systems which could be converted to differential equations can be solved by this method. Furthermore, it has been for the first time that these methods have been used to solve system of fractional partial differential equation. The application of our method yields rational and soliton solution to these equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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