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Research Article

Periodic, Cross-Kink, and Interaction between Stripe and Periodic Wave Solutions for Generalized Hietarinta Equation: Prospects for Applications in Environmental Engineering

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In the current work, the modified $(2+1)$-dimensional Hietarinta model is considered by employing Hirota's bilinear scheme. Likewise, the bilinear formalism is obtained for the considered model. In addition, the periodic-solitary, periodic wave, cross-kink wave, and interaction between stripe and periodic wave solutions of the mentioned equation by particular coefficients are offered. The obtained results may be used in the description of the model in fruitful way. Finally, by using the available situations, the physical demeanor of solutions is discussed in the given method. We demonstrated that these solutions validated the program using Maple and found them correct. Moreover, a lot of graphs in some sections to determine the analysis of obtained findings for the aforementioned equation are given. The achieved solutions are also verified by using the Maple software. These periodic wave solutions suggest that these three methods are useful, easy to use, and effective than other methods.

1. Introduction

Research in the field of nonlinear wave theory has been become very interesting due to its applications in sciences and engineering. Many physical phenomena are represented as models in the structure of nonlinear PDEs, mostly in the form of nonlinear integrable equations. These models clearly indicate the parameters that affect the phenomenon that are not seen directly by observing the phenomenon. Various models have been made in the field of science and engineering that are representing the different phenomenon. For example, mostly naturally occurring phenomenons are modeled as modified $\tan(\varphi/2)$-expansion technique [1], the homotopy perturbation scheme [2], the csch-function method [3], the Lie symmetry analysis [4], the Bäcklund transformation method [5], the sine-Gordon expansion approach [6], a new nonlinear mathematical programming model for dynamic cell formation [7], the $(G'/G, 1/G)$, the modified (G'/G^2) and $(1/G')$-expansion schemes [8], and imperialist competitive algorithm [9]. Except of these methods, there are other powerful methods such as the multiple exp-function method [10–13], a new fuzzy classification algorithm [14], Hirota's bilinear method [15–21], a deterministic mathematical mixed integer linear programming model [22], the coupled modified Korteweg-de Vries equation with nonzero boundary conditions at infinity [23], the high-order rogue wave of generalized non-linear Schrödinger equation with nonzero boundary [24], the supersymmetric constrained B type and C type KP hierarchies of Manin-Radul and Jacobian types [25], the $(3+1)$-

dimensional extended Jimbo–Miwa equations [26], understanding by design model as useful tool for a meaningful and permanent learning [27], and the first integral method for constructing the exact solutions of the time-fractional Wu-Zhang system [28].

In Ref. [29], the authors suggested the fuzzy clustering to discover the optimal number of clusters as an innovation clustering algorithm in marketing to determine the best group of customers, similar items, and products. In a valuable research, the Bayer–Hanck cointegration test, wavelet coherence, Fourier Toda–Yamamoto, and Breitung–Candelon frequency-domain spectral causality tests were investigated the causal relationships among carbon emissions, economic growth, and life expectancy in [30]. Adinda et al. [31] studied students' metacognitive awareness failures about solving absolute value problems (AVPs) in mathematics education, and they found that there was a significant failure, and three students were sampled from who had experienced different metacognitive awareness failures in solving AVPs. In [32], the residual power series method to solve the $(3+1)$-dimensional nonlinear conformable Schrödinger equation with cubic-quintic-septic nonlinearities along with three test applications was considered subject to different initial conditions. Two classes of lump and line rogue wave solutions for a new $(2+1)$-dimensional extension of the Hietarinta equation were obtained by means of the Hirota bilinear scheme by Manukure and Zhou [32]. In [33], the authors showed the existence of the three-periodic wave solutions numerically for the Hietarinta equation by using the direct method. Both Dirichlet and Neumann data on some part of the domain boundary for a family of quasilinear inverse problems to the Laplace equation coupled with a sequence of nonlinear scalar equations were recovered [34]. A novel integral transform involving the product of the Whittaker function and two Bessel functions of the first kind was employed to Bessel-Circular-Gaussian beam to generate a new laser beam called Exton-Gaussian beams [35]. The complete discrimination system method was used to construct the exact traveling wave solutions for fractional coupled Boussinesq equations in the sense of conformable fractional derivatives by Han and Li [36]. The periodic, cross-kink wave solutions were obtained by the authors of [37] by the help of Hirota bilinear operator, and also, the semi-inverse variational principle was utilized for the $(2+1)$-dimensional generalized Hirota–Satsuma–Ito equation. In [38], the effects of Mobile Ad Wearout on irritation, intrusiveness, engagement, and loyalty via social media outlets were studied. Author of [39] studied the mathematical models for global solar radiation intensity estimation at Shakardara area which is to estimate atmospheric transparency percentage. Fauzi and Respati [40] analyzed and studied the differences in students' critical thinking skills utilizing the guided discovery learning model and the problem-based learning model including both theoretical and practical knowledge and skills, and also, they used quantitative methods through an experimental approach. The present research focuses on the Hirota bilinear scheme to getting the analytical solutions of nonlinear $(2+1)$-dimensional wave equation. In this considered scheme, the solutions are written as a combination of trigonometric and hyperbolic waves and also a combination of trigonometric and exponential waves so that the solutions can adapt easily made by symbolic estimations.

The fundamental work of this paper is to extract new analytical findings of $(2+1)$-D generalized Hietarinta model. For the purpose, determining the solutions of the shown model by powerful technique has been made. Many kinds of schemes have been used to determine the new kinds of solitons of this model, such as, two good papers in references [41, 42]. According to used algorithm in reference [41] the bilinear shape can be driven as follows

$$\left(D_x^4 - D_x D_t^3 + h_1 D_x^2 + h_2 D_x D_t + h_3 D_t^2\right)\phi.\phi = 0, \tag{1}$$

in which $u = u(x, y, t)$ is a unfamiliar solution and $h_s\,(s = 1, 2, 3)$ are all free quantities. According to expansion and generalization of the Hietarinta-type model, [43] was studied with the below bilinear model form:

$$\left(D_x^4 + D_x D_t^3 + h_1 D_x^2 + h_2 D_x D_t + h_3 D_t^2 - D_t D_y\right)\phi.\phi = 0. \tag{2}$$

In addition, by using the following relations

$$u = 2\,(\ln\phi)_x,\ v = 2\,(\ln\phi), \tag{3}$$

the following nonlinear equation will be arisen as

$$\begin{aligned} &6u_x u_{xx} + u_{xxxx} + 3u_t u_{tt} + 3u_{tx} v_{tt} + u_{xttt} \\ &\quad + h_1 u_{xx} + h_2 u_{tx} + h_3 u_{tt} - u_{ty} = 0, \end{aligned} \tag{4}$$

in which $v_x = u$, and h_1, h_2, and h_3 are arbitrary quantities. Besides, a new $(2+1)$-D extension of equation (4) was proposed in [44]. On the basis of Hirota bilinear method, a few nonlinear models have been investigated as the valuable researches, for example, the coupled nonlinear Schrödinger equations [45]; the modified coupled Hirota equation by help of Riemann-Hilbert approach [46]; an extended $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff-like equation by using the generalized bilinear operators [47]; a generalized $(3+1)$ shallow water-like equation through the Hirota bilinear method and the Cole–Hopf transformation [48]; a new $(3+1)$-dimensional weakly coupled B-type Kadomtsev–Petviashvili equation by constructing the symmetric positive semidefinite matrix technique [49]. Wave solutions have been used for different purposes as modeling of contaminant distribution or biodegradation in environmental engineering [50–52]. Specifically, Janssen et al. modeled the biodegradation of contaminants in heterogeneous aquifers using a semianalytical traveling wave solution for the one-dimensional reactive transport [50], and Wang et al. suggested a multimedia fate model to evaluate the fate of an organic contaminant by a one-dimensional network kinematic wave equation [51]. Moreover, wave equations have also been exploited in the analysis of transient flow in large distribution systems like groundwater [52]. In this regard, Jaradat et al. analyzed the health risks from the intrusion of contaminants into the distribution system from pressure transients. In [53], the multiple-kink solutions and singular-kink solutions for $(2+1)$-D coupled Burgers system with time variable coefficients were obtained by Jaradat and coworkers.

This paper investigates new the periodic-solitary, periodic wave, cross-kink wave, and interaction between stripe and periodic wave solutions for the generalized Hietarinta equation. We seek to explore two types of soliton solutions using two different formulas according to trigonometric, hyperbolic, and rational functions. In addition, we establish singular and dark soliton findings according to trigonometric and hyperbolic, respectively.

The fundamental work of this paper is to extract new exact findings of equation (4), and the paper is organized as follows: in Section 2, the analysis of the governing system via bilinear form polynomials is formulated to the generalized $(2+1)$-dimensional nonlinear model. In Sections 3–6, we obtain the periodic-solitary, periodic wave, cross-kink wave, and interaction between stripe and periodic wave solutions, respectively, for the generalized $(2+1)$-dimensional Hietarinta equation. Some conclusions that be gained throughout the paper have been presented in Section 7.

2. The Bilinear Formalism Euations

Through ref. [21], take $\lambda = \lambda(x_1, x_2, \dots, x_n)$ be a C^∞ function with multivariables as follows:

$$\Upsilon_{n_1 x_1,\dots,n_j x_j}(\lambda) \equiv \Upsilon_{n_1,\dots,n_j}\left(\lambda_{d_1 x_1,\dots,d_j x_j}\right) = e^{-\lambda}\partial_{x_1}^{n_1}\cdots\partial_{x_j}^{n_j}e^{\lambda}, \quad (5)$$

with the below formalism (BBPs [21])

$$\begin{aligned}\lambda_{d_1 x_1,\dots,d_j x_j} &= \partial_{x_1}^{d_1}\cdots\partial_{x_j}^{d_j}\lambda,\ \lambda_{0x_i} \equiv \lambda,\ d_1 \\ &= 0,\dots,n_1;\dots;d_j = 0,\dots,n_j,\end{aligned} \quad (6)$$

and we have

$$\begin{aligned}\Upsilon_1(\lambda) &= \lambda_x, \Upsilon_2(\lambda) = \lambda_{2x} + \lambda_x^2, \Upsilon_3(\lambda) \\ &= \lambda_{3x} + 3\lambda_x\lambda_{2x} + \lambda_x^3, \dots, \lambda = \lambda(x,t), \\ \Upsilon_{x,t}(\lambda) &= \lambda_{x,t} + \lambda_x\lambda_t, \Upsilon_{2x,t}(\lambda) \\ &= \lambda_{2x,t} + \lambda_{2x}\lambda_t + 2\lambda_{x,t}\lambda_x + \lambda_x^2\lambda_t, \dots.\end{aligned} \quad (7)$$

The multidimensional binary Bell polynomial can be written as

$$\Sigma_{n_1 x_1,\dots,n_j x_j}(\mu_1,\mu_2) = \Upsilon_{n_1,\dots,n_j}(\lambda)\Big|_{\lambda_{d_1 x_1,\dots,d_j x_j} = \begin{cases}\mu_{1 d_1 x_1,\dots,d_j x_j}, & d_1 + d_2 + \cdots + d_j, \text{ is odd} \\ \mu_{2 d_1 x_1,\dots,d_j x_j}, & d_1 + d_2 + \cdots + d_j, \text{ is even.}\end{cases}} \quad (8)$$

We have the following conditions as

$$\begin{aligned}\Sigma_x(\mu_1) &= \mu_{1x}, \Sigma_{2x}(\mu_1,\mu_2) = \mu_{2x} + \mu_{1x}^2, \Sigma_{x,t}(\mu_1,\mu_2) \\ &= \mu_{2x,t} + \mu_{1x}\mu_{1t}, \dots.\end{aligned} \quad (9)$$

Proposition 1. *Let $\mu_1 = ln(\Omega_1/\Omega_2)$, $\mu_2 = ln(\Omega_1\Omega_2)$, then the relations between binary Bell polynomials and Hirota D-operator reads*

$$\begin{aligned}&\Sigma_{n_1 x_1,\dots,n_j x_j}(\mu_1,\mu_2)\big|_{\mu_1=\ln(\Omega_1/\Omega_2),\mu_2=\ln(\Omega_1\Omega_2)} \\ &\quad = (\Omega_1\Omega_2)^{-1}D_{x_1}^{n_1}\dots D_{x_j}^{n_j}\Omega_1\Omega_2,\end{aligned} \quad (10)$$

with Hirota operator

$$\begin{aligned}\prod_{i=1}^{j} D_{\xi_i}^{n_i} g.\eta &= \prod_{i=1}^{j}\left(\frac{\partial}{\partial\xi_i} - \frac{\partial}{\partial\xi_i'}\right)^{n_i}\Omega_1(\xi_1,\dots,\xi_j)\Omega_2 \\ &\quad \cdot(\xi_1',\dots,\xi_j')\big|_{\xi_1=\xi_1',\dots,\xi_j=\xi_j'}\end{aligned}. \quad (11)$$

$$\begin{aligned}(\Omega_1\Omega_2)^{-1}D_{\xi_1}^{n_1}\dots D_{\xi_j}^{n_j}\Omega_1\Omega_2 &= \Sigma_{n_1\xi_1,\dots,n_j\xi_j}(\mu_1,\mu_2)\big|_{\mu_1=\ln(\Omega_1/\Omega_2),\mu_2=\ln(\Omega_1\Omega_2)} \\ &= \Sigma_{n_1\xi_1,\dots,n_j\xi_j}(\mu_1,\mu_1+\gamma)\big|_{\mu_1=\ln(\Omega_1/\Omega_2),\gamma=\ln(\Omega_1\Omega_2)} \\ &= \sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{i=1}^{j}\binom{n_i}{k_i}\mathfrak{P}_{k_1\xi_1,\dots,k_j\xi_j}(\gamma)\Upsilon_{(n_1-k_1)\xi_1,\dots,(n_j-k_j)\xi_j}(\mu_1).\end{aligned} \quad (14)$$

Proposition 2. *Considering $\Xi(\gamma) = \sum_i \delta_i \mathfrak{P}_{d_1\xi_1,\dots,d_j\xi_j} = 0$ and $\mu_1 = ln(\Omega_1/\Omega_2)$, $\mu_1 = ln(\Omega_1\Omega_2)$, we receive*

$$\begin{cases}\sum_i \delta_{1i}\Upsilon_{n_1\xi_1,\dots,n_j\xi_j}(\mu_1,\mu_2) = 0, \\ \sum_i \delta_{1i}\Upsilon_{d_1\xi_1,\dots,d_j\xi_j}(\mu_1,\mu_2) = 0,\end{cases} \quad (12)$$

which need to satisfy

$$\mathfrak{R}(\gamma',\gamma) = \mathfrak{R}(\gamma') - \mathfrak{R}(\gamma) = \mathfrak{R}(\mu_2+\mu_1) - \mathfrak{R}(\mu_2-\mu_1) = 0. \quad (13)$$

The generalized Bell polynomials $\Upsilon_{n_1 x_1,\dots,n_j x_j}(\xi)$ is as

The Cole–Hopf relation is as follows:

$$\Upsilon_{k_1\xi_1,\ldots,k_j\xi_j}(\mu_1=\ln(\tau))=\frac{\tau_{n_1\xi_1,\ldots,n_j\xi_j}}{\tau},$$
$$\left(\Omega_1\Omega_2^{-1}D_{x_1}^{n_1}\ldots D_{x_j}^{n_j}\Omega_1\Omega_2|\right)_{\Omega_2=\exp(\gamma/2),\Omega_1/\Omega_2=\tau}$$
$$=\tau^{-1}\sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{l=1}^{j}\begin{pmatrix}n_l\\ k_l\end{pmatrix}\mathfrak{P}_{k_1\xi_1,\ldots,k_l\xi_l}(\gamma)\tau_{(n_1-k_1)\xi_1,\ldots,(n_d-k_l)\xi_l}, \tag{15}$$

with

$$\Upsilon_t(\mu_1)=\frac{\tau_t}{\tau},\Upsilon_{2x}(\mu_1,\beta)=\gamma_{2x}+\frac{\tau_{2x}}{\tau},\Upsilon_{2x,y}(\mu_1,\mu_2)$$
$$=\frac{\gamma_{2x}\tau_y}{\tau}+\frac{2\gamma_{x,y}\tau_x}{\tau}+\frac{\tau_{2x,y}}{\tau}. \tag{16}$$

Also, based on the above writings, the bilinear frame to the aforementioned nonlinear model will be as

$$\left(D_x^4+D_xD_t^3+h_1D_x^2+h_2D_xD_t+h_3D_t^2-D_tD_y\right)\phi.\phi$$
$$=2\left[\left(\phi_{xxxx}\phi-4\phi_{xxx}\phi_x+3\phi_{xx}^2\right)\right.$$
$$+\left(\phi_{xttt}\phi-4\phi_{ttt}\phi_x+6\phi_{tt}\phi_{tx}-4\phi_t\phi_{ttx}+\phi_x\phi_{ttt}\right)$$
$$+h_1\left(\phi_{xx}\phi-\phi_x^2\right)+h_2\left(\phi_{xt}\phi-\phi_x\phi_t\right)$$
$$\left.+h_3\left(\phi_{tt}\phi-\phi_t^2\right)-\left(\phi_{yt}\phi-\phi_y\phi_t\right)\right]=0. \tag{17}$$

3. Periodic-Solitary Solutions

Here, we utilize to formulate the new exact solutions to the $(2+1)$-dimensional generalized Hietarinta equation. Consider the following function for studying the periodic-solitary solutions as

$$\phi=\epsilon_3\sin(\tau_1)+\epsilon_4\sinh(\tau_2)+\epsilon_5,\tau_s$$
$$=\alpha_sx+\beta_sy+\delta_st+\epsilon_s,s=1,2. \tag{18}$$

Afterwards, the values $\alpha_s,\beta_s,\delta_s,\epsilon_s\,(s=1:5)$ will be found. By making use of equation (18) into equation (17) and taking the coefficients, each powers of $\sin(x,y,t)$ and $\sinh(x,y,t)$ to zero, a system of equations (algebraic) (these are not collected here for minimalist) for $\alpha_s,\beta_s,\delta_s,\epsilon_s\,(s=1:5)$ is yielded. These algebraic equations by using the emblematic computation software like, Maple, give the following solutions with using $u=2(\ln\phi)_x$ and $v=2(\ln\phi)$.

3.1. Set I Solutions

$$\begin{cases}\beta_1=\dfrac{3\left(\alpha_1^2+\alpha_2^2\right)^2\left(\alpha_1^2\varepsilon_3^2-\alpha_2^2\varepsilon_4^2\right)+3\left(\delta_1^2+\delta_2^2\right)\left(\alpha_1^2\varepsilon_3^2-\alpha_2^2\varepsilon_4^2\right)(\alpha_1\delta_1+\alpha_2\delta_2)+\left(\varepsilon_3^2+\varepsilon_4^2\right)(\alpha_1\delta_2-\alpha_2\delta_1)(-\alpha_1\delta_2h_3+\alpha_2\delta_1h_3+\alpha_1\beta_2)}{(\alpha_1\delta_2-\alpha_2\delta_1)\alpha_2\left(\varepsilon_3^2+\varepsilon_4^2\right)},\\[2ex]
h_1=\dfrac{\left(\delta_1^2+\delta_2^2\right)\left(3\alpha_1\delta_1^3\varepsilon_3^2+4\alpha_1\delta_1\delta_2^2\varepsilon_3^2+\alpha_1\delta_1\delta_2^2\varepsilon_4^2-\alpha_2\delta_1^2\delta_2\varepsilon_3^2-4\alpha_2\delta_1^2\delta_2\varepsilon_4^2-3\alpha_2\delta_2^3\varepsilon_4^2\right)+\Phi_1}{\left(\varepsilon_3^2+\varepsilon_4^2\right)(\alpha_1\delta_2-\alpha_2\delta_1)^2},\\[2ex]
\Phi_1=\delta_1^2\left(3\varepsilon_3^2\alpha_1^4+6\varepsilon_3^2\alpha_1^2\alpha_2^2-\varepsilon_3^2\alpha_2^4-4\varepsilon_4^2\alpha_2^4\right)+\delta_2^2\left(4\varepsilon_3^2\alpha_1^4+\alpha_1^4\varepsilon_4^2-6\alpha_1^2\alpha_2^2\varepsilon_4^2-3\varepsilon_4^2\alpha_2^4\right)-\\[2ex]
4\alpha_1\alpha_2\delta_1\delta_2\left(\varepsilon_3^2+\varepsilon_4^2\right)\left(\alpha_1^2-\alpha_2^2\right),h_2=\dfrac{\Phi_2}{\alpha_2\left(\varepsilon_3^2+\varepsilon_4^2\right)(\alpha_1\delta_2-\alpha_2\delta_1)^2},\\[2ex]
\Phi_2=\left(\alpha_1^2+\alpha_2^2\right)\left(\alpha_1\varepsilon_3^2\left(3\alpha_1^3\delta_2-6\alpha_1^2\alpha_2\delta_1-\alpha_1\alpha_2^2\delta_2-2\alpha_3^2\delta_1\right)-\alpha_2^2\varepsilon_4^2\left(\alpha_1^2\delta_2-4\alpha_1\alpha_2\delta_1-3\alpha_2^2\delta_2\right)\right)+\\[2ex]
\left(\varepsilon_3^2+\varepsilon_4^2\right)(\alpha_1\delta_2-\alpha_2\delta_1)^2(-\delta_2h_3+\beta_2)-\alpha_1^2\alpha_2\left(6\delta_1^4\varepsilon_3^2+3\delta_1^2\delta_2^2\varepsilon_3^2+\delta_2^4\varepsilon_3^2+4\delta_2^4\varepsilon_4^2\right)+3\alpha_1^3\delta_1\delta_2\varepsilon_3^2\left(\delta_1^2+\delta_2^2\right)\\[2ex]
-\alpha_1\alpha_2^2\delta_1\delta_2\left(10\delta_1^2\varepsilon_3^2+\delta_1^2\varepsilon_4^2+2\delta_2^2\varepsilon_3^2-7\delta_2^2\varepsilon_4^2\right)+\alpha_2^3\left(4\delta_1^4\varepsilon_3^2+4\delta_1^4\varepsilon_4^2+3\delta_1^2\delta_2^2\varepsilon_4^2+3\delta_2^4\varepsilon_4^2\right),\varepsilon_5=0.\end{cases} \tag{19}$$

Here, α_d, δ_d, and ε_k for $d = 1: 2, k = 1: 4$, β_2 are the unknown parameters. By considering the necessary assumption:

$$\alpha_2\left(\varepsilon_3^2 + \varepsilon_4^2\right)\left(\alpha_1\delta_2 - \alpha_2\delta_1\right)^2 \neq 0, \tag{20}$$

by substituting the received above parameters into equation (18), we obtain an analytical form of rational equation:

$$u = 2\left(\ln\phi_1\right)_x = 2\frac{\epsilon_3 \cos\left(t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1\right)\alpha_1 + \epsilon_4\cosh\left(t\delta_2 + x\alpha_2 + y\beta_2 + \epsilon_2\right)\alpha_2}{\epsilon_3 \sin\left(t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1\right) + \epsilon_4\sinh\left(t\delta_2 + x\alpha_2 + y\beta_2 + \epsilon_2\right)},$$
$$\phi_1 = \epsilon_3 \sin\left(t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1\right) + \epsilon_4\sinh\left(t\delta_2 + x\alpha_2 + y\beta_2 + \epsilon_2\right). \tag{21}$$

If $\tau_2 \longrightarrow \infty$, ϕ_1 will be constant with any time Figure 1 shows the analysis of treatment of periodic and progress of soliton wave as hyperbolic function with graphs of ϕ_1 with the following selected parameters:

$$\delta_1 = 1, \delta_2 = 0.5, \alpha_1 = 0.1, \alpha_2 = 0.5, \beta_2 = 1, h_3 = 2, \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 4, \varepsilon_4 = 2, t = 1, \tag{22}$$

in equation (21).

3.2. Set II Solutions

$$\begin{cases} \beta_1 = \dfrac{\left(\alpha_1^2 + \alpha_2^2\right)\left(3\varepsilon_3^2\alpha_1^4 - \varepsilon_3^2\alpha_1^2\alpha_2^2 - \alpha_1^2\alpha_2^2\varepsilon_4^2 + 3\varepsilon_4^2\alpha_2^2\right) + \alpha_2^2\delta_1\left(\varepsilon_3^2 + \varepsilon_4^2\right)\left(\alpha_1 h_2 + \delta_1 h_3\right) + \alpha_1\delta_1^3\left(3\alpha_1^2\varepsilon_3^2 - 4\alpha_2^2\varepsilon_3^2 - \alpha_2^2\varepsilon_4^2\right)}{\alpha_2^2\delta_1\left(\varepsilon_3^2 + \varepsilon_4^2\right)}, \\ \beta_2 = \dfrac{2\alpha_1\left(\alpha_1^2 + \alpha_2^2\right)\left(3\alpha_1^2\varepsilon_3^2 + \alpha_2^2\varepsilon_3^2 - 2\alpha_2^2\varepsilon_4^2\right) + 2\delta_1^3\left(3\alpha_1^2\varepsilon_3^2 - 2\alpha_2^2\varepsilon_3^2 - 2\alpha_2^2\varepsilon_4^2\right) + \alpha_2^2\delta_1 h_2\left(\varepsilon_3^2 + \varepsilon_4^2\right)}{\alpha_2\left(\varepsilon_3^2 + \varepsilon_4^2\right)\delta_1}, \\ h_1 = \dfrac{3\varepsilon_3^2\alpha_1^4 + 6\varepsilon_3^2\alpha_1^2\alpha_2^2 + 3\alpha_1\delta_1^3\varepsilon_3^2 - \varepsilon_3^2\alpha_2^4 - 4\varepsilon_4^2\alpha_2^4}{\alpha_2^2\left(\varepsilon_4^2 + \varepsilon_4^2\right)}, \delta_2 = \varepsilon_5 = 0. \end{cases} \tag{23}$$

Here, α_d, ϵ_k for $d = 1: 2, k = 1: 4$, δ_1 are the unknown parameters. By considering the necessary assumption,

$$\alpha_2^2\left(\varepsilon_3^2 + \varepsilon_4^2\right)\delta_1 \neq 0, \tag{24}$$

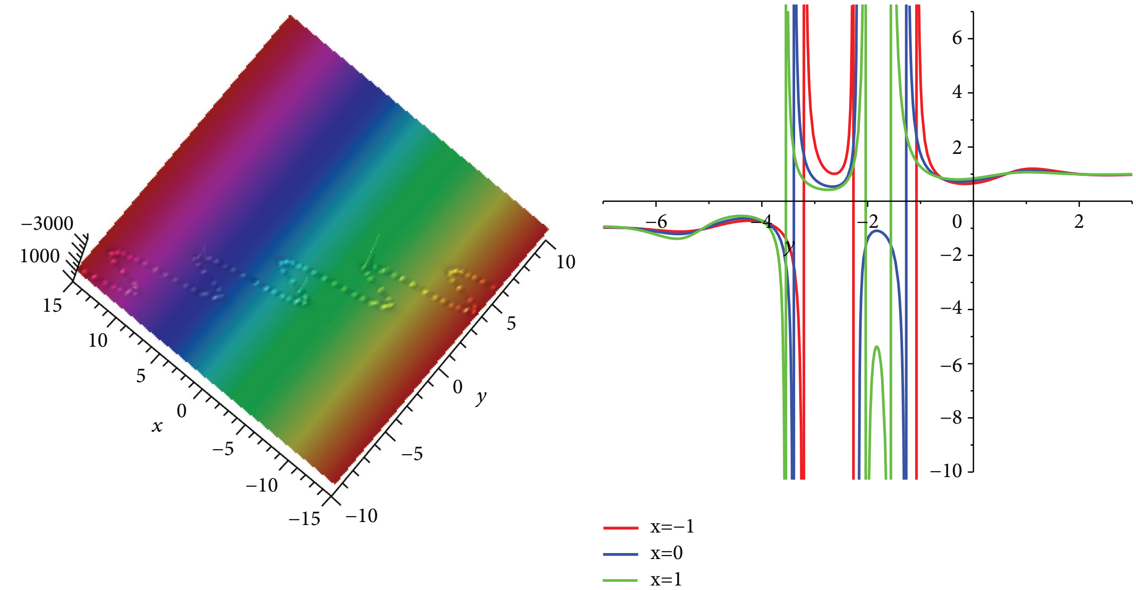
by substituting the above parameters into equation (18), we obtain an analytical form of rational equation:

$$u_2 = 2\left(\ln\phi_2\right)_x = 2\frac{\epsilon_3 \cos\left(t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1\right)\alpha_1 + \epsilon_4\cosh\left(x\alpha_2 + y\beta_2 + \epsilon_2\right)\alpha_2}{\epsilon_3 \sin\left(t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1\right) + \epsilon_4\sinh\left(x\alpha_2 + y\beta_2 + \epsilon_2\right) + \epsilon_5}. \tag{25}$$

If $\tau_2 \longrightarrow \infty$, the periodic-solitary wave outputs $u \longrightarrow 2\alpha_2$ at every time. Figure 2 shows the analysis of treatment of periodic and progress of soliton wave as hyperbolic function with graphs of ϕ_2 with the following selected parameters:

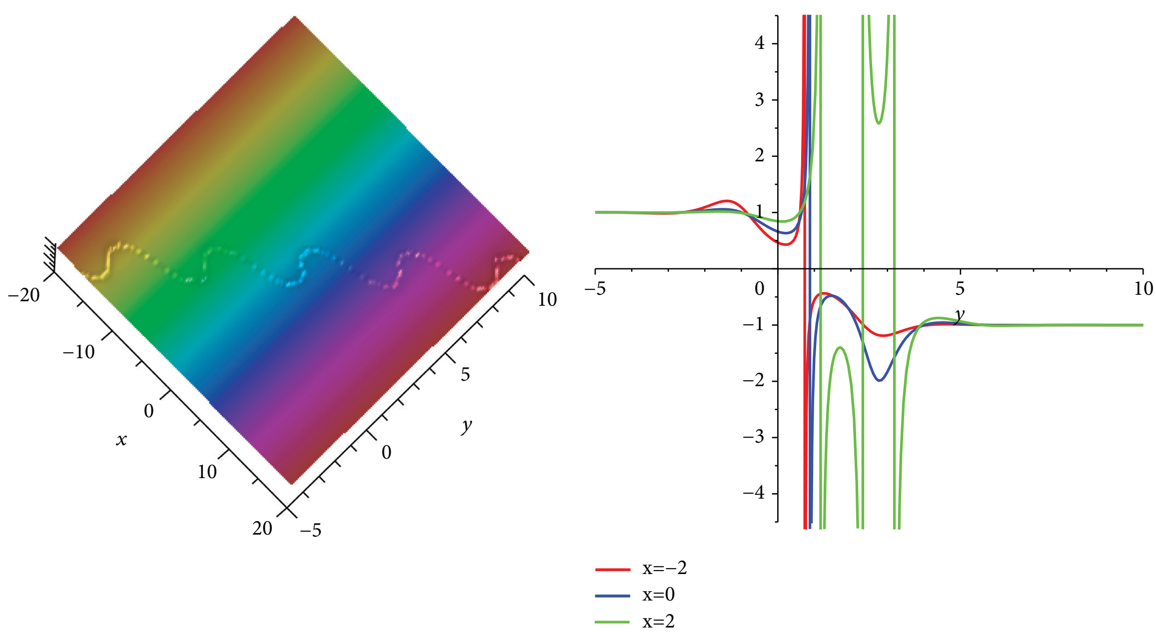
$$\delta_1 = 1, \delta_2 = 0.5, \alpha_1 = 0.1, \alpha_2 = 0.5, \beta_2 = 1, h_2 = 1, h_3 = 2, \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 4, \varepsilon_4 = 2, t = 1, \tag{26}$$

in equation (25).



(a) (b)

Figure 1: Periodic-solitary solution (24) such that (a) 3-D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1$ and $x = -1, 0, 1$.



(a) (b)

Figure 2: Periodic-solitary solution (26) such that (a) 3-D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1, x = -2, 0, 2$.

3.3. Set III Solutions

$$\begin{cases} \alpha_2 = \varepsilon_5 = 0, \beta_1 = \dfrac{\alpha_1\left(3\alpha_1^3\delta_1\varepsilon_3^2 + 3\delta_1^4\varepsilon_3^2 + 3\delta_1^2\delta_2^2\varepsilon_3^2 + 4\delta_2^4\varepsilon_3^2 + 4\delta_2^4\varepsilon_4^2\right) + \delta_2^2\left(\varepsilon_3^2 + \varepsilon_4^2\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\delta_2^2\left(\varepsilon_3^2 + \varepsilon_4^2\right)}, \\ \beta_2 = -\dfrac{\alpha_1\varepsilon_3^2\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right) - \delta_2^2 h_3\left(\varepsilon_3^2 + \varepsilon_4^2\right)}{\delta_2\left(\varepsilon_3^2 + \varepsilon_4^2\right)}, h_1 = \dfrac{\alpha_1^3\left(3\delta_1^2\varepsilon_3^2 + 4\delta_2^2\varepsilon_3^2 + \delta_2^2\varepsilon_4^2\right) + \delta_1\left(\delta_1^2 + \delta_2^2\right)\left(3\delta_1^2\varepsilon_3^2 + 4\delta_2^2\varepsilon_3^2 + \delta_2^2\varepsilon_4^2\right)}{\alpha_1\delta_2^2\left(\varepsilon_3^2 + \varepsilon_4^2\right)}. \end{cases} \tag{27}$$

Here, δ_d and ϵ_k for $d = 1{:}\ 2, k = 1{:}\ 4$, α_1 are the unknown parameters. By considering the necessary assumption,

$$\alpha_1 \delta_2^2\left(\varepsilon_3^2 + \varepsilon_4^2\right) \neq 0, \tag{28}$$

and by substituting the above parameters into equation (18), we obtain an analytical form of rational equation:

$$u_3 = 2\left(\ln \phi_3\right)_x = \frac{2\varepsilon_3 \cos\left(t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1\right)\alpha_1}{\varepsilon_3 \sin\left(t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1\right) + \varepsilon_4 \sinh\left(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2\right)}. \tag{29}$$

If $\tau_3 \longrightarrow \infty$, the periodic-solitary solution $u \longrightarrow 0$ at every time.

3.4. Set IV Solutions

$$\begin{cases} \alpha_1 = \dfrac{\alpha_2\varepsilon_4}{\varepsilon_3}, \beta_1 = \dfrac{\varepsilon_4\left(2\varepsilon_3^2\alpha_2^4 + 2\varepsilon_4^2\alpha_2^4 + 2\alpha_2\delta_2^3\varepsilon_3^2 + 2\alpha_2\delta_2^3\varepsilon_4^2 + \beta_2\delta_2\varepsilon_4^2\right)}{\varepsilon_3^3\delta_2}, \delta_1 = \dfrac{\delta_2\varepsilon_4}{\varepsilon_3}, \\ h_2 = -\dfrac{\varepsilon_3^2\alpha_2^4 - 3\varepsilon_4^2\alpha_2^4 + \alpha_2\varepsilon_2^3\varepsilon_3^2 - 3\alpha_2\delta_2^3\varepsilon_4^2 + \alpha_2^2 h_1\varepsilon_4^2 + \delta_2^2 h_3\varepsilon_3^2 - \beta_2\delta_2\delta_3^2}{\delta_3^2\delta_2\alpha_2}, \varepsilon_5 = 0. \end{cases} \tag{30}$$

Here, ε_k for $k = 1{:}\ 4$, and α_2 and δ_2 are the unknown parameters. By considering the necessary assumption,

$$\varepsilon_3^2\delta_2\alpha_2 \neq 0, \tag{31}$$

and by substituting the above parameters into equation (18), we obtain an analytical form of rational equation:

$$u_4 = 2\left(\ln \phi_4\right)_x = 2\frac{\cos\left(t\delta_2\varepsilon_4/\varepsilon_3 + x\alpha_2\varepsilon_4/\varepsilon_3 + y\beta_1 + \varepsilon_1\right)\alpha_2\varepsilon_4 + \varepsilon_4 \cosh\left(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2\right)\alpha_2}{\varepsilon_3 \sin\left(t\delta_2\varepsilon_4/\varepsilon_3 + x\alpha_2\varepsilon_4/\varepsilon_3 + y\beta_1 + \varepsilon_1\right) + \varepsilon_4 \sinh\left(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2\right)}. \tag{32}$$

If $\tau_2 \longrightarrow \infty$, the periodic-solitary solution $u \longrightarrow 2\alpha_2$ at every time. Figure 3 offers the analysis of treatment of periodic and progress of soliton as hyperbolic function with graphs of ϕ_4 with the following selected parameters:

$$\delta_1 = 1, \delta_2 = 2, \alpha_1 = 0.1, \alpha_2 = 0.25, \beta_2 = 1, h_2 = 1, h_3 = 1.5, \epsilon_1 = 1, \epsilon_2 = 2, \epsilon_3 = 4, \epsilon_4 = 2, t = 1, \tag{33}$$

in equation (31).

3.5. Collection V Findings

$$\left\{ \alpha_i = -\delta_i, i = 1, 2, \beta_1 = \frac{\beta_2\delta_1}{\delta_2}, h_2 = \frac{\delta_2^2 h_1 + \delta_2^2 h_3 - \beta_2\delta_2}{\delta_2^2}, \varepsilon_5 = 0. \right. \tag{34}$$

Here, δ_d, ϵ_j for $d = 1, 2, k = 1{:}\ 4$, and β_2 are the unknown parameters. By considering the necessary assumption

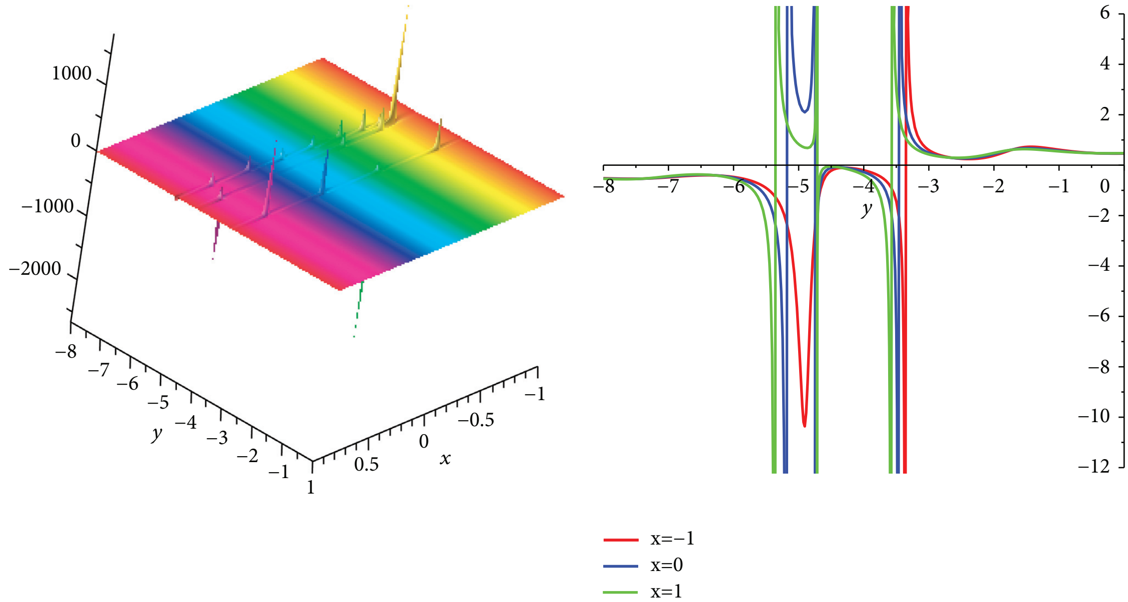
$$\delta_2 \neq 0, \tag{35}$$

and by substituting the above parameters into equation (18), we obtain an analytical form of rational equation:

$$u_5 = 2\left(\ln \phi_5\right)_x = 2\frac{-\varepsilon_3 \cos\left(t\delta_1 - x\delta_1 + y\beta_2\delta_1/\delta_2 + \varepsilon_1\right)\delta_1 - \varepsilon_4 \cosh\left(t\delta_2 - x\delta_2 + y\beta_2 + \varepsilon_2\right)\delta_2}{\varepsilon_3 \sin\left(t\delta_1 - x\delta_1 + y\beta_2\delta_1/\delta_2 + \varepsilon_1\right) + \varepsilon_4 \sinh\left(t\delta_2 - x\delta_2 + y\beta_2 + \varepsilon_2\right)}. \tag{36}$$

3.6. Set VI Solutions

$$\left\{ \alpha_i = -\delta_i, i = 1, 2, \beta_1 = \delta_1\left(h_1 - h_2 + h_3\right), \beta_2 = \delta_2\left(h_1 - h_2 + h_3\right). \right. \tag{37}$$



(a) (b)

FIGURE 3: Periodic-solitary solution (33) such that (a) 3-D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1, x = -2, 0, 2$.

Here, δ_d, ϵ_k for $d = 1, 2, k = 1{:}\ 5$ are the unknown parameters. By considering the necessary assumptions and by substituting the above parameters into equation (18), we obtain an analytical form of rational equation

$$u_6 = 2(\ln \phi_6)_x = 2\frac{-\varepsilon_3 \cos(\zeta_1)\delta_1 - \varepsilon_4 \cosh(t\delta_2 - x\delta_2 + y\delta_2(h_1 - h_2 + h_3) + \varepsilon_2)\delta_2}{\varepsilon_3 \sin(\zeta_1) + \varepsilon_4 \sinh(t\delta_2 - x\delta_2 + y\delta_2(h_1 - h_2 + h_3) + \varepsilon_2) + \varepsilon_5}, \zeta 1 = t\delta_1 - x\delta_1 + y\delta_1(h_1 - h_2 + h_3) + \varepsilon_1. \tag{38}$$

4. Periodic Wave Solutions

In this paragraph, we find out some advanced exact periodic wave soliton solutions to the $(2+1)$-dimensional generalized Hietarinta equation. Assume the stated function for studying the periodic wave solutions which is as follows:

$$\phi = \varepsilon_3 e^{\tau_1} + \varepsilon_4 e^{-\tau_1} + \varepsilon_5 \cos(\tau_2), \tau_s = \alpha_s x + \beta_s y + \delta_s t + \varepsilon_s, s = 1, 2. \tag{39}$$

Afterwards, the values $\alpha_s, \beta_s, \delta_s, \varepsilon_s (s = 1{:}\ 5)$ will be found. By making use of equation (39) into equation (17) and taking the coefficients, each powers of $e^{\Phi_1(x,y,t)}, e^{\Phi_2(x,y,t)}$, and trigonometric function $\cos(\Phi(x, y, t))$ to zero yield a system of equations (algebraic) (these are not collected here for minimalist) for $\alpha_s, \beta_s, \delta_s, \varepsilon_s (s = 1{:}\ 5)$. These algebraic equations by using the emblematic computation software like, Maple, give the solutions as follows with using $u = 2(\ln\phi)_x$ and $v = 2(\ln\phi)$.

4.1. Set I Findings

$$\{\alpha_l = -\delta_l, \beta_l = \delta_l(h_1 - h_2 + h_3), l = 1, 2, \epsilon_3 = 0. \tag{40}$$

Here, δ_d, ϵ_k for $d = 1, 2, k = 1{:}\ 4$ are the unknown parameters and by substituting the above parameters into equation (39), we obtain an analytical form of rational equation:

$$u = 2(\ln \phi_1)_x = 2\frac{\varepsilon_4 \delta_1 e^{\zeta_1} + \varepsilon_5 \sin(t\delta_2 - x\delta_2 + y\delta_2(h_1 - h_2 + h_3) + \varepsilon_2)\delta_2}{\varepsilon_4 e^{\zeta_1} + \varepsilon_5 \cos(t\delta_2 - x\delta_2 + y\delta_2(h_1 - h_2 + h_3) + \varepsilon_2)}, \zeta_1 = -t\delta_1 + x\delta_1 - y\delta_1(h_1 - h_2 + h_3) - \varepsilon_1. \tag{41}$$

If $\tau_1 \longrightarrow \infty$, the breather outputs $u \longrightarrow 2\delta_1$ at every time.

4.2. Set II Solutions

$$\begin{cases} \beta_1 = -\dfrac{4\alpha_1\varepsilon_3\varepsilon_4\left(3\alpha_1^5 + 2\alpha_1^3\alpha_2^2 + 3\alpha_1^2\delta_1^3 - \alpha_1\alpha_2^4 - 4\alpha_2^2\delta_1^3\right) + \alpha_1\alpha_2^2\varepsilon_5^2\left(\alpha_1^3 - 2\alpha_1\alpha_2^2 + \delta_1^3\right) - 3\alpha_2^6\varepsilon_5^2 - \alpha_2^2\delta_1\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\alpha_2^2\delta_1\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)}, \\ \beta_2 = -\dfrac{8\varepsilon_3\varepsilon_4\left(3\alpha_1^5 + 4\alpha_1^3\alpha_2^2 + 3\alpha_1^2\delta_1^3 + \alpha_1\alpha_2^4 - 2\alpha_2^2\delta_1^3\right) + 4\alpha_2^2\varepsilon_5^2\left(\alpha_1^3 + \alpha_1\alpha_2^2 + \delta_1^3\right) - \alpha_2^2\delta_1 h_2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)}{\alpha_2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)\delta_1}, \\ \delta_2 = 0, h_1 = -4\dfrac{3\alpha_1^4\varepsilon_3\varepsilon_4 + 6\alpha_1^2\alpha_2^2\varepsilon_3\varepsilon_4 + 3\alpha_1\delta_1^3\varepsilon_3\varepsilon_4 - \alpha_2^4\varepsilon_3\varepsilon_4 + \alpha_2^4\varepsilon_5^2}{\alpha_2^2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)}. \end{cases} \tag{42}$$

Here, α_d, ϵ_k for $d = 1, 2, k = 1: 5$, and δ_1 are the unknown parameters. By considering the necessary assumption,

$$\alpha_2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)\delta_1 \neq 0, \tag{43}$$

and by substituting the above parameters into the equation (36), we obtain an analytical form of rational equation:

$$u_2 = 2(\ln\phi_2)_x = 2\frac{\varepsilon_3\alpha_1 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} - \varepsilon_4\alpha_1 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} - \varepsilon_5\sin(x\alpha_2 + y\beta_2 + \varepsilon_2)\alpha_2}{\varepsilon_3 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_4 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_5\cos(x\alpha_2 + y\beta_2 + \varepsilon_2)}. \tag{44}$$

If $\tau_1 \longrightarrow \infty$, the breather outputs $u \longrightarrow 2\alpha_1$ at every time. Figure 4 shows the analysis of treatment of periodic and progress of breather-wave solutions as exponential and trigonometric functions with graphs of ϕ_1 with the following selected parameters:

$$\delta_1 = 1, \alpha_1 = 0.1, \alpha_2 = 0.5, h_2 = 1, h_3 = 2, \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 4, \varepsilon_4 = 2, \varepsilon_5 = 3, t = 1, \tag{45}$$

in equation (41).

4.3. Collection III Outputs

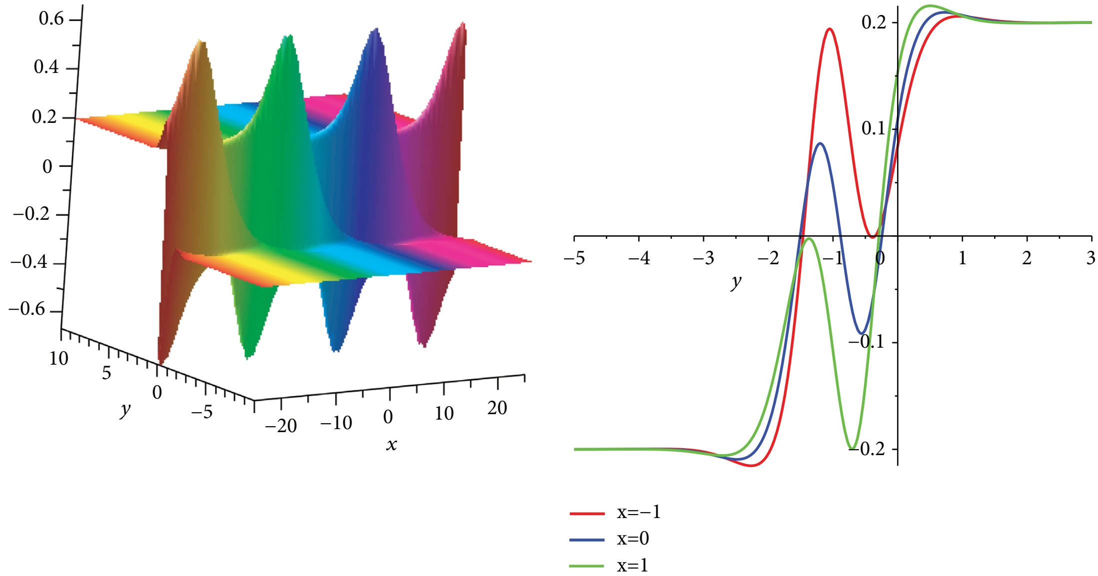
$$\begin{cases} \alpha_2 = 0, \beta_1 = -\dfrac{4\alpha_1\varepsilon_3\varepsilon_4\left(3\alpha_1^3\delta_1 + 3\delta_1^4 + 3\delta_1^2\delta_2^2 + 4\delta_2^4\right) - 4\alpha_1\delta_2^4\varepsilon_5^2 - \delta_2^2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\delta_2^2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)}, \\ \beta_2 = \dfrac{4\varepsilon_3\varepsilon_4\left(3\alpha_1^4 + 3\alpha_1\delta_1^3 + 3\alpha_1\delta_1\delta_2^2 + \delta_2^2 h_3\right) - \delta_2^2 h_3\varepsilon_5^2}{\delta_2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)}, \\ h_1 = -\dfrac{4\varepsilon_3\varepsilon_4\left(3\delta_1^2 + 4\delta_2^2\right)\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right) - \delta_2^2\varepsilon_5^2\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right)}{\alpha_1\delta_2^2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right)}. \end{cases} \tag{46}$$

Here, δ_d, ϵ_k for $d = 1, 2, k = 1: 5$, α_1, and β_2 are the unknown parameters. By considering the necessary assumption,

$$\alpha_1\delta_2^2\left(4\varepsilon_3\varepsilon_4 - \varepsilon_5^2\right) \neq 0, \tag{47}$$

and by substituting the above parameters into equation (39), we obtain an analytical form of rational equation:

$$u_3 = 2(\ln\phi_3)_x = 2\frac{\varepsilon_3\alpha_1 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} - \varepsilon_4\alpha_1 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1}}{\varepsilon_3 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_4 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_5\cos(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2)}. \tag{48}$$



(a) (b)

Figure 4: Periodic-wave solution (41) such that (a) 3D design of $u(x,y,t)$ at $t=1$ and (b) 2D plot of $u(x,y,t)$ at $t=1, x=-1,0,1$.

If $\tau_1 \longrightarrow \infty$, the periodic outputs $u \longrightarrow 2\alpha_1$ at any time. Figure 5 shows the analysis of treatment of periodic and progress of periodic wave solutions as exponential and trigonometric functions with graphs of ϕ_3 with the following selected parameters:

$$\delta_1 = 1, \delta\_2 = 2, \alpha_1 = 0.1, h_2 = 1, h_3 = 2, \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 4, \varepsilon_4 = 2, \varepsilon_5 = 3, t = 1, \tag{49}$$

in equation (48).

4.4. Set IV Solutions

$$\begin{cases} \alpha_1 = \dfrac{\alpha_2\delta_1}{\delta_2}, \beta_1 = -\dfrac{\delta_1\left(2\alpha_2^4\delta_1^2 + 2\alpha_2^4\delta_2^2 + 2\alpha_2\delta_1^2\delta_2^3 + 2\alpha_2\delta_2^5 - \beta_2\delta_2^3\right)}{\delta_2^4}, \\ h_2 = -\dfrac{3\alpha_2^4\delta_1^2 - \alpha_2^4\delta_2^2 + 3\alpha_2\delta_1^2\delta_2^3 - \alpha_2\delta_2^5 + \alpha_2^2\delta_2^2h_1 + \delta_2^4h_3 - \beta_2\delta_2^3}{\alpha_2\delta_2^3}, \varepsilon_3 = -\dfrac{1}{4}\dfrac{\delta_2^2\delta_5^2}{\delta_1^2\varepsilon_4}. \end{cases} \tag{50}$$

Here, δ_d, ϵ_k for $d = 1, 2, k = 1{:}\ 5$, α_2, and β_2 are the unknown parameters. By considering the necessary assumption,

$$\alpha_2\delta_2 \neq 0, \tag{51}$$

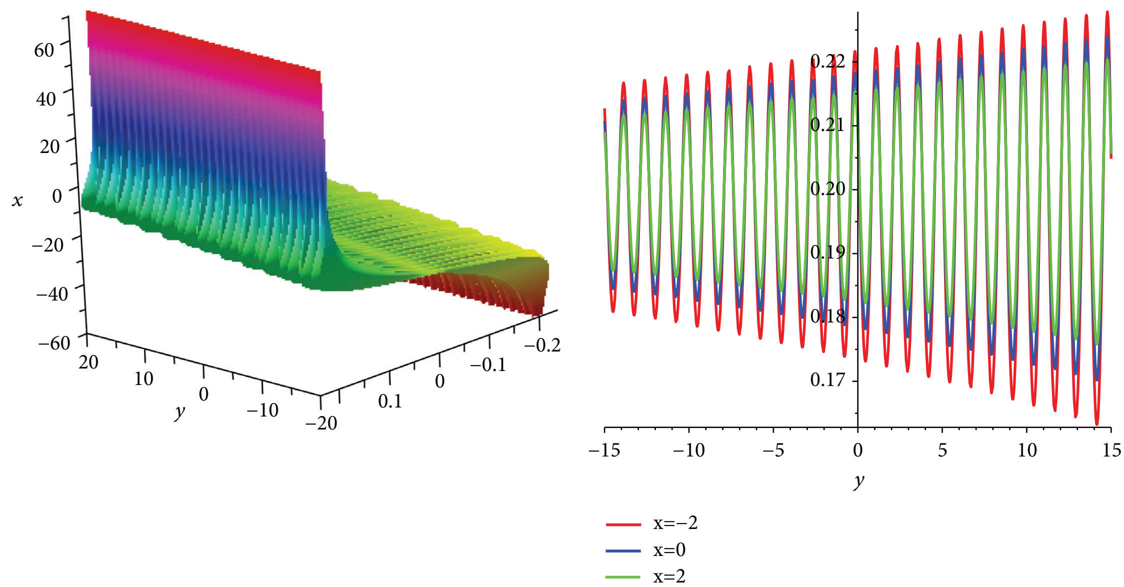
and by substituting the above parameters into equation (39), we obtain an analytical form of rational equation:

$$u_4 = 2(\ln\phi_4)_x = 2\frac{-1/4\delta_2^2\varepsilon_5^2\alpha_1/\delta_1^2\varepsilon_4 e^{t\delta_1 + x\alpha_2\delta_1/\delta_2 + y\beta_1 + \varepsilon_1} - \varepsilon_4\alpha_1 e^{-t\delta_1 - x\alpha_2\delta_1/\delta_2 - y\beta_1 - \varepsilon_1} - \varepsilon_5\sin(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2)\alpha_2}{-1/4\delta_2^2\varepsilon_5^2/\delta_1^2\varepsilon_4 e^{t\delta_1 + x\alpha_2\delta_1/\delta_2 + y\beta_1 + \varepsilon_1} + \varepsilon_4 e^{-t\delta_1 - x\alpha_2\delta_1/\delta_2 - y\beta_1 - \varepsilon_1} + \varepsilon_5\cos(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2)}.. \tag{52}$$

If $\tau_1 \longrightarrow \infty$, the breather outputs $u \longrightarrow 2\alpha_2\delta_1/\delta_2$ at every time. Figure 6 shows the analysis of treatment of periodic and progress of periodic wave solutions as exponential and trigonometric functions with graphs of ϕ_4 with the following selected parameters:

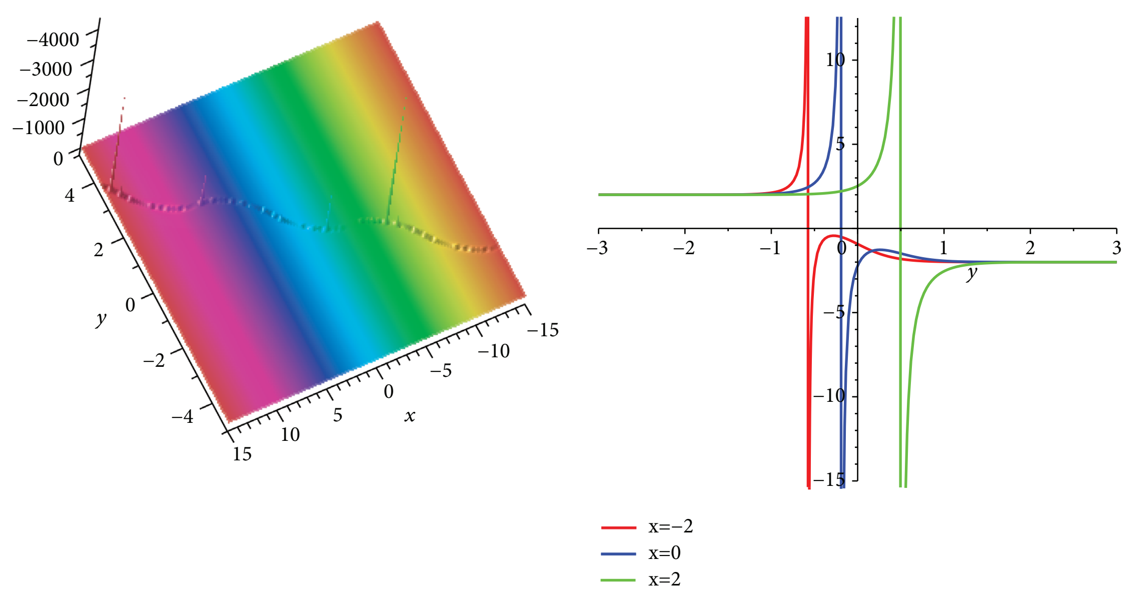
$$\delta_1 = 1, \delta_2 = 0.5, \alpha_2 = 0.5, \beta_2 = 0.2, h_1 = 1, h_3 = 2, \varepsilon_1 = 1, \varepsilon_2 = 0.1, \varepsilon_4 = 2, \varepsilon_5 = 3, t = 0.5, \tag{53}$$

in equation (48).



(a) (b)

Figure 5: Periodic-wave solution (48) such that (a) 3D design of $u(x,y,t)$ at $t=1$ and (b) 2D plot of $u(x,y,t)$ at $t=1, x=-2,0,2$.



(a) (b)

Figure 6: Periodic-wave solution (48) such that (a) 3D design of $u(x,y,t)$ at $t=1$ and (b) 2D plot of $u(x,y,t)$ at $t=1, x=-2,0,2$.

4.5. Collection V Outputs

$$\left\{ \alpha_l = -\delta_l, l = 1,2, \beta_1 = \frac{\beta_2 \delta_1}{\delta_2}, h_2 = -\frac{-\delta_2 h_1 - \delta_2 h_3 + \beta_2}{\delta_2}. \right. \tag{54}$$

Here, δ_d, ϵ_k for $d = 1,2, k = 1{:}\ 5$, and β_2 are the unknown parameters. By considering the necessary assumption,

$$\delta_2 \neq 0, \tag{55}$$

and by substituting the above parameters into equation (39), we obtain an analytical form of rational equation:

$$u_5 = 2(\ln\phi_5)_x = 2\frac{-\varepsilon_3\delta_1 e^{t\delta_1 - x\delta_1 + y\beta_2\delta_1/\delta_2 + \varepsilon_1} + \varepsilon_4\delta_1 e^{-t\delta_1 + x\delta_1 - y\beta_2\delta_1/\delta_2 - \varepsilon_1} + \varepsilon_5\sin(t\delta_2 - x\delta_2 + y\beta_2 + \varepsilon_2)\delta_2}{\varepsilon_3 e^{t\delta_1 - x\delta_1 + y\beta_2\delta_1/\delta_2 + \varepsilon_1} + \varepsilon_4 e^{-t\delta_1 + x\delta_1 - y\beta_2\delta_1/\delta_2 - \varepsilon_1} + \varepsilon_5\cos(t\delta_2 - x\delta_2 + y\beta_2 + \varepsilon_2)}. \tag{56}$$

4.6. Set VI Solutions.

$$\begin{cases} \alpha_1 = \theta, \alpha_2 = 0, \beta_1 = -\dfrac{4\theta\delta_2^3 - \theta\delta_2 h_2 - 2\delta_1\delta_2 h_3 + \beta_2\delta_1}{\delta_2}, h_1 \\ = -\dfrac{-\delta_1^2\delta_2 h_3 - \delta_2^3 h_3 + \beta_2\delta_1^2 + \beta_2\delta_2^2}{\theta^2\delta_2}, \varepsilon_3 = \dfrac{1}{4}\dfrac{\varepsilon_5^2}{\varepsilon_4}. \end{cases} \tag{57}$$

Here, $\theta = \sqrt{[3]} - \delta_1^4 - \delta_1\delta_2^2$, δ_d, ε_k for $d = 1, 2, k = 1{:}\ 5$ and β_2 are the unknown parameters. By considering the necessary assumption,

$$\delta_2\theta \neq 0, \tag{58}$$

and by substituting the above parameters into equation (39), we obtain an analytical form of rational equation:

$$u_6 = 2(\ln\phi_6)_x = 2\frac{1/4\varepsilon_5^2\theta e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1}/\varepsilon_4 - \varepsilon_4\alpha_1 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1}}{1/4\varepsilon_5^2 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1}/\varepsilon_4 + \varepsilon_4 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_5\cos(t\delta_2 + y\beta_2 + \varepsilon_2)}. \tag{59}$$

4.7. Set VII Solutions

$$\begin{cases} \beta_1 = \dfrac{-8\alpha_1^2\alpha_2^2 + \theta^2 h_3 + \alpha_1\theta h_2 + \alpha_1^2 h_1 - \alpha_2^2 h_1}{\theta}, \beta_2 = \dfrac{\alpha_2\left(-12\alpha_1^2\alpha_2^2 - 4\alpha_2^4 + \alpha_1\theta h_2 + 2\alpha_1^2 h_1\right)}{\alpha_1\theta}, \\ \delta_1 = \theta, \delta_2 = 0, \theta = \dfrac{\sqrt{[3]}\left(-\alpha_1^4 - 2\alpha_1^2\alpha_2^2 - \alpha_2^4\right)\alpha_1^2}{\alpha_1}, \varepsilon_3 = \dfrac{1}{4}\dfrac{\varepsilon_5^2}{\varepsilon_4}. \end{cases} \tag{60}$$

Here, α_d, ϵ_k for $d = 1, 2, k = 1{:}\ 5$ are the unknown parameters. By considering the necessary assumption,

$$\alpha_1\theta \neq 0, \tag{61}$$

and by substituting the above parameters into equation (39), we obtain an analytical form of rational equation:

$$u_7 = 2(\ln\phi_7)_x = 2\frac{1/4\epsilon_5^2\alpha_1 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1}/\varepsilon_4 - \epsilon_4\alpha_1 e^{-t\theta - x\alpha_1 - y\beta_1 - \epsilon_1} - \epsilon_5\sin(x\alpha_2 + y\beta_2 + \epsilon_2)\alpha_2}{1/4\epsilon_5^2 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \epsilon_1}/\varepsilon_4 + \epsilon_4 e^{-t\theta - x\alpha_1 - y\beta_1 - \epsilon_1} + \epsilon_5\cos(x\alpha_2 + y\beta_2 + \epsilon_2)}. \tag{62}$$

5. Cross-Kink Wave Solutions

In this segment, we utilize to formulate the new exact solutions to the (2+1)-dimensional generalized Hietarinta equation. Consider the following function for studying the cross-kink wave solutions as

$$\begin{aligned} \phi &= e^{-\tau_1} + \varepsilon_4 e^{\tau_1} + \varepsilon_5\sin(\tau_2) + \varepsilon_6\sinh(\tau_3), \tau_s \\ &= \alpha_s x + \beta_s y + \delta_s t + \varepsilon_s, s = 1{:}\ 3. \end{aligned} \tag{63}$$

Afterwards, the values $\alpha_s, \beta_s, \delta_s, \epsilon_s$ $(s = 1{:}\ 3)$ will be found. By making use of equation (63) into (17) and taking the coefficients, each powers of $e^{\Phi(x,y,t)}$, $\sin(x, y, t)$, and $\sinh(x, y, t)$ to zero yield a system of equations (algebraic) (these are not collected here for minimalist) for $\alpha_s, \beta_s, \delta_s, \epsilon_s$ $(s = 1{:}\ 3)$. These algebraic equations by using the emblematic computation software like, Maple, give the following solutions with using $u = 2(\ln\phi)_x$ and $v = 2(\ln\phi)$.

5.1. Set I Solutions

$$\begin{cases} \beta_1 = \dfrac{\alpha_3^2\varepsilon_6^2\left(\alpha_1^4 + 2\alpha_1^2\alpha_3^2 + \alpha_1\delta_1^3 - 3\alpha_3^4\right) + 4\alpha_1\varepsilon_4\left(3\alpha_1^5 - 2\alpha_1^3\alpha_3^2 + 3\alpha_1^2\delta_1^3 - \alpha_1\alpha_3^4 + 4\alpha_3^2\delta_1^3\right) + \alpha_3^2\delta_1\left(\varepsilon_6^2 + 4\varepsilon_4\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\alpha_3^2\delta_1\left(\varepsilon_6^2 + 4\varepsilon_4\right)}, \\ \beta_3 = \dfrac{4\alpha_3^2\varepsilon_6^2\left(\alpha_1^3 - \alpha_1\alpha_3^2 + \delta_1^3\right) + 8\varepsilon_4\left(3\alpha_1^5 - 4\alpha_1^3\alpha_3^2 + 3\alpha_1^2\delta_1^3 + \alpha_1\alpha_3^4 + 2\alpha_3^2\delta_1^3\right) + \alpha_3^2\delta_1 h_2\left(\varepsilon_6^2 + 4\varepsilon_4\right)}{\alpha_3\left(\varepsilon_6^2 + 4\varepsilon_4\right)\delta_1}, \\ \delta_3 = 0, h_1 = 4\dfrac{-\alpha_3^4\varepsilon_6^2 + \varepsilon_4\left(3\alpha_1^4 - 6\alpha_1^2\alpha_3^2 + 3\alpha_1\delta_1^3 - \alpha_3^4\right)}{\alpha_3^2\left(\varepsilon_6^2 + 4\varepsilon_4\right)}, \varepsilon_5 = 0. \end{cases} \tag{64}$$

Here, α_d, ϵ_k for $d = 1{:}\ 2, k = 1{:}\ 6$, β_2, δ_1, and δ_2 are the unknown parameters. By considering the necessary assumption,

$$\alpha_3^2\delta_1\left(\varepsilon_6^2 + 4\varepsilon_4\right) \neq 0, \tag{65}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_1 = 2\left(\ln \phi_1\right)_x = 2\frac{-\alpha_1 e^{-\zeta_1} + \varepsilon_4\alpha_1 e^{\zeta_1} + \varepsilon_6\cosh\left(x\alpha_3 + y\beta_3 + \varepsilon_3\right)\alpha_3}{e^{-\zeta_1} + \varepsilon_4 e^{\zeta_1} + \varepsilon_6\sinh\left(x\alpha_3 + y\beta_3 + \varepsilon_3\right)}, \zeta_1 = t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1. \tag{66}$$

If $\tau_1 \longrightarrow \infty$, the breather outputs $u \longrightarrow 2\alpha_3$ at every time.

5.2. Set II Solutions

$$\begin{cases} \alpha_3 = \varepsilon_5 = 0, \beta_1 = \dfrac{4\alpha_1\delta_3^4\varepsilon_6^2 + 4\alpha_1\varepsilon_4\left(3\alpha_1^3\delta_1 + 3\delta_1^4 - 3\delta_1^2\delta_3^2 + 4\delta_3^4\right) + \delta_3^2\left(\varepsilon_6^2 + 4\varepsilon_4\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\delta_3^2\left(\varepsilon_6^2 + 4\varepsilon_4\right)}, \\ \beta_3 = -\dfrac{12\alpha_1\varepsilon_4\left(\alpha_1^3 + \delta_1^3 - \delta_1\delta_3^2\right) - \delta_3^2 h_3\left(\varepsilon_6^2 + 4\varepsilon_4\right)}{\delta_3\left(\varepsilon_6^2 + 4\varepsilon_4\right)}, \\ h_1 = \dfrac{-\delta_3^2\varepsilon_6^2\left(\alpha_1^3 + \delta_1^3 - \delta_1\delta_3^2\right) + 4\varepsilon_4\left(3\delta_1^2 - 4\delta_3^2\right)\left(\alpha_1^3 + \delta_1^3 - \delta_1\delta_3^2\right)}{\alpha_1\delta_3^2\left(\varepsilon_6^2 + 4\varepsilon_4\right)}. \end{cases} \tag{67}$$

Here, δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 6$, α_1, α_2, and β_2 are the unknown parameters. By considering the necessary assumption,

$$\alpha_1\delta_3^2\left(\varepsilon_6^2 + 4\varepsilon_4\right) \neq 0, \tag{68}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_2 = 2\left(\ln \phi_2\right)_x = 2\frac{-\alpha_1 e^{-\zeta_1} + \varepsilon_4\alpha_1 e^{\zeta_1}}{e^{-\zeta_1} + \varepsilon_4 e^{\zeta_1} + \varepsilon_6\sinh\left(t\delta_3 + y\beta_3 + \varepsilon_3\right)}, \zeta_1 = t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1. \tag{69}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 0$ at every t, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at every time.

5.3. Set III Solutions

$$\begin{cases} \beta_1 = \dfrac{8\alpha_3^4 + 8\alpha_3\delta_3^3 + 2\alpha_3^2 h_1 + 2\alpha_3\delta_3 h_2 + 2\delta_3^2 h_3 - \beta_3\delta_3}{\delta_3}, \varepsilon_4 \\ = -\dfrac{1}{4}\varepsilon_6^2, \varepsilon_5 = 0. \end{cases} \tag{70}$$

Here $\alpha_d, \delta_d, \epsilon_k$ for $d = 1{:}\ 3, k = 1{:}\ 6$, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\delta_3 \neq 0, \tag{71}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_3 = 2(\ln \phi_3)_x = 2\frac{-\alpha_1 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} - 1/4\varepsilon_6^2 \alpha_1 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_6 \cosh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)\alpha_3}{e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} - 1/4\varepsilon_6^2 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_6 \sinh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)}. \tag{72}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_3$ at every time, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at any t.

5.4. Set IV Solutions

$$\begin{cases} \alpha_1 = \theta, \alpha_3 = 0, \beta_1 = \dfrac{4\theta\delta_3^3 + \theta\delta_3 h_2 + 2\delta_1\delta_3 h_3 - \beta_3\delta_1}{\delta_3}, h_1 = -\dfrac{-\delta_1^2\delta_3 h_3 + \delta_3^3 h_3 + \beta_3\delta_1^2 - \beta_3\delta_3^2}{\theta^2\delta_3}, \\ \varepsilon_4 = -\dfrac{1}{4}\varepsilon_6^2, \varepsilon_5 = 0. \end{cases} \tag{73}$$

Here, $\theta = \sqrt{[3]} - \delta_1^3 + \delta_1\delta_3^2, \alpha_d, \delta_d, \epsilon_k$ for $d = 1{:}\ 3, k = 1{:}\ 6$, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\delta_3 \neq 0, \tag{74}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_4 = 2(\ln \phi_4)_x = 2\frac{-\alpha_1 e^{-\zeta_1} - 1/4\varepsilon_6^2 \alpha_1 e^{\zeta_1} + \varepsilon_6 \cosh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)\alpha_3}{e^{-\zeta_1} - 1/4\varepsilon_6^2 e^{\zeta_1} + \varepsilon_6 \sinh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)}, \zeta_1 = t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1. \tag{75}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_3$ at every time, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at every time.

5.5. Set V Solutions.

$$\begin{cases} \alpha_1 = \theta, \alpha_3 = 0, \beta_1 = \dfrac{4\theta\delta_3^3 + \theta\delta_3 h_2 + 2\delta_1\delta_3 h_3 - \beta_3\delta_1}{\delta_3}, \\ h_1 = -\dfrac{-\delta_1^2\delta_3 h_3 + \delta_3^3 h_3 + \beta_3\delta_1^2 - \beta_3\delta_3^2}{\theta^2\delta_3}, \varepsilon_4 = -\dfrac{1}{4}\varepsilon_6^2, \varepsilon_5 = 0. \end{cases} \tag{76}$$

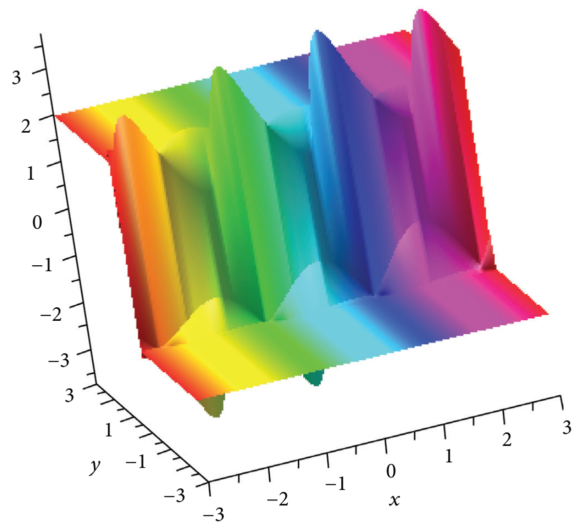
Here, $\theta = \sqrt{[3]} - \delta_1^3 + \delta_1\delta_3^2, \alpha_d, \delta_d, \epsilon_k$ for $d = 1{:}\ 3, k = 1{:}\ 6$, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\delta_3 \neq 0, \tag{77}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_5 = 2(\ln \phi_5)_x = 2\frac{-\alpha_1 e^{-\zeta_1} - 1/4\varepsilon_6^2 \alpha_1 e^{\zeta_1} + \varepsilon_6 \cosh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)\alpha_3}{e^{-\zeta_1} - 1/4\varepsilon_6^2 e^{\zeta_1} + \varepsilon_6 \sinh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)}, \zeta_1 = t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1. \tag{78}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_3$ at every time, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at any time. Figure 7 show the analysis of treatment of cross-kink wave as periodic and hyperbolic function with graphs of ϕ_5 with the following selected parameters:



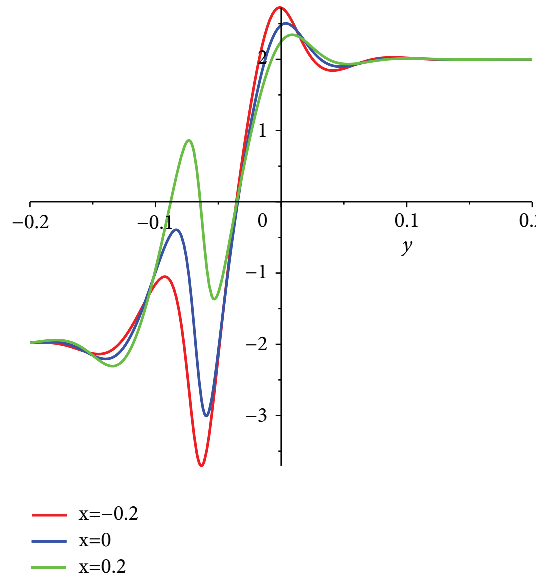


FIGURE 7: Cross-kink solution (78) such that (a) 3D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1, x = -0.2, 0, 0.2$.

$$\delta_1 = 1, \delta_3 = 0.5, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \beta_3 = 0.2, h_2 = 1, h_3 = 2, \varepsilon_1 = 1, \varepsilon_2 = 0.1, \varepsilon_3 = 2, \varepsilon_4 = 3, \varepsilon_5 = 2, t = 0.5, \tag{79}$$

in equation (74).

5.6. Set VI Solutions

$$\begin{cases} \alpha_2 = \varepsilon_6 = 0, \beta_1 = -\dfrac{-4\alpha_1\delta_2^4\varepsilon_5^2 + 4\alpha_1\varepsilon_4\left(3\alpha_1^3\delta_1 + 3\delta_1^4 + 3\delta_1^2\delta_2^2 + 4\delta_2^4\right) - \delta_2^2\left(-\varepsilon_5^2 + 4\varepsilon_4\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\delta_2^2\left(-\varepsilon_5^3 + 4\varepsilon_4\right)}, \\[2ex] \beta_2 = \dfrac{12\alpha_1\varepsilon_4\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right) - \delta_2^2 h_3\varepsilon_5^2 + 4\delta_2^2 h_3\varepsilon_4}{\delta_2\left(-\varepsilon_5^3 + 4\varepsilon_4\right)}, \\[2ex] h_1 = -\dfrac{-\delta_2^2\varepsilon_5^2\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right) + 4\varepsilon_4\left(3\delta_1^2 + 4\delta_2^2\right)\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right)}{\alpha_1\delta_2^2\left(-\varepsilon_5^2 + 4\varepsilon_4\right)}. \end{cases} \tag{80}$$

Here, δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 6$, α_1, α_3, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\alpha_1\delta_2^2\left(-\varepsilon_5^2 + 4\varepsilon_4\right) \neq 0, \tag{81}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_6 = 2\left(\ln f_6\right)_x = 2\frac{-\alpha_1 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_4\alpha_1 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1}}{e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_4 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_5 \sin\left(t\delta_2 + y\beta_2 + \varepsilon_2\right)}. \tag{82}$$

If $\tau_2 > \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 0$ at every time, but if $\tau_2 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at every time.

5.7. Set VII Solutions

$$\begin{cases} \alpha_1 = \dfrac{\alpha_2\delta_1}{\delta_2}, \beta_1 = -\dfrac{\delta_1\left(2\alpha_2^4\delta_1^2 + 2\alpha_2^4\delta_2^2 + 2\alpha_2\delta_1^2\delta_2^3 + 2\alpha_2\delta_2^5 - \beta_2\delta_2^3\right)}{\delta_2^4}, \\ h_2 = -\dfrac{3\alpha_2^4\delta_1^2 - \alpha_2^4\delta_2^2 + 3\alpha_2\delta_1^2\delta_2^3 - \alpha_2\delta_2^5 + \alpha_2^2\delta_2^2 h_1 + \delta_2^4 h_3 - \beta_2\delta_2^3}{\alpha_2\delta_2^3}, \varepsilon_4 = -\dfrac{1}{4}\dfrac{\delta_2^2\varepsilon_5^2}{\delta_1^2}. \end{cases} \tag{83}$$

Here δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 6$, $\alpha_2, \alpha_3, \beta_2$, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\alpha_2\delta_2 \neq 0, \tag{84}$$

and by substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_7 = 2(\ln f_7)_x = 2\frac{-\alpha_1 e^{-t\delta_1 - x\alpha_2\delta_1/\delta_2 - y\beta_1 - \varepsilon_1} - 1/4\delta_2\varepsilon_5^2\alpha_2/\delta_1 e^{t\delta_1 + x\alpha_2\delta_1/\delta_2 + y\beta_1 + \varepsilon_1} + \varepsilon_5\cos(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2)\alpha_2}{e^{-t\delta_1 - x\alpha_2\delta_1/\delta_2 - y\beta_1 - \varepsilon_1} - 1/4\delta_2^2\varepsilon_5^2/\delta_1^2 e^{t\delta_1 + x\alpha_2\delta_1/\delta_2 + y\beta_1 + \varepsilon_1} + \varepsilon_5\sin(t\delta_2 + x\alpha_2 + y\beta_2 + \varepsilon_2)}. \tag{85}$$

If $\tau_1 \longrightarrow \infty$, the cross-kink wave outputs $u \longrightarrow 2\alpha_2\delta_1/\delta_2$ at any time.

5.8. Set VIII Solutions

$$\begin{cases} \alpha_l = -\delta_l, l = 1, 2, \beta_1 = \dfrac{\beta_2\delta_1}{\delta_2}, h_2 = -\dfrac{-\delta_2 h_1 - \delta_2 h_3 + \beta_2}{\delta_2}, \\ \varepsilon_6 = 0, \theta = \dfrac{\sqrt{[3]}\left(-\alpha_1^4 - 2\alpha_1^2\alpha_2^2 - \alpha_2^4\right)\alpha_1^2}{\alpha_1}. \end{cases} \tag{86}$$

Here δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 5$, α_3, β_2, and β_3 are arbitrary inputs, and the following case is considered as

$$\delta_2 \neq 0. \tag{87}$$

By substituting the above parameters into equation (63), we obtain an analytical form of rational equation:

$$u_8 = 2(\ln\phi_8)_x = 2\frac{-\alpha_1 e^{-t\theta - x\alpha_1 - y\beta_1 - \varepsilon_1} + 1/4\varepsilon_5^2\alpha_1 e^{t\theta + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_5\cos(x\alpha_2 + y\beta_2 + \varepsilon_2)\alpha_2}{e^{-t\theta - x\alpha_1 - y\beta_1 - \varepsilon_1} + 1/4\varepsilon_5^2 e^{t\theta + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_5\sin(x\alpha_2 + y\beta_2 + \varepsilon_2)}. \tag{88}$$

If $\tau_1 \longrightarrow \infty$, the cross-kink wave outputs $u \longrightarrow 2\alpha_1$ at any time.

6. Interaction between Stripe and Periodic Wave Solutions

In this paragraph, we find out some advanced exact interaction between stripe and periodic wave solutions to the (2 + 1)-dimensional generalized Hietarinta equation. Assume the stated function for studying the interaction of solutions as

$$\phi = e^{-\tau_1} + \varepsilon_4 e^{\tau_1} + \varepsilon_5\cos(\tau_2) + \varepsilon_6\cosh(\tau_3), \tau_s = \alpha_s x + \beta_s y + \delta_s t + \varepsilon_s, s = 1{:}\ 3. \tag{89}$$

Afterwards, the values $\alpha_s, \beta_s, \delta_s, \epsilon_s$ $(s = 1{:}\ 3)$ will be found. By making use of equation (18) into (17) and taking the coefficients, each powers of $\cos(x, y, t)$ and $\cosh(x, y, t)$ and exponential function to zero yield a system of equations (algebraic) (these are not collected here for minimalist) for $\alpha_s, \beta_s, \delta_s, \epsilon_s$ $(s = 1{:}\ 3)$. These algebraic equations by using the emblematic computation software like, Maple, give the following solutions with using $u = 2(\ln\phi)_x$ and $v = 2(\ln\phi)$.

6.1. Set I Solutions

$$\begin{cases} \beta_1 = \dfrac{-\alpha_3^2\varepsilon_6^2\left(\alpha_1^3 + 2\alpha_1^2\alpha_3^2 + \alpha_1\delta_1^3 - 3\alpha_3^4\right) + 4\alpha_1\varepsilon_4\left(3\alpha_1^5 - 2\alpha_1^3\alpha_3^2 + 3\alpha_1^2\delta_1^3 - \alpha_1\alpha_3^4 + 4\alpha_3^2\delta_1^3\right) + \alpha_3^2\delta_1\left(-\varepsilon_6^2 + 4\varepsilon_4\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\alpha_3^2\delta_1\left(-\varepsilon_6^2 + 4\varepsilon_4\right)}, \\ \beta_3 = \dfrac{-4\alpha_3^2\varepsilon_6^2\left(\alpha_1^3 - \alpha_1\alpha_3^2 + \delta_1^3\right) + 8\varepsilon_4\left(3\alpha_1^5 - 4\alpha_1^3\alpha_3^2 + 3\alpha_1^2\delta_1^3 + \alpha_1\alpha_3^4 + 2\alpha_3^2\delta_1^3\right) + \alpha_3^2\delta_1 h_2\left(-\varepsilon_6^2 + 4\varepsilon_4\right)}{\alpha_3\left(-\varepsilon_6^2 + 4\varepsilon_4\right)\delta_1}, \\ \delta_3 = 0, h_1 = 4\dfrac{\alpha_3^4\varepsilon_6^2 + 3\alpha_1^4\varepsilon_4 - 6\alpha_1^2\alpha_3^2\varepsilon_4 + 3\alpha_1\delta_1^3\varepsilon_4 - \alpha_3^4\varepsilon_4}{\alpha_3^2\left(-\varepsilon_6^2 + 4\varepsilon_4\right)}, \varepsilon_5 = 0. \end{cases} \tag{90}$$

Here, α_d, ϵ_k for $d = 1{:}\ 2, k = 1{:}\ 6$, β_2, δ_1, and δ_2 are the unknown parameters. By considering the necessary assumption,

$$\alpha_3^2\delta_1\left(\varepsilon_6^2 + 4\varepsilon_4\right) \neq 0, \tag{91}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_1 = 2\left(\ln\phi_1\right)_x = 2\frac{-\alpha_1 e^{-\zeta_1} + \varepsilon_4\alpha_1 e^{\zeta_1} + \varepsilon_6\sinh\left(x\alpha_3 + y\beta_3 + \varepsilon_3\right)\alpha_3}{e^{-\zeta_1} + \varepsilon_4 e^{\zeta_1} + \varepsilon_6\cosh\left(x\alpha_3 + y\beta_3 + \varepsilon_3\right)}, \zeta_1 = t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1. \tag{92}$$

If $\tau_1 \longrightarrow \infty$, the breather outputs $u \longrightarrow 2\alpha_3$ at very time.

6.2. Set II Solutions

$$\begin{cases} \alpha_3 = \varepsilon_5 = 0, \beta_1 = \dfrac{-4\alpha_1\delta_3^4\varepsilon_6^2 + 4\alpha_1\varepsilon_4\left(3\alpha_1^3\delta_1 + 3\delta_1^4 - 3\delta_1^2\delta_3^2 + 4\delta_3^4\right) + \delta_3^2\left(-\varepsilon_6^2 + 4\varepsilon_4\right)\left(\alpha_1 h_2 + \delta_1 h_3\right)}{\left(-\varepsilon_6^2 + 4\varepsilon_4\right)\delta_3^2}, \\ \beta_3 = -\dfrac{12\alpha_1^4\varepsilon_4 + 12\alpha_1\delta_1^3\varepsilon_4 - 12\alpha_1\delta_1\delta_3^2\varepsilon_4 + \delta_3^2 h_3\varepsilon_6^2 - 4\delta_3^2 h_3\varepsilon_4}{\delta_3\left(-\varepsilon_6^2 + 4\varepsilon_4\right)}, \\ h_1 = \dfrac{\delta_3^2\varepsilon_6^2\left(\alpha_1^3 + \delta_1^3 - \delta_1\delta_3^2\right) + 4\varepsilon_4\left(3\delta_1^2 - 4\delta_3^2\right)\left(\alpha_1^4 + \delta_1^3 - \delta_1\delta_3^2\right)}{\alpha_1\delta_3^2\left(-\varepsilon_6^2 + 4\varepsilon_4\right)}. \end{cases} \tag{93}$$

Here δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 6$, $\alpha_1, \alpha_2, \beta_2$ are the unknown parameters. By considering the necessary assumption,

$$\alpha_1\delta_3^2\left(\varepsilon_6^2 + 4\varepsilon_4\right) \neq 0, \tag{94}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_2 = 2\left(\ln\phi_2\right)_x = 2\frac{-\alpha_1 e^{-\zeta_1} + \varepsilon_4\alpha_1 e^{\zeta_1}}{e^{-\zeta_1} + \varepsilon_4 e^{\zeta_1} + \varepsilon_6\cosh\left(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3\right)}, \zeta_1 = t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1. \tag{95}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the interaction between stripe and periodic wave outputs $u \longrightarrow 0$ at every time, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at every time.

6.3. Set III Solutions

$$\begin{cases} \beta_1 = \dfrac{4\alpha_3^4 + 4\alpha_3\delta_3^3 + \alpha_3^2 h_1 + \alpha_3\delta_3 h_2 + \delta_3^2 h_3}{\delta_3}, \beta_3 \\ = \dfrac{4\alpha_3^4 + 4\alpha_3\delta_3^3 + \alpha_3^2 h_1 + \alpha_3\delta_3 h_2 + \delta_3^2 h_3}{\delta_3}, \varepsilon_5 = 0. \end{cases} \tag{96}$$

Here, $\alpha_d, \delta_d, \epsilon_k$ for $d = 1: 3, k = 1: 6$, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\delta_3 \neq 0, \tag{97}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_3 = 2(\ln \phi_3)_x = 2\frac{-\alpha_1 e^{-\zeta_1} + \varepsilon_4 \alpha_1 e^{\zeta_1} + \varepsilon_6 \sinh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)\alpha_3}{e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_4 e^{\zeta_1} + \varepsilon_6 \cosh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)}. \tag{98}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_3$ at every time, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the cross-kink outputs $u \longrightarrow 2\alpha_1$ at any time.

6.4. Set IV Solutions

$$\left\{ \alpha_1 = -\alpha_3, \beta_1 = 2\delta_3 h_3 - \beta_3, h_1 = -4\alpha_3^2, \varepsilon_4 = \frac{1}{4}\varepsilon_6^2, \varepsilon_5 = 0. \right. \tag{99}$$

Here, δ_d, ϵ_k for $d = 1: 3, k = 1: 6$, $\alpha_2, \alpha_3, \beta_2$, and β_3 are the unknown parameters, and by considering the necessary assumption and also by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_4 = 2(\ln \phi_4)_x = 2\frac{-\alpha_3 e^{-t\delta_1 + x\alpha_3 - y(2\delta_3 h_3 - \beta_3) - \varepsilon_1} - 1/4\alpha_3 \varepsilon_6^2 e^{t\delta_1 - x\alpha_3 + y(2\delta_3 h_3 - \beta_3) + \varepsilon_1} + \varepsilon_6 \sinh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)\alpha_3}{e^{t\delta_1 - x\alpha_3 + y(2\delta_3 h_3 - \beta_3) + \varepsilon_1} + 1/4\varepsilon_6^2 e^{t\delta_1 - x\alpha_3 + y(2\delta_3 h_3 - \beta_3) + \varepsilon_1} + \varepsilon_6 \cosh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)}. \tag{100}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the interaction of outputs $u \longrightarrow 2\alpha_3$ at any t, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the the interaction of outputs $u \longrightarrow 2\alpha_1$ at any time.

6.5. Set V Solutions

$$\begin{cases} \alpha_1 = -\alpha_3, \beta_1 = -\dfrac{4\alpha_3^4 + 4\alpha_3\delta_3^3 + \alpha_3^2 h_1 + \alpha_3\delta_3 h_2 + \delta_3^2 h_3}{\delta_3}, \\ \beta_3 = \dfrac{4\alpha_3^4 + 4\alpha_3\delta_3^3 + \alpha_3^2 h_1 + \alpha_3\delta_3 h_2 + \delta_3^2 h_3}{\delta_3}, \delta_1 = -\delta_3, \varepsilon_5 = 0. \end{cases} \tag{101}$$

Here, $\alpha_d, \delta_d, \epsilon_k$ for $d = 1: 3, k = 1: 6$, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\delta_3 \neq 0, \tag{102}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation equation:

$$u_5 = 2(\ln \phi_5)_x = 2\frac{-\alpha_1 e^{t\delta_3 + x\alpha_3 - y\beta_1 - \varepsilon_1} + \varepsilon_4 \alpha_1 e^{-t\delta_3 - x\alpha_3 + y\beta_1 + \varepsilon_1} + \varepsilon_6 \sinh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)\alpha_3}{e^{t\delta_3 + x\alpha_3 - y\beta_1 - \varepsilon_1} + \varepsilon_4 e^{-t\delta_3 - x\alpha_3 + y\beta_1 + \varepsilon_1} + \varepsilon_6 \cosh(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3)}. \tag{103}$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the interaction of solution $u \longrightarrow 2\alpha_3$ at any t, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the interaction of solution $u \longrightarrow 2\alpha_1$ at any time.

6.6. Set VI Solutions

$$\begin{cases} \alpha_2 = \varepsilon_6 = 0, \beta_1 = -\dfrac{-4\alpha_1\delta_2^4\varepsilon_5^2 + 4\alpha_1\varepsilon_4\left(3\alpha_1^3\delta_1 + 3\delta_1^4 + 3\delta_1^2\delta_2^2 + 4\delta_2^4\right) - \delta_2^2\left(-\varepsilon_5^2 + 4\varepsilon_4\right)(\alpha_1 h_2 + \delta_1 h_3)}{\delta_2^2\left(-\varepsilon_5^2 + 4\varepsilon_4\right)}, \\ \beta_2 = \dfrac{12\alpha_1\varepsilon_4\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right) + \delta_2^2 h_3\left(-\varepsilon_5^2 + 4\varepsilon_4\right)}{\delta_2\left(-\varepsilon_5^2 + 4\varepsilon_4\right)}, \\ h_1 = -\dfrac{-\delta_2^2\varepsilon_5^2\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right) + 4\varepsilon_4\left(3_1^2 + 4\delta_2^2\right)\left(\alpha_1^3 + \delta_1^3 + \delta_1\delta_2^2\right)}{\alpha_1\delta_2^2\left(-\varepsilon_5^2 + 4\varepsilon_4\right)}. \end{cases} \tag{104}$$

Here, δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 6$, α_2, α_3, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\alpha_1 \delta_2^2 \left(-\varepsilon_5^2 + 4\varepsilon_4\right) \neq 0, \tag{105}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_6 = 2\left(\ln \phi_6\right)_x = 2\frac{-\alpha_1 e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_4 \alpha_1 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1}}{e^{-t\delta_1 - x\alpha_1 - y\beta_1 - \varepsilon_1} + \varepsilon_4 e^{t\delta_1 + x\alpha_1 + y\beta_1 + \varepsilon_1} + \varepsilon_5 \cos\left(t\delta_2 + y\beta_2 + \varepsilon_2\right)}. \tag{106}$$

If $\tau_1 \longrightarrow \infty$, the interaction between stripe and periodic wave solution $u \longrightarrow 2\alpha_1$ at any t. Figure 8 shows the analysis of treatment of interaction of solutions as periodic and hyperbolic functions with graphs of ϕ_6 with the following selected parameters:

$$\begin{aligned} &\delta_1 = 0.3, \delta_2 = 2, \delta_3 = 1, \alpha_1 = 0.1, \alpha_3 = 0.5, \beta_3 = 1, h_2 = 2, \\ &h_3 = 3, \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 4, \varepsilon_4 = 2, \varepsilon_5 = 1, t = 0.1, \end{aligned} \tag{107}$$

in equation (102).

6.7. Set VII Solutions

$$\begin{cases} \alpha_1 = \theta, \alpha_2 = \varepsilon_6 = 0, \beta_1 = -\dfrac{4\theta\delta_2^3 - \theta\delta_2 h_2 - 2\delta_1\delta_2 h_3 + \beta_2\delta_1}{\delta_2}, \\ h_1 = -\dfrac{-\delta_1^2\delta_2 h_3 - \delta_2^3 h_3 + \beta_2\delta_1^2 + \beta_2\delta_2^2}{\theta^2\delta_2}, \varepsilon_4 = \dfrac{1}{4}\varepsilon_5^2. \end{cases} \tag{108}$$

Here, $\theta = \sqrt{[3]} - \delta_1^3 - \delta_1\delta_2^2, \delta_d, \epsilon_k$ for $d = 1{:}\ 3, k = 1{:}\ 3$, α_3, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

$$\delta_2 \neq 0, \tag{109}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_7 = 2\left(\ln \phi_7\right)_x = 2\frac{-\alpha_1 e^{-t\delta_1 - x\sqrt{[3]} - \delta_1^3 - \delta_1\delta_2^2 - y\beta_1 - \varepsilon_1} + 1/4\varepsilon_5^2 \alpha_1 e^{t\delta_1 + x\sqrt{[3]} - \delta_1^3 - \delta_1\delta_2^2 + y\beta_1 + \varepsilon_1}}{e^{-t\delta_1 - x\sqrt{[3]} - \delta_1^3 - \delta_1\delta_2^2 - y\beta_1 - \varepsilon_1} + 1/4\varepsilon_5^2 e^{t\delta_1 + x\sqrt{[3]} - \delta_1^3 - \delta_1\delta_2^2 + y\beta_1 + \varepsilon_1} + \varepsilon_5 \cos\left(t\delta_2 + y\beta_2 + \varepsilon_2\right)}. \tag{110}$$

If $\tau_1 \longrightarrow \infty$, the interaction between stripe and periodic wave solution $u \longrightarrow 2\alpha_1$ at any t. Figure 9 shows the analysis of treatment of interaction of solutions as periodic and hyperbolic functions with graphs of ϕ_7 with the following selected parameters:

$$\begin{aligned} \delta_1 &= -1.2, \delta_2 = 1, \delta_3 = 1, \alpha_3 = 0.5, \beta_3 = 1, h_2 = 2, h_3 \\ &= 3, \varepsilon_1 = 1, \varepsilon_2 = 0.2, \varepsilon_3 = 4, \varepsilon_5 = 1, t = 0.01, \end{aligned} \tag{111}$$

in equation (106).

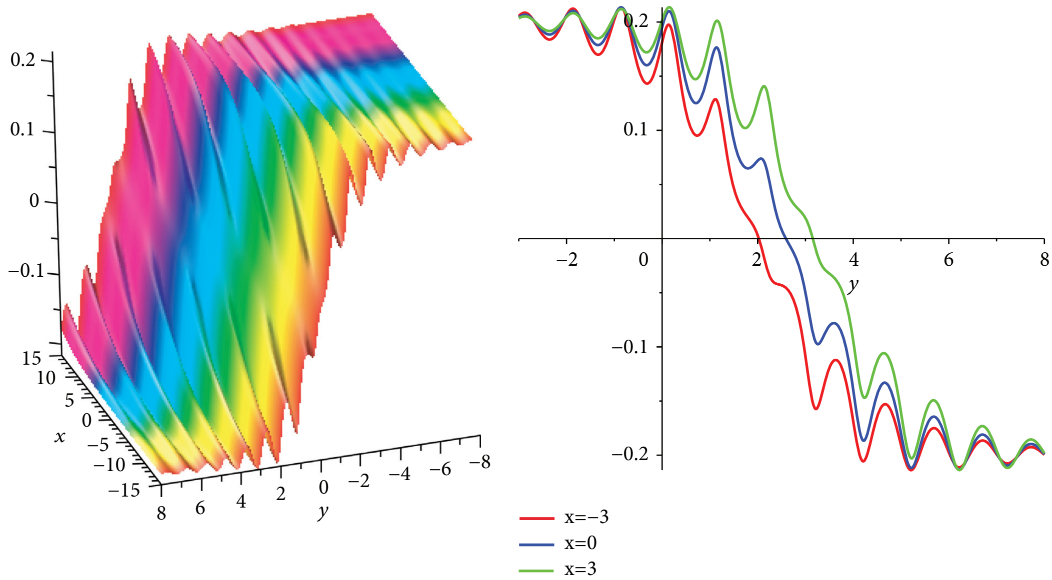
6.8. Set VIII Solutions

$$\begin{cases} \alpha_i = -\delta_i, i = 1, 2, \beta_1 = \dfrac{\beta_2\delta_1}{\delta_2}, h_2 \\ = -\dfrac{-\delta_2 h_1 - \delta_2 h_3 + \beta_2}{\delta_2}, \varepsilon_6 = 0. \end{cases} \tag{112}$$

Here, δ_d, ϵ_k for $d = 1{:}\ 3, k = 1{:}\ 5$, α_3, β_2, and β_3 are the unknown parameters. By considering the necessary assumption,

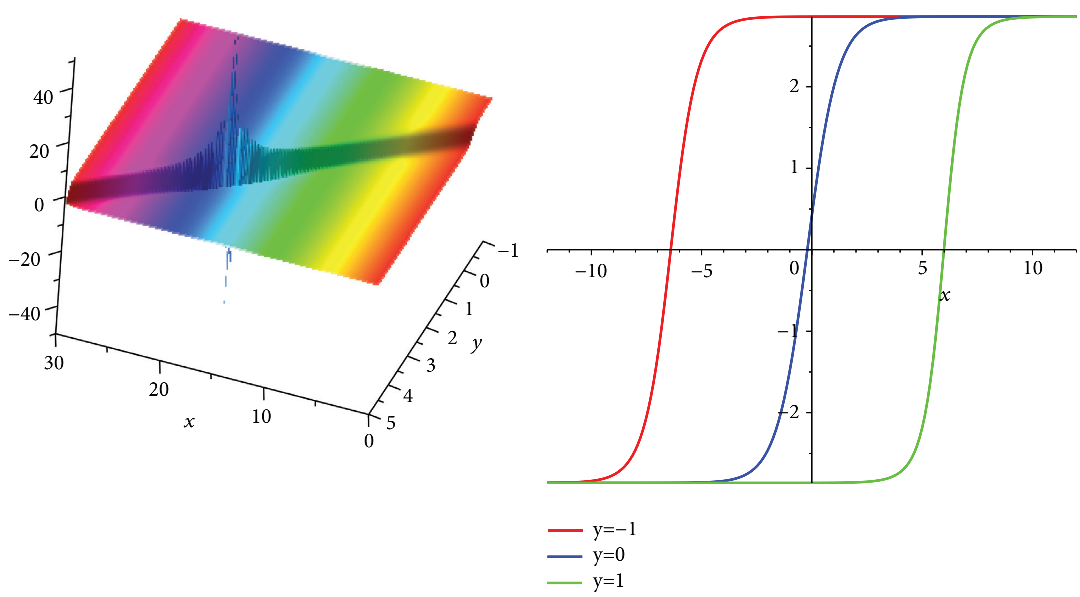
$$\delta_2 \neq 0, \tag{113}$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:



(a) (b)

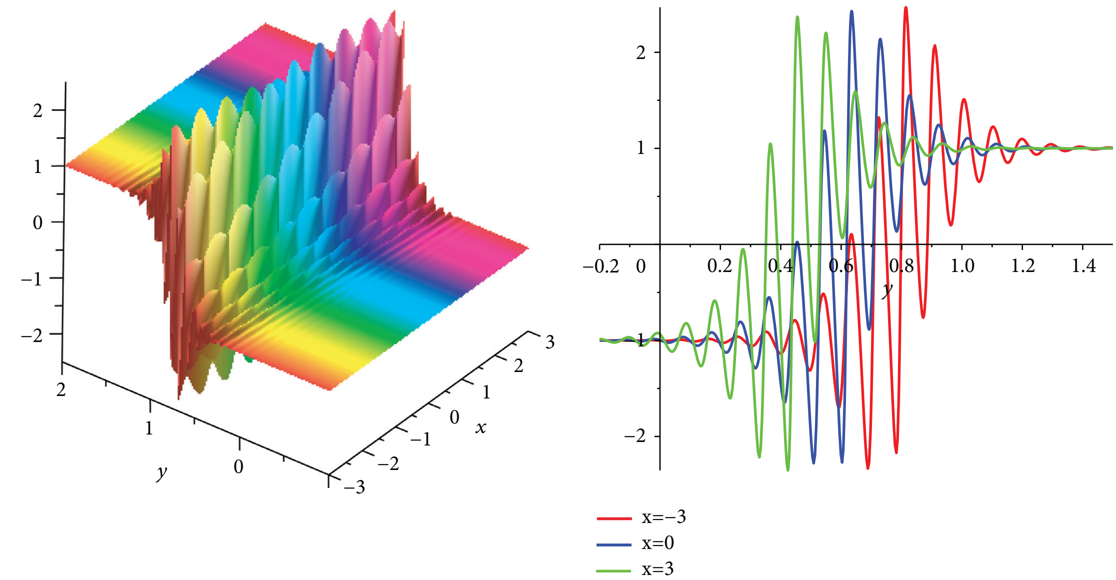
FIGURE 8: Interaction between stripe and periodic wave solution (107) such that (a) 3D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1, x = -3, 0, 3$.



(a) (b)

FIGURE 9: Interaction between stripe and periodic wave solution (111) such that (a) 3D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1, y = -1, 0, 1$.

$$u_8 = 2(\ln \phi_8)_x = 2\frac{\delta_1 e^{-t\delta_1 + x\delta_1 - y\beta_2\delta_1/\delta_2 - \varepsilon_1} - \varepsilon_4 \delta_1 e^{t\delta_1 - x\delta_1 + y\beta_2\delta_1/\delta_2 + \varepsilon_1} + \varepsilon_5 \sin(t\delta_2 - x\delta_2 + y\beta_2 + \varepsilon_2)\delta_2}{e^{-t\delta_1 + x\delta_1 - y\beta_2\delta_1/\delta_2 - \varepsilon_1} + \varepsilon_4 e^{t\delta_1 - x\delta_1 + y\beta_2\delta_1/\delta_2 + \varepsilon_1} + \varepsilon_5 \cos(t\delta_2 - x\delta_2 + y\beta_2 + \varepsilon_2)}. \quad (114)$$



(a) (b)

Figure 10: Interaction between stripe and periodic wave solution (117) such that (a) 3D design of $u(x, y, t)$ at $t = 1$ and (b) 2D plot of $u(x, y, t)$ at $t = 1, x = -3, 0, 3$.

6.9. Set IX Solutions

$$\begin{cases} \alpha_2 = \theta, \beta_1 = -\dfrac{4\alpha_3^6 + 4\alpha_3^3\delta_3^3 + \theta^2\left(4\alpha_3^4 - 4\alpha_3\delta_3^3 - \alpha_3\delta_3 h_2 - \delta_3^2 h_3\right)}{\theta^2\delta_3}, \\ \beta_2 = -\dfrac{4\alpha_3^5 + 4\alpha_3^2\delta_3^3 + \theta^2\left(4\alpha_3^3 - 4\delta_3^3 - \delta_3 h_2\right)}{\theta\delta_3}, \\ \beta_3 = -\dfrac{4\alpha_3^6 + 4\alpha_3^3\delta_3^3 + \theta^2\left(4\alpha_3^4 - 4\alpha_3\delta_3^3 - \alpha_3\delta_3 h_2 - \delta_3^2 h_3\right)}{\theta^2\delta_3}, \\ \delta_2 = 0, h_1 = -4\dfrac{\alpha_3\left(2\alpha_3\theta^2 + \alpha_3^3 + \delta_3^3\right)}{\theta^2}, \alpha_1 = \alpha_3, \delta_1 = \delta_3. \end{cases} \quad (115)$$

Here, ϵ_k for $k = 1$: 6, α_3, and δ_3 are free values. Also, θ solves the $\theta^4 + 2\theta^2\alpha_3^2 + \alpha_3^4 + \alpha_3\delta_3^3 = 0$. By considering the necessary assumption,

$$\theta\delta_3 \neq 0, \quad (116)$$

and by substituting the above parameters into equation (89), we obtain an analytical form of rational equation:

$$u_9 = 2\frac{-\alpha_3 e^{-t\delta_3 - x\alpha_3 - y\beta_1 - \varepsilon_1} + \varepsilon_4\alpha_3 e^{t\delta_3 + x\alpha_3 + y\beta_1 + \varepsilon_1} - \varepsilon_5 \sin\left(x\theta + y\beta_2 + \varepsilon_2\right)\theta + \varepsilon_6 \sinh\left(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3\right)\alpha_3}{e^{-t\delta_3 - x\alpha_3 - y\beta_1 - \varepsilon_1} + \varepsilon_4 e^{t\delta_3 + x\alpha_3 + y\beta_1 + \varepsilon_1} + \varepsilon_5 \cos\left(x\theta + y\beta_2 + \varepsilon_2\right) + \varepsilon_6 \cosh\left(t\delta_3 + x\alpha_3 + y\beta_3 + \varepsilon_3\right)}. \quad (117)$$

If $\tau_3 > \tau_1 \longrightarrow \infty$, the the interaction between stripe and periodic wave solution $u \longrightarrow 2\alpha_3$ at any t, but if $\tau_3 < \tau_1 \longrightarrow \infty$, the interaction between stripe and periodic wave solution $u \longrightarrow 2\alpha_1$ at any t. Figure 10 shows the analysis of treatment of interaction of solutions as periodic and hyperbolic functions with graphs of ϕ_9 with the following selected parameters:

$$\begin{aligned}\delta_1 &= -1.2, \delta_2 = 1, \delta_3 = 3, \alpha_3 = -0.5, \beta_2 \\ &= 1, \beta_3 = 1, h_2 = 2, h_3 = 3, \\ \varepsilon_1 &= 1, \varepsilon_2 = 0.1, \varepsilon_3 = 3, \varepsilon_4 \\ &= 4, \varepsilon_5 = 5, \varepsilon_6 = 4, t = 1,\end{aligned} \tag{118}$$

in equation (117).

7. Conclusion

This article investigated the soliton and periodic solutions of the generalized Hietarinta equation. The Cole–Hopf algorithm has been described by means of binary Bell polynomials. The governing equation is translated to nonlinear ODE using Hirota transformation. Various types of soliton, breather, and periodic solutions have been constructed in terms of exponential, hyperbolic, trigonometric, and rational functions. The dynamic features of different types of traveling waves are analyzed in detail through numerical simulation. Meanwhile, the profiles of the surface for the deduced solutions have been depicted in 2D and 3D for the obtained solutions. The gained solutions may be applied to explain the model in simple and straight forward way. At the end, it is concluded that, to handle nonlinear partial differential equations, Hirota bilinear technique suggested an effective and well-built mathematical tools. These solutions are also verified by using Maple software.

Data Availability

The datasets supporting the conclusions of this article are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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