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Research Article

Global Existence and Blow-Up for a Weakly Dissipative Modified Two-Component Camassa-Holm System

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In this paper, we consider a weakly dissipative modified two-component Camassa-Holm system. We demonstrate a simple sufficient condition on initial data to guarantee blow-up of solutions in finite time, and the solutions exist globally in time.

1. Introduction

The well-known Camassa-Holm (CH) equation

$$m_t + um_x + 2u_x m = 0,\ m = u - u_{xx}, \tag{1}$$

which was first derived by Fokas and Fuchssteiner [1] and later derived as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [2]. After the birth of the CH equation, a lot of works have been carried out to it. For example, the CH equation has travelling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the travelling waves of largest amplitude [3–6]. It is shown in [7] that the blow-up occurs in the form of breaking waves; namely, the solution remains bounded but its slope becomes unbounded in finite time.

In general, it is difficult to avoid energy dissipation mechanisms in our real world. Ott and Sudan [8] studied how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on solitary solution of the KdV equation. Ghidaglia [9] investigated the long-time behavior of solutions to a weakly dissipative KdV equation as a finite-dimensional dynamical system. Inspired by the above works, Wu and Yin consider the following weakly dissipative CH equation [10, 11]:

$$m_t + um_x + 2u_x m + \lambda m = 0, \quad m = u_{xx}, \tag{2}$$

where λm is the weakly dissipative term and $\lambda > 0$ is a dissipative parameter. Wu and Yin show that if the initial momentum $m_0 = u_0 - u_{0xx}$ at some point $x_0 \in \mathbb{R}$ satisfies some sign condition, then the corresponding solution to Equation (2) exists globally in time and blows up in finite time. Novruzov and Hagverdiyev [12] derived a condition of the changing of the sign of m_0 at some point $x_0 \in \mathbb{R}$ to guarantee blow-up in finite time.

The two-component Camassa-Holm system is as follows:

$$\begin{cases} m_t + um_x + 2u_x m + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{3}$$

where $m = u - u_{xx}$, $u(t, x)$ describes the horizontal velocity of the fluid, and $\rho(t, x)$ describes the horizontal deviation of the surface from equilibrium. The system (3) appears initially in [13]; then, Constantin and Ivanov [14] give a demonstration about its derivation in view of the shallow water theory from the hydrodynamic point of view. Local well-posedness, blow-up, global existence, stability, and other mathematical properties can be seen in [15–22] and references therein.

The modified two-component Camassa-Holm system is as follows:

$$\begin{cases} m_t + um_x + 2u_x m + \rho\bar{\rho}_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{4}$$

where $m = u - u_{xx}$, $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, u denotes the velocity field, and $\bar{\rho}_0$ is taken to be a constant. The system (4) does admit peaked solutions in the velocity and average density; we refer this to Ref. [23] for details. In Ref. [23], the authors analytically identified the steepening mechanism that allows the singular solutions to emerge from smooth spatially confined initial data. They found that wave breaking in the fluid velocity does not imply singularity in the pointwise density ρ at the point of vertical slope. Some related work can be found in [24–29]. Let $\gamma = \bar{\rho} - \bar{\rho}_0$; then, the system (4) is equivalent to the following one, where $m = u - u_{xx}$ and $\rho = \gamma - \gamma_{xx}$.

In this paper, we are interested in the effect of the weakly dissipative term on the system (5) as follows:

$$\begin{cases} m_t + um_x + 2u_x m + \lambda m + \rho\gamma_x = 0, \\ \rho_t + (\rho u)_x + \lambda\rho = 0, \end{cases} \tag{5}$$

where $m = u - u_{xx}$, $\rho = \gamma - \gamma_{xx}$, and $\lambda > 0$ is a dissipative parameter. The main difference between the systems (4) and (5) is that the system (5) does not have conservation law.

$$\int_{\mathbb{R}} \left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right) dx. \tag{6}$$

In fact, for the system (5), $E(t)$ decays to zero as time goes to infinity (see Lemma 7). Recently, a blow-up result for the system (5) is presented in [30]. Similar to [10, 11], Ref. [30] shows that if the initial momentum $m_0 = u_0 - u_{0xx}$ at some point $x_0 \in \mathbb{R}$ satisfies some sign condition, then the corresponding solution to the system (5) blows up in finite time.

The main goal of the present paper is to demonstrate a simple condition guaranteeing blow-up of solutions in finite time and guaranteeing the solutions exist globally in time by using some properties of the solution generated by initial data. Our results could be stated as follows:

Theorem 1. *Suppose that $(u_0, \gamma_0) \in H^s(\mathbb{R})$ with $s \geq 5/2$, if*

$$\left(\|m_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2\right)^{1/2} < \frac{4\lambda}{3}. \tag{7}$$

Then, the corresponding solution $u(t, x)$ to the system (5) exists globally in time.

Theorem 2. *Suppose that $(u_0, \gamma_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 5/2$. Assume that $\rho_0(x_0) = 0$, $\rho_0(x) \geq 0$ on $(-\infty, x_0)$, and $\rho_0(x) \leq 0$ on (x_0, ∞), and assume further that*

$$2\left(\lambda + \frac{1}{\sqrt{2}}\left(\|u_0\|_{H^1}^2 + \|\gamma_0\|_{H^1}^2\right)^{1/2}\right) < u_0(x_0) - u_{0x}(x_0), \tag{8}$$

for some point x_0. Then, the corresponding solution to the system (5) blows up in finite time.

Remark 3. Note that our theorems do not need to assume that the initial momentum $m_0 = u_0 - u_{0xx}$ at some point $x_0 \in \mathbb{R}$ satisfies some sign condition, so our theorems improve the results in [30]. If $\rho = 0$, our theorems improve the global existence and blow-up results in [11] and cover the results in [12].

The rest of this paper is organized as follows. In Section 2, we recall several useful results which are crucial in the proof of Theorem 1 and Theorem 2. In Section 3 and Section 4, we complete the proof of our results.

2. Preliminaries

In this section, we recall several useful results to pursue our goal. First, we recall local well-posedness for the system (5).

Theorem 4 (see [30]). *Given $(u_0, \gamma_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 5/2$, there exist a maximal existence time $T = T(\|u_0\|_{H^s(\mathbb{R})}, \|\gamma_0\|_{H^{s-1}(\mathbb{R})})$ and a unique solution (u, γ) to the system (5) such that*

$$\begin{aligned}(u, \gamma) \in C\left([0, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\right) \\ \cap C^1\left([0, T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})\right).\end{aligned} \tag{9}$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping $(u_0, \gamma_0) \mapsto (u, \gamma)$:

$$\begin{aligned}H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \mapsto C\left([0, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\right) \\ \cap C^1\left([0, T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})\right)\end{aligned} \tag{10}$$

is continuous.

Next, we state the following precise blow-up scenario.

Theorem 5 (see [30]). *Let $(u_0, \gamma_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 5/2$ and $T > 0$ be the maximal existence time of the solution (u, γ) to the system (5) with initial data (u_0, γ_0). Then, the corresponding solution blows up in a finite time $T < \infty$ if and only if*

$$\liminf_{t \longrightarrow T} \inf_{x \in \mathbb{R}} \{u_x(t, x)\} = -\infty. \tag{11}$$

Consider the following initial value problem of ordinary differential equation (ODE):

$$\begin{cases} q_t = u(t, q(t, x)), & (t, x) \in (0, T) \times \mathbb{R}, \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \tag{12}$$

The following lemma will be used to prove our theorem.

Lemma 6 (see [30]). *Let $(u_0, \gamma_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 5/2$ and $T > 0$ be the maximal existence time of the solution (u, γ) to the system (5) with initial data (u_0, γ_0). Then, we have*

$$\rho(t, q(t, x)) q_x(t, x) = \rho_0(x) e^{-\lambda t}, \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{13}$$

Moreover, if there exist $x_0 \in \mathbb{R}$ such that $\rho_0(x_0) = 0$, then $\rho(t, q(t, x_0)) = 0$ for all $t \in [0, T)$.

Lemma 7 (see [30]). *Let $(u_0, \gamma_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 5/2$ and $T > 0$ be the maximal existence time of the solution (u, γ) to the system (5) with initial data (u_0, γ_0). Then, we have*

$$E(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right) dx = e^{-2\lambda t} \int_{\mathbb{R}} \left(u_0^2 + u_{0x}^2 + \gamma_0^2 + \gamma_{0x}^2\right) dx \tag{14}$$

or

$$E(t) = \|u\|_{H^1}^2 + \|\gamma\|_{H^1}^2 = e^{-2\lambda t} \left(\|u_0\|_{H^1}^2 + \|\gamma_0\|_{H^1}^2\right) = e^{-2\lambda t} E(0). \tag{15}$$

Lemma 8. *If $\rho_0(x_0) = 0$, $\rho_0(x) \geq 0$ on $(-\infty, x_0)$, and $\rho_0(x) \leq 0$ on (x_0, ∞), then for any fixed t,*

$$\gamma_x^2(t, x) - \gamma^2(t, x) \leq \gamma_x^2(t, q(t, x_0)) - \gamma^2(t, q(t, x_0)), \tag{16}$$

for all $x \in \mathbb{R}$.

Proof. If $x \leq q(t, x_0)$, then

$$\begin{aligned} \gamma_x^2(t, x) - \gamma^2(t, x) &= -\left(\int_{-\infty}^{q(t,x_0)} e^{\xi} \rho(t, \xi) d\xi - \int_{x}^{q(t,x_0)} e^{\xi} \rho(t, \xi) d\xi\right) \\ &\quad \times \left(\int_{q(t,x_0)}^{\infty} e^{-\xi} \rho(t, \xi) d\xi + \int_{x}^{q(t,x_0)} e^{-\xi} \rho(t, \xi) d\xi\right) \\ &= \gamma_x^2(t, q(t, x_0)) - \gamma^2(t, q(t, x_0)) \\ &\quad - \int_{-\infty}^{x} e^{\xi} \rho(t, \xi) d\xi \int_{x}^{q(t,x_0)} e^{-\xi} \rho(t, \xi) d\xi \\ &\quad + \int_{x}^{q(t,x_0)} e^{\xi} \rho(t, \xi) d\xi \int_{q(t,x_0)}^{\infty} e^{-\xi} \rho(t, \xi) d\xi \\ &\leq \gamma_x^2(t, q(t, x_0)) - \gamma^2(t, q(t, x_0)). \end{aligned} \tag{17}$$

Similarly, if $x \geq q(t, x_0)$, we also have

$$\gamma_x^2(t, x) - \gamma^2(t, x) \leq \gamma_x^2(t, q(t, x_0)) - \gamma^2(t, q(t, x_0)). \tag{18}$$

Therefore, we complete the proof of Lemma 8. □

3. Proof of Theorem 1

First, multiplying the first equation of the system (5) by $2m$ and integrating by parts, we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} m^2 dx &= -2 \int_{\mathbb{R}} u m m_x dx - 4 \int_{\mathbb{R}} m^2 u_x dx \\ &\quad - 2\lambda \int_{\mathbb{R}} m^2 dx - 2 \int_{\mathbb{R}} m \rho \gamma_x dx \\ &= -3 \int_{\mathbb{R}} m^2 u_x dx - 2\lambda \int_{\mathbb{R}} m^2 dx - 2 \int_{\mathbb{R}} m \rho \gamma_x dx. \end{aligned} \tag{19}$$

Similarly,

$$\frac{d}{dt} \int_{\mathbb{R}} \rho^2 dx = -\int_{\mathbb{R}} \rho^2 u_x dx - 2\lambda \int_{\mathbb{R}} \rho^2 dx. \tag{20}$$

Adding (19) and (20), we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(m^2 + \rho^2\right) dx + 2\lambda \int_{\mathbb{R}} \left(m^2 + \rho^2\right) dx \\ &\quad = -3 \int_{\mathbb{R}} m^2 u_x dx - \int_{\mathbb{R}} \rho^2 u_x dx - 2 \int_{\mathbb{R}} m \rho \gamma_x dx. \end{aligned} \tag{21}$$

Multiplying (21) by $e^{2\lambda t}$ yields

$$\begin{aligned} \frac{d}{dt} \left(e^{2\lambda t} \int_{\mathbb{R}} \left(m^2 + \rho^2\right) dx\right) &= -3e^{2\lambda t} \int_{\mathbb{R}} m^2 u_x dx - e^{2\lambda t} \int_{\mathbb{R}} \rho^2 u_x dx \\ &\quad - 2e^{2\lambda t} \int_{\mathbb{R}} m \rho \gamma_x dx. \end{aligned} \tag{22}$$

Note that $p(x) \stackrel{\Delta}{=} (1/2) e^{-|x|}$; we have $\left(1 - \partial_x^2\right)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * m = u$, where we denote by $ * $ the convolution. Then, taking the Young inequality, one gets

$$\begin{aligned} &\|u_x\|_{L^\infty} = \|p_x * m\|_{L^\infty} \leq \|p_x\|_{L^2} \|m\|_{L^2} \leq \frac{1}{2} \|m\|_{L^2}, \\ &\int_{\mathbb{R}} \rho^2 u_x dx \leq \|u_x\|_{L^\infty} \int_{\mathbb{R}} \rho^2 dx \leq \frac{1}{2} \|m\|_{L^2} \int_{\mathbb{R}} \rho^2 dx, \\ &\int_{\mathbb{R}} m \rho \gamma_x dx \leq \|m\|_{L^\infty} \left| \int_{\mathbb{R}} \rho \gamma_x dx \right| = 0. \end{aligned} \tag{23}$$

Hence, we have

$$\begin{aligned} \frac{d}{dt} \left(e^{2\lambda t} \int_{\mathbb{R}} \left(m^2 + \rho^2\right) dx\right) &\leq \frac{3}{2} e^{2\lambda t} \left(\int_{\mathbb{R}} m^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} m^2 dx\right) \\ &\quad + \frac{1}{2} e^{2\lambda t} \left(\int_{\mathbb{R}} m^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} \rho^2 dx\right) \\ &\leq \frac{3}{2} e^{2\lambda t} \left(\int_{\mathbb{R}} \left(m^2 + \rho^2\right) dx\right)^{3/2} \\ &= \frac{3}{2} e^{-\lambda t} \left(e^{2\lambda t} \int_{\mathbb{R}} \left(m^2 + \rho^2\right) dx\right)^{3/2}. \end{aligned} \tag{24}$$

It is easy to derive that

$$\frac{d}{dt}\left(e^{2\lambda t}\int_{\mathbb{R}}\left(m^2+\rho^2\right)dx\right)^{-(1/2)} \geq -\frac{3}{4}e^{-\lambda t}. \quad (25)$$

Integrating from 0 to t yields

$$\left(e^{2\lambda t}\int_{\mathbb{R}}\left(m^2+\rho^2\right)dx\right)^{-(1/2)} - \left(\int_{\mathbb{R}}\left(m_0^2+\rho_0^2\right)dx\right)^{-(1/2)} \geq \frac{3}{4\lambda}\left(e^{-\lambda t}-1\right) \geq -\frac{3}{4\lambda}. \quad (26)$$

Thus, it follows that

$$\left(\|m\|_{L^2}^2+\|\rho\|_{L^2}^2\right)^{1/2} \leq e^{-\lambda t}\left(\left(\|m_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)^{-(1/2)}-\frac{3}{4\lambda}\right)^{-1}, \quad (27)$$

due to $\left(\|m_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)^{1/2} < (4\lambda/3)$ and

$$\begin{aligned}\|u_x\|_{L^\infty} &\leq \frac{1}{2}\|m\|_{L^2} < \left(\|m\|_{L^2}^2+\|\rho\|_{L^2}^2\right)^{1/2}\\ &< e^{-\lambda t}\left(\left(\|m_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)^{-(1/2)}-\frac{3}{4\lambda}\right)^{-1}.\end{aligned} \quad (28)$$

According to Theorem 5, we have that the solution exists globally in time.

4. Proof of Theorem 2

First, we introduce the following notation:

$$I(t)=\int_{-\infty}^{q(t,x_0)} e^{\xi} m(t,\xi)\,d\xi. \quad (29)$$

Differentiating $I(t)$ with respect to x, we get

$$\frac{dI(t)}{dt}=\int_{-\infty}^{q(t,x_0)} e^{\xi} m_t(t,\xi)\,d\xi + e^{q(t,x_0)} m(t,q(t,x_0))q_t(t,x_0). \quad (30)$$

By using the first equation of the system (5) and integrating by parts, we have

$$\begin{aligned}&\int_{-\infty}^{q(t,x_0)} e^{\xi} m_t(t,\xi)\,d\xi\\ &=\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(-(um)_x-u_x m-\lambda m-\rho\gamma_x\right)d\xi\\ &=-e^{\xi}um\Big|_{-\infty}^{q(t,x_0)}+\int_{-\infty}^{q(t,x_0)} e^{\xi}um\,d\xi-\frac{1}{2}e^{\xi}\left(u^2-u_x^2\right)\Big|_{-\infty}^{q(t,x_0)}\\ &\quad+\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(u^2-u_x^2\right)d\xi-\lambda\int_{-\infty}^{q(t,x_0)} e^{\xi}m\,d\xi\\ &\quad+\frac{1}{2}e^{q(t,x_0)}\gamma_x^2(t,q(t,x_0))-\frac{1}{2}e^{q(t,x_0)}\gamma^2(t,q(t,x_0))\\ &\quad-\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(\gamma_x^2-\gamma^2\right)d\xi\\ &=-e^{\xi}um\Big|_{-\infty}^{q(t,x_0)}+\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(u^2+\frac{1}{2}u_x^2\right)d\xi-\lambda\int_{-\infty}^{q(t,x_0)} e^{\xi}m\,d\xi\\ &\quad+\frac{1}{2}e^{\xi}u_x^2\Big|_{-\infty}^{q(t,x_0)}-e^{\xi}uu_x\Big|_{-\infty}^{q(t,x_0)}+\frac{1}{2}e^{q(t,x_0)}\gamma_x^2(t,q(t,x_0))\\ &\quad-\frac{1}{2}e^{q(t,x_0)}\gamma^2(t,q(t,x_0))-\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(\gamma_x^2-\gamma^2\right)d\xi.\end{aligned} \quad (31)$$

Note that

$$\begin{aligned}&\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(u^2+\frac{1}{2}u_x^2\right)d\xi\\ &=\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}u^2\,d\xi+\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(u^2+u_x^2\right)d\xi\\ &\geq\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}u^2\,d\xi+\frac{1}{2}\int_{-\infty}^{q(t,x_0)} 2e^{\xi}uu_x\,d\xi\\ &\geq\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}u^2\,d\xi+\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(u^2\right)_x d\xi\\ &=\frac{1}{2}e^{q(t,x_0)}u^2(t,q(t,x_0)).\end{aligned} \quad (32)$$

Inserting (31), (32), and (12) into (30), we have

$$\begin{aligned}\frac{dI(t)}{dt}+\lambda\int_{-\infty}^{q(t,x_0)} e^{\xi}m\,d\xi &\geq \frac{1}{2}e^{q(t,x_0)}u^2(t,q(t,x_0))+\frac{1}{2}e^{\xi}u_x^2\Big|_{-\infty}^{q(t,x_0)}\\ &\quad-e^{\xi}uu_x\Big|_{-\infty}^{q(t,x_0)}-e^{\xi}um\Big|_{-\infty}^{q(t,x_0)}\\ &\quad+e^{q(t,x_0)}m(t,q(t,x_0))q_t(t,x_0)\\ &\quad+\frac{1}{2}e^{q(t,x_0)}\gamma_x^2(t,q(t,x_0))\\ &\quad-\frac{1}{2}e^{q(t,x_0)}\gamma^2(t,q(t,x_0))\\ &\quad-\frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi}\left(\gamma_x^2-\gamma^2\right)d\xi\\ &\geq\frac{1}{2}e^{q(t,x_0)}u^2(t,q(t,x_0))\\ &\quad-e^{q(t,x_0)}u(t,q(t,x_0))u_x(t,q(t,x_0))\\ &\quad+\frac{1}{2}e^{q(t,x_0)}u_x^2(t,q(t,x_0))\\ &=\frac{1}{2}e^{q(t,x_0)}\left(u(t,q(t,x_0))-u_x(t,q(t,x_0))\right)^2.\end{aligned} \quad (33)$$

Next, by integrating by parts, we have

$$\begin{aligned} I(t) &= \int_{-\infty}^{q(t,x_0)} e^{\xi} m(t,\xi)\,d\xi \\ &= \int_{-\infty}^{q(t,x_0)} e^{\xi} u\,d\xi + \int_{-\infty}^{q(t,x_0)} e^{\xi} u_x\,d\xi - e^{q(t,x_0)} u_x(t,q(t,x_0)) \\ &= e^{q(t,x_0)}\left(u(t,q(t,x_0)) - u_x(t,q(t,x_0))\right). \end{aligned} \quad (34)$$

Thus, from (33) and (34), it is obvious that

$$\frac{dI(t)}{dt} \geq \frac{1}{2} e^{-q(t,x_0)} I(t)^2 - \lambda I(t). \quad (35)$$

Multiplying (35) by $e^{-q(t,x_0)}$, we have

$$e^{-q(t,x_0)} \frac{dI(t)}{dt} \geq \frac{1}{2} e^{-2q(t,x_0)} I(t)^2 - \lambda e^{-q(t,x_0)} I(t). \quad (36)$$

Due to $q_t(t,x_0) = u(t,q(t,x_0))$, adding $-q_t(t,x_0)e^{-q(t,x_0)} I(t)$ to the left side of (36) and $-u(t,q(t,x_0))e^{-q(t,x_0)} I(t)$ to the right side of (36) yields

$$\begin{aligned} \frac{d}{dt}\left(e^{-q(t,x_0)} I(t)\right) &\geq \frac{1}{2}\left(e^{-q(t,x_0)} I(t)\right)^2 - (\lambda + u)\left(e^{-q(t,x_0)} I(t)\right) \\ &= \left(\frac{1}{2} e^{-q(t,x_0)} I(t) - \lambda - u\right)\left(e^{-q(t,x_0)} I(t)\right). \end{aligned} \quad (37)$$

Hence, $(d/dt)(e^{-q(t,x_0)} I(t)) > 0$ holds as long as $(1/2) e^{-q(t,x_0)} I(t) > \lambda + u$ and $e^{-q(t,x_0)} I(t) > 0$. In view of Lemma 7, we have

$$\begin{aligned} \|u\|_{L^\infty} &\leq \frac{1}{\sqrt{2}} \|u\|_{H^1} \leq \frac{1}{\sqrt{2}} \left(\|u\|_{H^1}^2 + \|\gamma\|_{H^1}^2\right)^{1/2} \\ &\leq \frac{1}{\sqrt{2}} e^{-\lambda t} \sqrt{E(0)}. \end{aligned} \quad (38)$$

Note that in (34) and the condition of Theorem 2, we get

$$\begin{aligned} \frac{1}{2} e^{-q_0(x_0)} I(0) = \frac{1}{2}\left(u_0(x_0) - u_{0x}(x_0)\right) &> \lambda + \frac{1}{\sqrt{2}} \sqrt{E(0)} \\ &\geq \lambda + \|u_0\|_{L^\infty} \geq \lambda + u_0(x_0). \end{aligned} \quad (39)$$

Therefore, we can conclude that $(d/dt)(e^{-q(t,x_0)} I(t)) = (d/dt)(u(t,q(t,x_0)) - u_x(t,q(t,x_0))) > 0$ holds. Since $u(t,q(t,x_0)) - u_x(t,q(t,x_0))$ is a continuous function, so we have

$$\begin{aligned} \frac{1}{2}\left(u(t,q(t,x_0)) - u_x(t,q(t,x_0))\right) &> \frac{1}{2}\left(u_0(x_0) - u_{0x}(x_0)\right) \\ &> \lambda + \frac{1}{\sqrt{2}} \sqrt{E(0)}. \end{aligned} \quad (40)$$

Lemma 7 implies that $u(t,q(t,x_0)) \longrightarrow 0$ as $t \longrightarrow \infty$, so there exist a $T_0 > 0$ such that for $t > T_0$

$$-\frac{1}{2} u_x(t,q(t,x_0)) > \lambda + \frac{1}{\sqrt{2}} \sqrt{E(0)}. \quad (41)$$

Thus, we have

$$\begin{aligned} &-\frac{1}{2} u_x^2(t,q(t,x_0)) - \lambda u_x(t,q(t,x_0)) \\ &= u_x(t,q(t,x_0))\left(-\frac{1}{2} u_x(t,q(t,x_0)) - \lambda\right) \\ &= -|u_x(t,q(t,x_0))|\left(-\frac{1}{2} u_x(t,q(t,x_0)) - \lambda\right) \\ &< -2\left(\lambda + \frac{1}{\sqrt{2}} \sqrt{E(0)}\right) \frac{1}{\sqrt{2}} \sqrt{E(0)} \\ &= -\sqrt{2}\left(\lambda + \frac{1}{\sqrt{2}} \sqrt{E(0)}\right) \sqrt{E(0)}. \end{aligned} \quad (42)$$

Note that $p(x) \overset{\Delta}{=} (1/2)e^{-|x|}$, we have $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * m = u$, where we denote by $*$ the convolution. Then, we can rewrite the first equation of the system (5) as follows:

$$u_t + u u_x + \partial_x p * \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \gamma^2 - \frac{1}{2} \gamma_x^2\right) + \lambda u = 0. \quad (43)$$

Differentiating (43) with respect to x, we get

$$\begin{aligned} u_{tx} + u u_{xx} = u^2 - \frac{1}{2} u_x^2 + \frac{1}{2} \gamma^2 - \frac{1}{2} \gamma_x^2 - p \\ * \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \gamma^2 - \frac{1}{2} \gamma_x^2\right) - \lambda u_x, \end{aligned} \quad (44)$$

then

$$\begin{aligned} \frac{d}{dt} u_x(t,q(t,x_0)) &= u_{tx}(t,q(t,x_0)) + u(t,q(t,x_0)) u_{xx}(t,q(t,x_0)) \\ &= u^2(t,q(t,x_0)) - \frac{1}{2} u_x^2(t,q(t,x_0)) \\ &\quad + \frac{1}{2} \gamma^2(t,q(t,x_0)) - \frac{1}{2} \gamma_x^2(t,q(t,x_0)) - p \\ &\quad * \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \gamma^2 - \frac{1}{2} \gamma_x^2\right) \\ &\quad - \lambda u_x(t,q(t,x_0)) \\ &\leq u^2(t,q(t,x_0)) - \frac{1}{2} u_x^2(t,q(t,x_0)) - p \\ &\quad * \left(u^2 + \frac{1}{2} u_x^2\right) - \lambda u_x(t,q(t,x_0)) \\ &\leq \frac{1}{2} u^2(t,q(t,x_0)) - \frac{1}{2} u_x^2(t,q(t,x_0)) \\ &\quad - \lambda u_x(t,q(t,x_0)), \end{aligned} \quad (45)$$

where we used Lemma 6 and the inequality $p * (u^2 + (1/2)u_x^2) \geq (1/2)u^2$. Then, by (42) and (45), we can infer that

$$\frac{du_x(t, q(t, x_0))}{dt} < \frac{1}{2}u^2(t, q(t, x_0)) - \sqrt{2}\left(\lambda + \frac{1}{\sqrt{2}}\sqrt{E(0)}\right)\sqrt{E(0)}. \quad (46)$$

Further, by Lemma 7, for sufficiently large t, we have

$$\frac{1}{2}u^2(t, q(t, x_0)) < \left(\sqrt{2} - \frac{1}{2}\right)\left(\lambda + \frac{1}{\sqrt{2}}\sqrt{E(0)}\right)\sqrt{E(0)}. \quad (47)$$

Then, there exists a T_1 such that for $t > T_1 > T_0$,

$$\frac{du_x(t, q(t, x_0))}{dt} < -\frac{1}{2}\left(\lambda + \frac{1}{\sqrt{2}}\sqrt{E(0)}\right)\sqrt{E(0)}. \quad (48)$$

Thus, integrating from T_1 to t, we get

$$u_x(t, q(t, x_0)) < u_x(T_1, q(T_1, x_0)) - \frac{1}{2}\left(\lambda + \frac{1}{\sqrt{2}}\sqrt{E(0)}\right)\sqrt{E(0)}(t - T_1). \quad (49)$$

This means that we can choose a $T_2 > T_1$ such that for all $t > T_2$,

$$-\lambda u_x(t, q(t, x_0)) < \frac{1}{8}u_x^2(t, q(t, x_0)), \quad (50)$$

and by Lemma 7, we have

$$\frac{1}{2}u^2(t, q(t, x_0)) < \frac{1}{8}u_x^2(t, q(t, x_0)), \quad (51)$$

for $t > T_2$.

Then, substituting (50) and (51) into (45) yields

$$\begin{aligned}\frac{d}{dt}u_x(t, q(t, x_0)) &\leq \frac{1}{2}u^2(t, q(t, x_0)) - \frac{1}{2}u_x^2(t, q(t, x_0)) - \lambda u_x(t, q(t, x_0)) \\ &< \frac{1}{8}u_x^2(t, q(t, x_0)) - \frac{1}{2}u_x^2(t, q(t, x_0)) + \frac{1}{8}u_x^2(t, q(t, x_0)) \\ &= -\frac{1}{4}u_x^2(t, q(t, x_0)),\end{aligned} \quad (52)$$

which leads to

$$-\frac{d}{dt}\left(\frac{1}{u_x(t, q(t, x_0))}\right) < -\frac{1}{4}, \quad (53)$$

and integrating from T_2 to t gives

$$0 < -\frac{1}{u_x(t, q(t, x_0))} < -\frac{1}{u_x(T_2, q(T_2, x_0))} - \frac{1}{4}(t - T_2). \quad (54)$$

Thus, if we suppose that the solution exists globally, then for sufficiently large $t^* > T_2$,

$$0 < -\frac{1}{u_x(t, q(t, x_0))} < -\frac{1}{u_x(T_2, q(T_2, x_0))} - \frac{1}{4}(t^* - T_2) < 0, \quad (55)$$

which contradicts that the solution blows up in finite time. Therefore, we complete the proof of Theorem 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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