

## Research Article

# Double Sumudu Transform Iterative Method for One-Dimensional Nonlinear Coupled Sine-Gordon Equation

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In this paper, the combined double Sumudu transform with iterative method is successfully implemented to obtain the approximate analytical solution of the one-dimensional coupled nonlinear sine-Gordon equation (NLSGE) subject to the appropriate initial and boundary conditions which cannot be solved by applying double Sumudu transform only. The solution of the nonlinear part of this equation was solved by a successive iterative method, the proposed technique has the advantage of producing an exact solution, and it is easily applied to the given problems analytically. Two test problems from mathematical physics were taken to show the liability, accuracy, convergence, and efficiency of the proposed method. Furthermore, the results indicate that the introduced method is promising for solving other types of systems of NLPDEs.

## 1. Introduction

The system of coupled sine-Gordon equations in the form

$$\begin{cases} u_{tt} - u_{xx} = -\delta^2 \sin(u - v), \\ v_{tt} - \alpha^2 v_{xx} = \sin(u - v), \end{cases} \quad \alpha > 0, \delta > 0, \quad (1)$$

was introduced by Khusnutdinova and Pelinovsky [1]. The coupled sine-Gordon equations generalize the Frenkel-Kontorova dislocation model [2, 3]. System (1) with  $\alpha = 1$  was also proposed to describe the open states in the DNA double helix model [4]. This type of equation has been receiving an enormous amount of attention due to the presence of soliton solutions, and it has applications in the propagation of fluxons in Josephson junctions between two superconductors [5], the motion of a rigid pendulum attached to a stretched wire [6], etc.

Recently, many researchers used various methods to get the solution of system (1). Li et al. [7] presented a lattice Boltzmann model for the two-component system of coupled sine-Gordon equations by using the coupled mesoscopic

Boltzmann equations and the Chapman-Enskog multiscale expansion, while the macroscopical governing evolution system can be recovered correctly by selecting suitable discrete equilibrium distribution functions and the amending functions. In paper [8], the authors used a rational exponential ansatz to derive the exact solutions of a coupled sine-Gordon equation. The simplest equation method has been used for finding the exact solutions of coupled sine-Gordon equations by Zhao [9]. In papers [10–12], Zhao et al. obtained some new solutions of coupled sine-Gordon equations including Jacobi elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions by the Jacobi elliptic function expansion method, the hyperbolic auxiliary function method, and the symbolic computation method. Sadighi et al. [13] employed the homotopy perturbation method (HPM) for solving both sine-Gordon and coupled sine-Gordon equations. In paper [14], the authors obtained numerical solution of inhomogeneous systems of sine-Gordon equation (1) by finite difference method with fixed-point iteration.

In 2018, the authors of [15] studied the coupled sine-Gordon equations in nonlinear optics, which describe the

propagation of an optical pulse in fiber waveguide, and they derived the new exact solutions of the problem through the use of the well-organized modified Kudryashov method. For more related research about solving the sine-Gordon equations, interested readers may refer to [16–18].

Since many application problems in science and engineering are formulated by nonlinear partial differential equations, new techniques are required to solve those problems efficiently. The Sumudu transform is an integral transform similar to the Laplace transform, introduced in the early 1990s by Watugala [19] to solve differential equations and control engineering problems. Recently, this method is used to solve differential equations like nonlinear boundary value problems [20] and the Cauchy problem for the wave equation in one dimensional space [21]. In [22], the Sumudu transform is applied to arbitrary powers Dumont bimodular Jacobi elliptic functions for arbitrary powers. Furthermore, Watugala [23] has extended the Sumudu transform to two variables with emphasis on solutions to partial differential equations. Ahmeda et al. [24] studied the convergence properties of double Sumudu transformation and used it to find the exact solution of the Volterra integropartial differential equation. In the paper [25], the authors' applied the double Sumudu transform linked with the Adomian decomposition method or with variational iteration method to find the analytical solution of nonlinear fractional partial differential equations. Kiwne and Sonawane [26] present the proofs of some theorems of double Sumudu transform with application in solving partial differential equations like one-dimensional heat equation, Poisson equation, and second-order hyperbolic equations of two variables. The advantage of the double Sumudu transform is it gives a rapid convergence of the exact solution without any restrictive assumption of the solution compared to other known methods (see [27]). Unfortunately, this transform fails to solve nonlinear partial differential equations like other integral transforms; to solve this problem, this transform is often combined with other methods like the variational method [28], reduced differential transform method [29], double integral transform (Laplace-Sumudu transform) method [30], Adomian decomposition method [25, 31], and homotopy perturbation method [20, 32].

The main aim of this work is to apply the double Sumudu transform method coupled with the new iterative method (NIT) proposed by Daftardar-Gejji and Jafari in [33] to find an exact/approximate solution of the nonlinear coupled sine-Gordon equation. The new iterative method (NIM) has been extensively used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order (see [33–37]). The method converges to the exact solution if it exists through successive approximations. The nonlinear term in the equation is expanded in terms of Daftardar-Gejji and Jafari polynomials (see [38, 39]).

In the present study, we consider the nonlinear initial boundary value problem (IBVP) for the one-dimensional coupled sine-Gordon equation [14] in the following form:

$$\begin{cases} u_{tt} - u_{xx} = -\sin(u - v) + f(x, t), \\ v_{tt} - v_{xx} = \sin(u - v) + h(x, t), \end{cases} \quad (2)$$

subject to the initial conditions

$$\begin{cases} u(x, 0) = \varphi_1(x), & u_t(x, 0) = \varphi_2(x), \\ v(x, 0) = \varphi_3(x), & v_t(x, 0) = \varphi_4(x), \end{cases} \quad (3)$$

and boundary conditions (Cauchy-type BCs)

$$\begin{cases} u(0, t) = g_1(t), & u_x(0, t) = g_2(t), \\ u(0, t) = g_3(t), & u_x(0, t) = g_4(t), \end{cases} \quad (4)$$

within the problem domain of  $\Omega = \{x : a \leq x \leq b\}$  and  $t > 0$ .

The remaining parts of this paper are structured as follows. In Section 2, we begin with some basic definitions, properties, and theorems of the double Sumudu transform method. Section 3 illustrates the details of the new iterative method and its convergence. Section 4 presents how to obtain an approximate analytical solution of the coupled SGE using the DSTIM. In Section 5, we apply the proposed method to two illustrative examples in order to show its liability, convergence, and efficiency. Finally, concluding remarks are given in Section 6.

## 2. A Brief Introduction of Double Sumudu Transform (DST)

*Definition 1* (see [24, 40]). Sumudu transform of a function  $f(t)$  is defined for all real numbers  $t > 0$  as the function  $S(p)$ , given by

$$S_t[f(t)] = F(p) = \frac{1}{p} \int_0^\infty e^{-(t/p)} f(t) dt = \lim_{j \rightarrow \infty} \frac{1}{p} \int_0^j e^{-(t/p)} f(t) dt. \quad (5)$$

*Definition 2* (see [30]). The inverse Sumudu transform of a function  $F(p)$  is denoted by the symbol  $S^{-1}[F(p)] = f(t)$  and is defined by the Bromwich contour integral

$$S_t^{-1}[F(p)] = f(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{t/p} F(p) dp. \quad (6)$$

*Definition 3* (see [24, 40, 41]). Let  $f(x, t)$  be a function that can be expressed as a convergent infinite series and let  $(x, t) \in \mathbb{R}_+^n$ , then the double Sumudu transform of the function  $f(x, t)$  in the positive quadrant of the  $xt$ -plane is given by

$$S_x S_t[f(x, t)] = F(k, p) = \frac{1}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} f(x, t) dp dk, \quad (7)$$

where  $x, t \geq 0$ , and  $k, p$  are the transform variables for  $x$  and  $t$ , respectively, whenever the improper integral is convergent.

**Definition 4** (see [42, 43]). The inverse double Sumudu transform  $S_x^{-1}S_t^{-1}[F(k, p)] = f(x, t)$  is defined by the following form:

$$S_x^{-1}S_t^{-1}[F(k, p)] = f(x, t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{k} e^{x/k} \left[ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{p} e^{t/p} F(k, p) dp \right] dk. \tag{8}$$

**Remark 5** (see [26, 40]). If  $f(x, t)$  is a continuous function having a second partial derivative, then double Sumudu transforms of partial derivative of the first and second are defined as follows:

Double Sumudu transform for the first partial derivative with respect to  $t$  is given by

$$S_x S_t \left[ \frac{\partial f(x, t)}{\partial t}; k, p \right] = \frac{1}{kp} \lim_{j \rightarrow \infty} \left[ \int_0^j \int_0^j e^{-(x/k+t/p)} f(x, t) dt dx \right] = \frac{1}{p} [F(k, p) - F(k, 0)]. \tag{9}$$

Double Sumudu transform for the first partial derivative with respect to  $x$  is given by

$$S_x S_t \left[ \frac{\partial f(x, t)}{\partial x}; k, p \right] = \frac{1}{kp} \lim_{j \rightarrow \infty} \left[ \int_0^j \int_0^j e^{-(x/k+t/p)} f(x, t) dt dx \right] = \frac{1}{k} [F(k, p) - F(0, p)]. \tag{10}$$

Double Sumudu transform for the second partial derivative with respect to  $t$  is given by

$$S_x S_t \left[ \frac{\partial^2 f(x, t)}{\partial t^2}; k, p \right] = \int_0^\infty \int_0^\infty \frac{\partial^2 f(x, t)}{\partial t^2} e^{-(x/k+t/p)} f(x, t) dt dx = \frac{1}{p^2} F(k, p) - \frac{1}{p^2} F(0, p) - \frac{1}{p} \frac{\partial F(0, p)}{\partial x}. \tag{11}$$

Double Sumudu transform for the second partial derivative with respect to  $x$  is given by

$$S_x S_t \left[ \frac{\partial^2 f(x, t)}{\partial x^2}; k, p \right] = \int_0^\infty \int_0^\infty \frac{\partial^2 f(x, t)}{\partial x^2} e^{-(x/k+t/p)} f(x, t) dt dx = \frac{1}{k^2} F(k, p) - \frac{1}{k^2} F(k, 0) - \frac{1}{k} \frac{\partial F(k, 0)}{\partial t}. \tag{12}$$

**2.1. Convergence Theorem of Double Sumudu Transform**

**Lemma 6** (see [24]). Let  $f(x, t)$  be a function of two variables continuous in the positive quadrant of the  $xt$ -plane. If the integral

$$\frac{1}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} f(x, t) dt dx, \tag{13}$$

converges at  $k = k_0, p = p_0$ , then the integral converges for  $k < k_0, p < p_0$ .

**2.2. Existence and Uniqueness of the Double Sumudu Transform.** If  $f(x, t)$  is an exponential order, then  $c$  and  $d$  as  $x \rightarrow \infty, t \rightarrow \infty$ , and if there exists a positive constant  $K$  such that for all  $x > X, t > T$ , then

$$|f(x, t)| = K e^{cx+dt}. \tag{14}$$

And we write  $f(x, t) = O e^{cx+dt}$  as  $x \rightarrow \infty, t \rightarrow \infty$ . Or, equivalently,

$$\lim_{x \rightarrow \infty, t \rightarrow \infty} e^{-(x/k)-t/p} |f(x, t)| = K \lim_{x \rightarrow \infty, t \rightarrow \infty} e^{-(x/k-c)x-(t/p-d)t} = 0, \quad \frac{1}{k} > c, \frac{1}{p} > d. \tag{15}$$

And we write this as  $f(x, t) = O(e^{ax+by+ct})$  as  $x \rightarrow \infty$  and  $t \rightarrow \infty$ .

The function  $f(x, t)$  is called an exponential order as  $x \rightarrow \infty, t \rightarrow \infty$ , and clearly, it does not grow faster than  $K e^{cx+dt}$  as  $x \rightarrow \infty, t \rightarrow \infty$ .

**Theorem 7** (existence). If a function  $f(x, t)$  is continuous in every finite intervals  $(0, X)$  and  $(0, T)$  and of exponential order  $e^{cx+dt}$ , then the double Sumudu transform of  $f(x, t)$  exists for all  $1/k$  and  $1/p$  provided that  $\text{Re}(1/k) > c$  and  $\text{Re}(1/p) > d$ .

For the proof, see [42, 43].

**Theorem 8** (uniqueness). Let  $f(x, t)$  and  $g(x, t)$  be continuous functions defined for  $x, t \geq 0$  and having double Sumudu transforms  $F(k, p)$  and  $G(k, p)$ , respectively. If  $F(k, p) = G(k, p)$ , then  $f(x, t) = g(x, t)$ .

**2.3. Some Properties of Double Sumudu Transform**

**Property 9** (linearity of TLT, [42, 43]). If  $f(x, t)$  and  $g(x, t)$  are two functions of  $x$  and  $t$  such that  $S_x S_t \{f(x, t)\} = F(k, p)$  and  $S_x S_t \{g(x, t)\} = G(k, p)$ , then

$$\begin{aligned} S_x S_t \{ \alpha f(x, t) + \beta g(x, t) \} &= \frac{1}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} [ \alpha f(x, t) + \beta g(x, t) ] dt dx \\ &= \frac{1}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} \alpha f(x, t) dt dx + \frac{1}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} \beta g(x, t) dt dx \\ &= \frac{\alpha}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} f(x, t) dt dx + \frac{\beta}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} g(x, t) dt dx \\ &= \alpha S_x S_t f(x, t) + \beta S_x S_t g(x, t) = \alpha F(k, p) + \beta G(k, p), \end{aligned} \tag{16}$$

where  $\alpha$  and  $\beta$  are constants.

*Property 10* (change of scale property, [44]). If  $f(x, t)$  and  $g(x, t)$  are two functions of  $x$  and  $t$  such that  $S_x S_t \{f(x, t)\} = F(k, p)$  and  $S_x S_t \{g(x, t)\} = G(k, p)$ , then

- (i)  $S_x S_t \{f(cx, dt)\} = (1/cd)F(ck, dp)$ ,
- (ii)  $S_x S_t \{f(cx)g(dt)\} = (1/cd)F(ck)G(dp)$ ,

where  $c$  and  $d$  are nonzero constants.

*Property 11* (first shifting property). If  $S_x S_t \{f(x, t)\} = F(u, v)$ , then

$$S_x S_t \left\{ e^{cx+dt} f(x, t) \right\} = \frac{1}{(1 - ck)(1 - dp)} F\left(\frac{k}{1 - ck}, \frac{p}{1 - dp}\right). \tag{17}$$

*Proof.* from (7), we have

$$\begin{aligned} S_x S_t \left[ e^{cx+dt} f(x, t) \right] &= \frac{1}{kp} \int_0^\infty \int_0^\infty e^{-(x/k+t/p)} e^{cx+dt} f(x, t) dt dx, \\ &= \frac{1}{k} \int_0^\infty e^{-(1/k-c)x} \left[ \int_0^\infty \frac{1}{p} e^{-(1/p-d)t} f(x, t) dt \right] dx, \\ &= \frac{1}{k} \int_0^\infty e^{-(1/k-c)x} \left[ \int_0^\infty \frac{1}{p} e^{-(1/p-d)t} f(x, t) dt \right] dx. \end{aligned} \tag{18}$$

By using the integration by substitution method for the integral inside the bracket and the integral outside the bracket, we get

$$S_x S_t \left\{ e^{cx+dt} f(x, t) \right\} = \frac{1}{(1 - ck)(1 - dp)} F\left(\frac{k}{1 - ck}, \frac{p}{1 - dp}\right). \tag{19}$$

Similarly,

$$L_x L_y L_t \left\{ e^{-cx-dt} \right\} f(x, t) = \frac{1}{(1 + ck)(1 + dp)} F\left(\frac{k}{1 - ck}, \frac{p}{1 - dp}\right). \tag{20}$$

□ □

*Property 12* (second shifting property, [42, 43]). If  $S_x S_t \{f(x, t)\} = F(k, p)$ , then

$$S_x S_t \{f(x - c, t - d)H(x - c, t - d)\} = e^{-(ck)-dp} F(k, p), \tag{21}$$

where  $H(x, t)$  is the Heaviside unit step function defined by

$$H(x - c, t - d) = \begin{cases} 1, & x > c, t > d \\ 0, & \text{otherwise} \end{cases}. \tag{22}$$

### 3. The New Iterative Method

Daftardar-Gejji and Jafari [33] have considered the following functional equation:

$$u = N(u) + f, \tag{23}$$

where  $N$  is a nonlinear operator in a Banach space such that  $N : B \rightarrow B$  and  $f$  is a given element of the Banach space  $B$ . Here,  $u$  is assumed to be the solution of equation (23) whose series expansion is given by

$$u = \sum_{i=0}^\infty u_i. \tag{24}$$

The nonlinear operation  $N$  can then be decomposed as

$$N\left(\sum_{i=0}^\infty u_i\right) = N(u_0) + \sum_{i=1}^\infty \left\{ N\left(\sum_{r=0}^i u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right) \right\}. \tag{25}$$

Using Equations (24) and (25), Equation (23) is equivalent to

$$\sum_{i=0}^\infty u_i = f + N(u_0) + \sum_{i=1}^\infty \left\{ N\left(\sum_{r=0}^i u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right) \right\}. \tag{26}$$

From Equation (26), we define the following recurrence relation:

$$\begin{cases} u_0 = f, & u_1 = N(u_0), \\ u_2 = N(u_0 + u_1) - N(u_0), \\ u_{(n+1)} = N(u_0 + \dots + u_n) - N(u_0 + \dots + u_{(n-1)}), & n = 1, 2, \dots \end{cases} \tag{27}$$

Thus,

$$\left(u_1 + \dots + u_{(n+1)}\right) = N\left(u_0 + \dots + u_{(n)}\right), \quad n = 1, 2, \dots, \tag{28}$$

and hence,

$$\sum_{i=0}^\infty u_i = f + N\left(\sum_{i=0}^\infty u_i\right). \tag{29}$$

Therefore, the  $n^{\text{th}}$  term approximate solution of equation (24) is given by

$$u = u_0 + u_1 + u_2 + u_3 + \dots + u_{n-1}, \quad n > 1. \tag{30}$$

#### 3.1. Convergence of the New Iterative Method

**Theorem 13.** If  $N$  is a continuously differentiable functional in a neighborhood of  $u_0$  and  $\|N^{(n)}(u_0)\| \leq M \leq e^{-1}$  for all  $n$ , then the series  $\sum_{n=0}^\infty u_{n+1}$  is absolutely convergent [33].

### 4. Double Sumudu Transform Coupled with Iterative Method

To solve the system of sine-Gordon equation (2), first, we decompose the source function  $f(x, t)$  into  $f_1(x, t)$  and  $f_2(x, t)$ . The part  $f_1(x, t)$  with the terms in equation (2) always leads to the simple algebraic expression while applying the inverse double Sumudu transform. The portion  $f_2(x, t)$  is combined with the nonlinear term of equation (2) to avoid noise terms in the iteration process. Similarly,  $h(x, t) = h_1(x, t) + h_2(x, t)$ .

Applying the double Sumudu transform to both sides of the system of equation (2) and by using Table 1 and Definition 4, we get

$$\begin{cases} \frac{1}{k^2} \bar{u}(k, p) - \frac{1}{k^2} \bar{u}(k, 0) - \frac{1}{k} \bar{u}_t(k, 0) - \frac{1}{p^2} \bar{u}(k, p) + \frac{1}{p^2} \bar{u}(0, p) + \frac{1}{p} \bar{u}_x(0, p) \\ = \bar{f}_1(k, s) - S_x S_t \{ \sin(u - v) - f_2(x, t) \}, \\ \frac{1}{k^2} \bar{v}(k, p) - \frac{1}{k^2} \bar{v}(k, 0) - \frac{1}{k} \bar{v}_t(k, 0) - \frac{1}{p^2} \bar{v}(k, p) + \frac{1}{p^2} \bar{v}(0, p) + \frac{1}{p} \bar{v}_x(0, p) \\ = \bar{h}_1(k, p) + S_x S_t \{ \sin(u - v) + h_2(x, t) \}, \end{cases} \quad (31)$$

where  $\bar{u}$ ,  $\bar{u}_t$ ,  $\bar{u}_x$ ,  $\bar{v}$ ,  $\bar{v}_t$ ,  $\bar{v}_x$ ,  $\bar{f}_1$ , and  $\bar{h}_1$  are Sumudu transforms of functions of two variables.

Multiplying both sides of each equation in system (31) by  $k^2 p^2$  yields

$$\begin{cases} p^2 \bar{u}(k, p) - p^2 \bar{u}(k, 0) - kp^2 \bar{u}_t(k, 0) - k^2 \bar{u}(k, p) + k^2 \bar{u}(0, p) + pk^2 \bar{u}_x(0, p) \\ = k^2 p^2 \bar{f}_1(k, s) - k^2 p^2 S_x S_t \{ \sin(u - v) - f_2(x, t) \}, \\ p^2 \bar{v}(k, p) - p^2 \bar{v}(k, 0) - kp^2 \bar{v}_t(k, 0) - k^2 \bar{v}(k, p) + k^2 \bar{v}(0, p) + pk^2 \bar{v}_x(0, p) \\ = k^2 p^2 \bar{h}_1(k, p) + k^2 p^2 S_x S_t \{ \sin(u - v) + h_2(x, t) \}. \end{cases} \quad (32)$$

Next, applying a single Sumudu transform to the systems of equations (3) and (4), we get

$$\begin{cases} \bar{u}(k, 0) = \bar{\varphi}_1(k), \quad \bar{u}_t(k, 0) = \bar{\varphi}_2(k), \\ \bar{v}(k, 0) = \bar{\varphi}_3(k), \quad \bar{v}_t(k, 0) = \bar{\varphi}_4(k), \end{cases} \quad (33)$$

$$\begin{cases} \bar{u}(0, p) = \bar{g}_1(p), \quad \bar{u}_t(0, s) = \bar{g}_2(p), \\ \bar{v}(0, p) = \bar{g}_3(p), \quad \bar{v}_t(0, s) = \bar{g}_4(p). \end{cases} \quad (34)$$

Here,  $\bar{\varphi}_i$  and  $\bar{g}_i$  ( $i = 1, 2, 3, 4$ ) are Sumudu transforms of functions of single variables.

By substituting the systems of equations (34) and (33) into the systems of equation (32) and simplifying, we obtain

$$\begin{cases} \bar{u}(k, p) = \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{u}(k, 0) + kp^2 \bar{u}_t(k, 0) - k^2 \bar{u}(0, p) - pk^2 \bar{u}_x(0, p) - k^2 p^2 \bar{f}_1(k, p) \right], \\ = -k^2 p^2 S_x S_t \{ \sin(u - v) - f_2(x, t) \}, \\ \bar{v}(k, p) = \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{v}(k, 0) + kp^2 \bar{v}_t(k, 0) - k^2 \bar{v}(0, p) - pk^2 \bar{v}_x(0, p) - k^2 p^2 \bar{h}_1(k, p) \right], \\ = k^2 p^2 S_x S_t \{ \sin(u - v) + h_2(x, t) \}. \end{cases} \quad (35)$$

Applying the inverse double Sumudu transform to the system of equation (35), we obtain

$$\begin{cases} u(x, t) = S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{u}(k, 0) + kp^2 \bar{u}_t(k, 0) - k^2 \bar{u}(0, p) - pk^2 \bar{u}_x(0, p) - k^2 p^2 \bar{f}_1(k, p) \right] \right\}, \\ = -k^2 p^2 S_x S_t \{ \sin(u - v) - f_2(x, t) \}, \\ v(x, t) = S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{v}(k, 0) + kp^2 \bar{v}_t(k, 0) - k^2 \bar{v}(0, p) - pk^2 \bar{v}_x(0, p) - k^2 p^2 \bar{h}_1(k, p) \right] \right\}, \\ = k^2 p^2 S_x S_t \{ \sin(u - v) + h_2(x, t) \}. \end{cases} \quad (36)$$

Now, we start an iterative process. We assume that

$$\begin{cases} u(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \\ v(x, t) = \sum_{i=0}^{\infty} v_i(x, t), \end{cases} \quad (37)$$

is the solution of the coupled nonlinear sine-Gordon equation (2).

Substituting the systems of equation (37) into the systems of equation (36), we obtain

$$\begin{cases} \sum_{i=0}^{\infty} u_i(x, t) = S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{u}(k, 0) + kp^2 \bar{u}_t(k, 0) - k^2 \bar{u}(0, p) - pk^2 \bar{u}_x(0, p) - k^2 p^2 \bar{f}_1(k, p) \right] \right\}, \\ = k^2 p^2 S_x S_t \left\{ f_2(x, t) - \sin \left\{ \sum_{i=0}^{\infty} [u_i(x, t) - v_i(x, t)] \right\} \right\}, \\ \sum_{i=0}^{\infty} v_i(x, t) = S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{v}(k, 0) + kp^2 \bar{v}_t(k, 0) - k^2 \bar{v}(0, p) - pk^2 \bar{v}_x(0, p) - k^2 p^2 \bar{h}_1(k, p) \right] \right\}, \\ = k^2 p^2 S_x S_t \left\{ h_2(x, t) + \sin \left\{ \sum_{i=0}^{\infty} [u_i(x, t) - v_i(x, t)] \right\} \right\}. \end{cases} \quad (38)$$

Now, by (25), the nonlinear term  $\sin(u(x, t) - v(x, t))$  in the systems of equation (38) is decomposed as

Substituting the system of equation (38) into the systems of equation (37), we get

$$\begin{aligned} \sin\left(\sum_{i=0}^{\infty}(u_i(x, t) - v_i(x, t))\right) &= \sin(u_0(x, t) - v_0(x, t)) \\ &+ \sum_{i=1}^{\infty}\left\{\sin\left(\sum_{r=0}^i(u_r(x, t) - v_r(x, t))\right) - \sin\left(\sum_{r=0}^{i-1}(u_r(x, t) - v_r(x, t))\right)\right\}. \end{aligned} \tag{39}$$

$$\left\{ \begin{aligned} \sum_{i=0}^{\infty} u_i(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{u}(k, 0) + k p^2 \bar{u}_t(k, 0) - k^2 \bar{u}(0, p) - p k^2 \bar{u}_x(0, p) - k^2 p^2 \bar{f}_1(k, p) \right] \right. \\ &= k^2 p^2 S_x S_t \left\{ \begin{aligned} &f_2(x, t) - \sin(u_0(x, t) - v_0(x, t)) \\ &\sin\left(\sum_{r=0}^i(u_r(x, t) - v_r(x, t))\right) \\ &-\sin\left(\sum_{r=0}^{i-1}(u_r(x, t) - v_r(x, t))\right) \end{aligned} \right\} \end{aligned} \right\}, \\ \sum_{i=0}^{\infty} v_i(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ p^2 \bar{v}(k, 0) + k p^2 \bar{v}_t(k, 0) - k^2 \bar{v}(0, p) - p k^2 \bar{v}_x(0, p) - k^2 p^2 \bar{h}_1(k, p) \right] \right. \\ &= k^2 p^2 S_x S_t \left\{ \begin{aligned} &h_2(x, t) + \sin(u_0(x, t) - v_0(x, t)) \\ &\sin\left(\sum_{r=0}^i(u_r(x, t) - v_r(x, t))\right) \\ &-\sin\left(\sum_{r=0}^{i-1}(u_r(x, t) - v_r(x, t))\right) \end{aligned} \right\} \end{aligned} \right\}. \end{aligned} \tag{40}$$

Now, we define the recurrence relation from the systems of equation (40) as follows:

$$\begin{aligned} \left\{ \begin{aligned} u_0(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} [p^2 \bar{u}(k, 0) + k p^2 \bar{u}_t(k, 0) - k^2 \bar{u}(0, p) - p k^2 \bar{u}_x(0, p) - k^2 p^2 \bar{f}_1(k, p)] \right\}, \\ v_0(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} [p^2 \bar{v}(k, 0) + k p^2 \bar{v}_t(k, 0) - k^2 \bar{v}(0, p) - p k^2 \bar{v}_x(0, p) - k^2 p^2 \bar{h}_1(k, p)] \right\}, \end{aligned} \right. \\ \left\{ \begin{aligned} u_1(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} [k^2 p^2 S_x S_t \{f_2(x, t) - \sin(u_0(x, t) - v_0(x, t))\}] \right\}, \\ v_1(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} [k^2 p^2 S_x S_t \{h_2(x, t) + \sin(u_0(x, t) - v_0(x, t))\}] \right\}, \end{aligned} \right. \\ \left\{ \begin{aligned} u_{n+1}(x, t) &= -S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ k^2 p^2 S_x S_t \left\{ \sum_{i=1}^{\infty} \left[ \begin{aligned} &\sin\left(\sum_{r=0}^i(u_r(x, t) - v_r(x, t))\right) \\ &-\sin\left(\sum_{r=0}^{i-1}(u_r(x, t) - v_r(x, t))\right) \end{aligned} \right] \right\} \right] \right\}, \\ v_{n+1}(x, t) &= S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ k^2 p^2 S_x S_t \left\{ \sum_{i=1}^{\infty} \left[ \begin{aligned} &\sin\left(\sum_{r=0}^i(u_r(x, t) - v_r(x, t))\right) \\ &-\sin\left(\sum_{r=0}^{i-1}(u_r(x, t) - v_r(x, t))\right) \end{aligned} \right] \right\} \right] \right\}, \end{aligned} \right. \quad n > 1. \end{aligned} \tag{41}$$

TABLE 1: Double Sumudu transforms of some functions of two variables [30, 41, 45, 46].

Functions $f(x, t)$	Double Sumudu transform $F(k, p)$
$ab$	$ab$
$xt$	$kp$
$x^m t^n$	$m!n!k^n p^m$
$e^{-ax-bt}$	$\frac{1}{(1+ak)(1+bp)}$
$e^{ax+bt}$	$\frac{1}{(1-ak)(1-bp)}$
$\cos(ax) \cos(bt)$	$\frac{1}{(1+a^2k^2)(1+b^2p^2)}$
$\sin(ax) \sin(bt)$	$\frac{abkp}{(1+a^2k^2)(1+b^2p^2)}$
$\cos h(ax) \cos h(bt)$	$\frac{1}{(1-a^2k^2)(1-b^2p^2)}$
$\sin h(ax) \sin h(bt)$	$\frac{abkp}{(1-a^2k^2)(1-b^2p^2)}$
$\sin(ax+bt)$	$\frac{ak+bp}{(1+a^2k^2)(1+b^2p^2)}$
$\cos(ax+bt)$	$\frac{abkp}{(1+a^2k^2)(1+b^2p^2)}$
$\cos h(ax+bt)$	$1/2[(1/(1-ak)(1-bp)) + (1/(1+ak)(1+bp))]$
$\sin h(ax+bt)$	$\frac{1}{2} \left[ \frac{1}{(1-ak)(1-bp)} - \frac{1}{(1+ak)(1+bp)} \right]$

Therefore, the solution of the systems of SGE equations (2)–(4) in series form is given by

$$\begin{cases} u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots + u_n(x, t) + \dots, \\ v(x, t) = v_0(x, t) + v_1(x, t) + v_2(x, t) + \dots + v_n(x, t) + \dots. \end{cases} \quad (42)$$

Finally, Theorem 13 will be applied to demonstrate the series solution obtained from the new iterative method which is convergent.

### 5. Illustrative Examples

In order to show the validity and effectiveness of the method under consideration, some examples are presented here.

*Example 1.* Consider the following IBVP for one-dimensional coupled sine-Gordon equation on  $\Omega = [0, \pi]$ ,  $t > 0$ :

$$\begin{cases} u_{tt} - u_{xx} = -\sin(u-v) + \sin(t \sin x - e^{-t} \sin x) + t \sin x, \\ v_{tt} - v_{xx} = \sin(u-v) - \sin(t \sin x - e^{-t} \sin x) + 2e^{-t} \sin x, \end{cases} \quad (43)$$

with initial condition

$$\begin{cases} u(x, 0) = 0, & u_t(x, 0) = \sin x, \\ v(x, 0) = \sin x, & v_t(x, 0) = -\sin x, \end{cases} \quad (44)$$

and boundary condition

$$\begin{cases} u(0, t) = 0, & u_x(0, t) = t, \\ u(\pi, t) = 0, & u_x(\pi, t) = e^{-t}. \end{cases} \quad (45)$$

*Solution.* Applying the properties of double Sumudu transform to both sides of the system of equation (43), we get

$$\begin{cases} \frac{1}{k^2} \bar{u}(k, p) - \frac{1}{k^2} \bar{u}(k, 0) - \frac{1}{k} \bar{u}_t(k, 0) - \frac{1}{p^2} \bar{u}(k, p) + \frac{1}{p^2} \bar{u}(0, p) + \frac{1}{p} \bar{u}_x(0, p) + S_x S_t (\sin(u-v)) \\ = \frac{pk}{1+k^2} + S_x S_t (\sin(t \sin x - e^{-t} \sin x)), \\ \frac{1}{k^2} \bar{v}(k, p) - \frac{1}{k^2} \bar{v}(k, 0) - \frac{1}{k} \bar{v}_t(k, 0) - \frac{1}{p^2} \bar{v}(k, p) + \frac{1}{p^2} \bar{v}(0, p) + \frac{1}{p} \bar{v}_x(0, p) - S_x S_t (\sin(u-v)) \\ = \frac{2}{(1+p)(1+k^2)} - S_x S_t (\sin(t \sin x - e^{-t} \sin x)). \end{cases} \quad (46)$$

Multiplying both sides of each equation in the system

(46) by  $k^2 p^2$ , we obtain

$$\begin{cases} p^2 \bar{u}(k, p) - p^2 \bar{u}(k, 0) - kp^2 \bar{u}_t(k, 0) - k^2 \bar{u}(k, p) + k^2 \bar{u}(0, p) + pk^2 \bar{u}_x(0, p) \\ = \frac{p^3 k^3}{1+k^2} - k^2 p^2 S_x S_t (\sin(u-v) - \sin(t \sin x - e^{-t} \sin x)), \\ p^2 \bar{v}(k, p) - p^2 \bar{v}(k, 0) - kp^2 \bar{v}_t(k, 0) - k^2 \bar{v}(k, p) + k^2 \bar{v}(0, p) + pk^2 \bar{v}_x(0, p) \\ = \frac{2k^2 p^2}{(1+p)(1+k^2)} + k^2 p^2 S_x S_t (\sin(u-v) - \sin(t \sin x - e^{-t} \sin x)). \end{cases} \quad (47)$$

Next, applying the single Sumudu transform to the systems of equations (44) and (45), we get

$$\begin{cases} \bar{u}(k, 0) = 0, \bar{u}_t(k, 0) = \frac{k}{1+k^2}, \\ \bar{v}(k, 0) = \frac{k}{1+k^2}, \bar{v}_t(k, 0) = -\frac{k}{1+k^2}, \end{cases} \quad (48)$$

$$\begin{cases} \bar{u}(0, p) = 0, \bar{u}_x(0, p) = p, \\ \bar{v}(0, p) = \frac{k}{1+k^2}, \bar{v}_x(0, p) = \frac{1}{1+p}. \end{cases} \quad (49)$$

By substituting the system of equations (49) and (48) into the system of equation (47) and simplifying, we obtain

$$\begin{cases} \bar{u}(k, p) = \frac{pk}{1+k^2} - \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u-v) - \sin(t \sin x - e^{-t} \sin x) \}], \\ \bar{v}(k, p) = \frac{2k}{(1+p)(1+k^2)} + \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u-v) - \sin(t \sin x - e^{-t} \sin x) \}]. \end{cases} \quad (50)$$

Applying the inverse double Sumudu transform to (50), we get

$$\begin{cases} u(x, t) = t \sin x - S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u-v) - \sin(t \sin x - e^{-t} \sin x) \}] \right], \\ v(x, t) = e^{-t} \sin x + S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u-v) - \sin(t \sin x - e^{-t} \sin x) \}] \right]. \end{cases} \quad (51)$$

Now, applying the new iterative method to a system of equations (51), we obtain the components of the solution as follows:

$$\begin{cases} u_0(x, t) = t \sin x, \\ v_0(x, t) = e^{-t} \sin x, \end{cases}$$

$$\begin{cases} u_1(x, t) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u_0(x, t) - v_0(x, t)) - \sin(t \sin x - e^{-t} \sin x) \}] \right], \\ v_1(x, t) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u_0(x, t) - v_0(x, t)) - \sin(t \sin x - e^{-t} \sin x) \}] \right], \\ \implies \begin{cases} u_1(x, t) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(t \sin x - e^{-t} \sin x) - \sin(t \sin x - e^{-t} \sin x) \}] \right] = 0, \\ v_1(x, t) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(t \sin x - e^{-t} \sin x) - \sin(t \sin x - e^{-t} \sin x) \}] \right] = 0, \end{cases} \end{cases} \quad (52)$$

$$\begin{cases} u_{n+1} u(x, t) = S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \sum_{i=1}^{\infty} \left\{ \begin{aligned} & \sin \left( \sum_{r=0}^i (u_r(x, t) - v_r(x, t)) \right) \\ & - \sin \left( \sum_{r=0}^{i-1} (u_r(x, t) - v_r(x, t)) \right) \end{aligned} \right\} \right\} \right\} \right\}, \\ v_{n+1} u(x, t) = -S_x^{-1} S_y^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \sum_{i=1}^{\infty} \left\{ \begin{aligned} & \sin \left( \sum_{r=0}^i (u_r(x, t) - v_r(x, t)) \right) \\ & - \sin \left( \sum_{r=0}^{i-1} (u_r(x, t) - v_r(x, t)) \right) \end{aligned} \right\} \right\} \right\} \right\}, \end{cases} \quad n > 1.$$



For  $n = 1$ ,

$$\begin{cases} u_2(x, t) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} \sin((u_0(x, t) - v_0(x, t)) + (u_1(x, t) - v_1(x, t))) \\ -\sin(u_0(x, t) - v_0(x, t)) \end{array} \right\} \right] \right] = 0, \\ v_2(x, t) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} \sin((u_0(x, t) - v_0(x, t)) + (u_1(x, t) - v_1(x, t))) \\ -\sin(u_0(x, t) - v_0(x, t)) \end{array} \right\} \right] \right] = 0, \end{cases} \quad (53)$$

in the same way, we obtain

$$\begin{cases} u_3(x, t) = 0, & u_4(x, t) = 0, \\ v_3(x, t) = 0, & v_4(x, t) = 0, \end{cases} \quad (54)$$

and so on.

Therefore, the solution of Example 1 in the sense of (42) is

$$\begin{cases} u(x, t) = t \sin x, \\ v(x, t) = e^{-t} \sin x, \end{cases} \quad (55)$$

which is the same as the result obtained by Yildirim et al. [14].

Let us now test the convergence of the obtained series solution. From the system of equation (51), we have

$$\begin{cases} u_0(x, t) = t \sin x, \\ v_0(x, t) = e^{-t} \sin x, \\ \begin{cases} N(u(x, t)) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u - v) - \sin(t \sin x - e^{-t} \sin x) \}] \right], \\ N(v(x, t)) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u - v) - \sin(t \sin x - e^{-t} \sin x) \}] \right]. \end{cases} \end{cases} \quad (56)$$

Thus, for all  $x, t \geq 0$ , we have

$$\begin{cases} N(u_0(x, t)) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u_0(x, t) - v_0(x, t)) - \sin(t \sin x - e^{-t} \sin x) \}] \right] = 0, \\ N(v_0(x, t)) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u_0(x, t) - v_0(x, t)) - \sin(t \sin x - e^{-t} \sin x) \}] \right] = 0. \end{cases} \quad (57)$$

Therefore,

$$\begin{cases} \|N(u_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N(v_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \end{cases} \quad (58)$$

$$\begin{cases} N\left(\frac{\partial}{\partial t} u(x, t)\right) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} \cos(u - v) \frac{\partial}{\partial t}(u - v) \\ -\cos(t \sin x - e^{-t} \sin x) (\sin x + e^{-t} \sin x) \end{array} \right\} \right] \right], \\ N\left(\frac{\partial}{\partial t} v(x, t)\right) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} \cos(u - v) \frac{\partial}{\partial t}(u - v) \\ -\cos(t \sin x - e^{-t} \sin x) (\sin x + e^{-t} \sin x) \end{array} \right\} \right] \right]. \end{cases} \quad (59)$$

Then,

$$\begin{cases} N\left(\frac{\partial}{\partial t}u_0(x,t)\right) = -S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} \cos(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x) \\ -c\cos(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x) \end{array}\right\}\right]\right] = 0, \\ N\left(\frac{\partial}{\partial t}v_0(x,t)\right) = S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} \cos(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x) \\ -\cos(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x) \end{array}\right\}\right]\right] = 0, \\ N\left(\frac{\partial}{\partial x}u(x,t)\right) = -S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} \cos(u-v)\frac{\partial}{\partial x}(u-v) \\ -\cos(t\sin x - e^{-t}\sin x)(t\cos x - e^{-t}\cos x) \end{array}\right\}\right]\right], \\ N\left(\frac{\partial}{\partial x}v(x,t)\right) = S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} \cos(u-v)\frac{\partial}{\partial x}(u-v) \\ -\cos(t\sin x - e^{-t}\sin x)(t\cos x - e^{-t}\cos x) \end{array}\right\}\right]\right]. \end{cases} \quad (60)$$

Then,

$$\begin{cases} N\left(\frac{\partial}{\partial x}u_0(x,t)\right) = -S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} \cos(t\sin x - e^{-t}\sin x)(t\cos x - e^{-t}\cos x) - \\ \cos(t\sin x - e^{-t}\sin x)(t\cos x - e^{-t}\cos x) \end{array}\right\}\right]\right] = 0, \\ N\left(\frac{\partial}{\partial x}v_0(x,t)\right) = S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} \cos(t\sin x - e^{-t}\sin x)(t\cos x - e^{-t}\cos x) - \\ \cos(t\sin x - e^{-t}\sin x)(t\cos x - e^{-t}\cos x) \end{array}\right\}\right]\right] = 0. \end{cases} \quad (61)$$

Therefore,

$$\begin{cases} \|N'(u_0(x,t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N'(v_0(x,t))\| = \|0\| = 0 < \frac{1}{e}, \end{cases} \quad (62)$$

$$\begin{cases} N\left(\frac{\partial^2}{\partial t^2}u_0(x,t)\right) = -S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} -\sin(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x)^2 \\ +\cos(t\sin x - e^{-t}\sin x)(-e^{-t}\sin x) \\ \sin(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x)^2 \\ -\cos(t\sin x - e^{-t}\sin x)(-e^{-t}\sin x) \end{array}\right\}\right]\right] = 0, \\ N\left(\frac{\partial^2}{\partial t^2}v_0(x,t)\right) = S_x^{-1}S_t^{-1}\left[\frac{k^2p^2}{k^2-p^2}\left[S_xS_t\left\{\begin{array}{l} -\sin(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x)^2 \\ +\cos(t\sin x - e^{-t}\sin x)(-e^{-t}\sin x) \\ \sin(t\sin x - e^{-t}\sin x)(\sin x + e^{-t}\sin x)^2 \\ -\cos(t\sin x - e^{-t}\sin x)(-e^{-t}\sin x) \end{array}\right\}\right]\right] = 0, \end{cases}$$

$$\left\{ \begin{array}{l} N\left(\frac{\partial^2}{\partial x^2} u_0(x, t)\right) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} -\sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)^2 \\ +\cos(t \sin x - e^{-t} \sin x)(-t \sin t + e^{-t} \sin x) \\ \sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)^2 \\ -\cos(t \sin x - e^{-t} \sin x)(-t \sin t + e^{-t} \sin x) \end{array} \right\} \right] \right] = 0, \\ N\left(\frac{\partial^2}{\partial x^2} v_0(x, t)\right) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} -\sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)^2 \\ +\cos(t \sin x - e^{-t} \sin x)(-t \sin t + e^{-t} \sin x) \\ \sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)^2 \\ -\cos(t \sin x - e^{-t} \sin x)(-t \sin t + e^{-t} \sin x) \end{array} \right\} \right] \right] = 0, \\ N\left(\frac{\partial^2}{\partial t \partial x} u_0(x, t)\right) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} -\sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)(\sin x + e^{-t} \sin x) \\ +\cos(t \sin x - e^{-t} \sin x)(\cos x + e^{-t} \cos x) \\ \sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)(\sin x + e^{-t} \sin x) \\ -\cos(t \sin x - e^{-t} \sin x)(\cos x + e^{-t} \cos x) \end{array} \right\} \right] \right] = 0, \\ N\left(\frac{\partial^2}{\partial t \partial x} v_0(x, t)\right) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{array}{l} -\sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)(\sin x + e^{-t} \sin x) \\ +\cos(t \sin x - e^{-t} \sin x)(\cos x + e^{-t} \cos x) \\ \sin(t \sin x - e^{-t} \sin x)(t \cos x - e^{-t} \cos x)(\sin x + e^{-t} \sin x) \\ -\cos(t \sin x - e^{-t} \sin x)(\cos x + e^{-t} \cos x) \end{array} \right\} \right] \right] = 0. \end{array} \right. \quad (63)$$

Therefore,

$$\left\{ \begin{array}{l} \|N''(u_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N''(v_0(x, t))\| = \|0\| = 0 < \frac{1}{e}. \end{array} \right. \quad (64)$$

Similarly, we have

$$\left\{ \begin{array}{l} \|N^{(3)}(u_0(x, t))\| = \|N^{(4)}(u_0(x, t))\| = \dots = \|N^{(k)}(u_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N^{(3)}(v_0(x, t))\| = \|N^{(4)}(v_0(x, t))\| = \dots = \|N^{(k)}(v_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \end{array} \right. \quad (65)$$

for all  $k \geq 0$  by the principle of mathematical induction.

As the conditions of Theorem 13 are satisfied, the series solution obtained by the new iterative method is convergent on the domain of interest.

*Example 2.* Consider the following IBVP for one-dimensional coupled sine-Gordon equation on  $\Omega = [0, \pi]$ ,  $t > 0$ :

$$\begin{cases} u_{tt} - u_{xx} = -\sin(u - v) + \sin(\cos(x + 2t) - \sin x \sinh(t)) - 3 \cos(x + 2t), \\ v_{tt} - v_{xx} = \sin(u - v) - \sin(\cos(x + 2t) - \sin x \sinh(t)) + 2 \sin x \sinh(t), \end{cases} \quad (66)$$

with initial condition

$$\begin{cases} u(x, 0) = \cos x, & u_t(x, 0) = -2 \sin x, \\ v(x, 0) = 0, & v_t(x, 0) = \sin x, \end{cases} \quad (67)$$

and boundary condition

$$\begin{cases} u(0, t) = \cos(2t), & u_x(0, t) = -\sin(2t), \\ u(0, t) = 0, & u_x(0, t) = \sin h(t). \end{cases} \quad (68)$$

*Solution.* Applying the properties of the double Sumudu transform to both sides of the system of equations (66), we get

$$\begin{cases} \frac{1}{k^2} \bar{u}(k, p) - \frac{1}{k^2} \bar{u}(k, 0) - \frac{1}{k} \bar{u}_t(k, 0) - \frac{1}{p^2} \bar{u}(k, p) + \frac{1}{p^2} \bar{u}(0, p) + \frac{1}{p} \bar{u}_x(0, p) + S_x S_t (\sin(u - v)) = -\frac{6kp}{(1+k^2)(1+4p^2)} + S_x S_t (\sin(\cos(x+2t) - \sin x \sinh(t))), \\ \frac{1}{k^2} \bar{v}(k, p) - \frac{1}{k^2} \bar{v}(k, 0) - \frac{1}{k} \bar{v}_t(k, 0) - \frac{1}{p^2} \bar{v}(k, p) + \frac{1}{p^2} \bar{v}(0, p) + \frac{1}{p} \bar{v}_x(0, p) - S_x S_t (\sin(u - v)) = \frac{2kp}{(1+k^2)(1-p^2)} + S_x S_t (\sin(\cos(x+2t) - \sin x \sinh(t))). \end{cases} \tag{69}$$

Multiplying both sides of each equation in the system of equation (69) by  $k^2 p^2$ , we get

$$\begin{cases} p^2 \bar{u}(k, p) - p^2 \bar{u}(k, 0) - kp^2 \bar{u}_t(k, 0) - k^2 \bar{u}(k, p) + k^2 \bar{u}(0, p) + pk^2 \bar{u}_x(0, p) \\ = -\frac{6p^3 k^3}{(1+k^2)(1+4p^2)} - k^2 p^2 S_x S_t (\sin(u - v) - \sin(\cos(x+2t) - \sin x \sinh(t))), \\ p^2 \bar{v}(k, p) - p^2 \bar{v}(k, 0) - kp^2 \bar{v}_t(k, 0) - k^2 \bar{v}(k, p) + k^2 \bar{v}(0, p) + pk^2 \bar{v}_x(0, p) \\ = \frac{2p^3 k^3}{(1+k^2)(1-p^2)} + k^2 p^2 S_x S_t (\sin(u - v) - \sin(\cos(x+2t) - \sin x \sinh(t))). \end{cases} \tag{70}$$

Next, applying a single Sumudu transform to the systems of equations (67) and (68), we obtain

$$\begin{cases} \bar{u}(k, 0) = \frac{1}{1+k^2}, \quad \bar{u}_t(k, 0) = \frac{-2k}{1+k^2}, \\ \bar{v}(k, 0) = 0, \quad \bar{v}_t(k, 0) = \frac{k}{1+k^2}, \end{cases} \tag{71}$$

$$\begin{cases} \bar{u}(0, p) = \frac{1}{1+4p^2}, \quad \bar{u}_x(0, p) = \frac{-2p}{1+4p^2}, \\ \bar{v}(0, p) = \frac{k}{1+k^2}, \quad \bar{v}_x(0, p) = \frac{p}{1-4p^2}. \end{cases} \tag{72}$$

By substituting the systems of equations (72) and (71) into the system of equation (70) and simplifying, we obtain

$$\begin{cases} \bar{u}(k, p) = \frac{2kp}{(1+k^2)(1+4p^2)} - \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(u - v) - \sin(\cos(x+2t) - \sin x \sinh(t))\}], \\ \bar{v}(k, p) = \frac{kp}{(1+k^2)(1-p^2)} + \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(u - v) - \sin(\cos(x+2t) - \sin x \sinh(t))\}]. \end{cases} \tag{73}$$

Applying inverse double Sumudu transform to (73), we obtain

$$\begin{cases} u(x, t) = \cos(x+2t) - S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(u - v) - \sin(\cos(x+2t) - \sin x \sinh(t))\}] \right], \\ v(x, t) = \sin x \sinh(t) + S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(u - v) - \sin(\cos(x+2t) - \sin x \sinh(t))\}] \right]. \end{cases} \tag{74}$$

Now, applying the new iterative method to the system of equations (74), we obtain the components of the solution as

follows:

$$\begin{cases} u_0(x, t) = \cos(x+2t), \\ v_0(x, t) = \sin x \sinh(t), \end{cases}$$

$$\begin{cases} u_1(x, t) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(u_0(x, t) - v_0(x, t)) - \sin(\cos(x+2t) - \sin x \sinh(t))\}] \right], \\ v_1(x, t) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(u_0(x, t) - v_0(x, t)) - \sin(\cos(x+2t) - \sin x \sinh(t))\}] \right], \\ \implies \begin{cases} u_1(x, t) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(ts \sin x - e^{-t} \sin x) - \sin(ts \sin x - e^{-t} \sin x)\}] \right] = 0, \\ v_1(x, t) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{\sin(ts \sin x - e^{-t} \sin x) - \sin(ts \sin x - e^{-t} \sin x)\}] \right] = 0, \end{cases} \end{cases}$$

$$\begin{cases} u_{n+1}(x, t) = S_x^{-1} S_t^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \sum_{i=1}^{\infty} \left\{ \sin \left( \sum_{r=0}^i (u_r(x, t) - v_r(x, t)) \right) \right\} \right\} \right\} \right\}, \\ v_{n+1}(x, t) = -S_x^{-1} S_t^{-1} \left\{ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \sum_{i=1}^{\infty} \left\{ \sin \left( \sum_{r=0}^i (u_r(x, t) - v_r(x, t)) \right) \right\} \right\} \right\} \right\}, \end{cases} \quad n > 1. \tag{75}$$

For  $n = 1$ ,

$$\begin{cases} u_2(x, t) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{matrix} \sin((u_0(x, t) - v_0(x, t)) + (u_1(x, t) - v_1(x, t))) \\ -\sin(u_0(x, t) - v_0(x, t)) \end{matrix} \right\} \right] \right] = 0, \\ v_2(x, t) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{matrix} \sin((u_0(x, t) - v_0(x, t)) + (u_1(x, t) - v_1(x, t))) \\ -\sin(u_0(x, t) - v_0(x, t)) \end{matrix} \right\} \right] \right] = 0, \end{cases} \tag{76}$$

in the same way, we obtain

$$\begin{cases} u_3(x, t) = 0, & \begin{cases} u_4(x, t) = 0, \\ v_3(x, t) = 0, & \begin{cases} v_4(x, t) = 0, \end{cases} \end{cases} \end{cases} \tag{77}$$

and so on.

Therefore, the solution of Example 2 in the sense of (40) is

$$\begin{cases} u(x, t) = \cos(x+2t), \\ v(x, t) = \sin x \sinh(t). \end{cases} \tag{78}$$

Let us now test the convergence of the obtained series

solution. From the system of equation (74), we have

$$\begin{cases} u_0(x, t) = \cos(x + 2t), \\ v_0(x, t) = \sin x \sinh(t), \end{cases} \tag{79}$$

$$\begin{cases} N(u(x, t)) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u - v) - \sin(\cos(x + 2t) - \sin x \sinh(t)) \}] \right], \\ N(v(x, t)) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \{ \sin(u - v) - \sin(\cos(x + 2t) - \sin x \sinh(t)) \}] \right]. \end{cases} \tag{80}$$

Thus, for all  $x, t \geq 0$ , we have

$$\begin{cases} N(u_0(x, t)) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \left\{ \begin{matrix} \sin(\cos(x + 2t) - \sin x \sinh(t)) \\ -\sin(\cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\}] \right] = 0, \\ N(v_0(x, t)) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} [S_x S_t \left\{ \begin{matrix} \sin(\cos(x + 2t) - \sin x \sinh(t)) \\ -\sin(\cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\}] \right] = 0. \end{cases} \tag{81}$$

Therefore,

$$\begin{cases} \|N(u_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N(v_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \end{cases} \tag{82}$$

$$\begin{cases} N\left(\frac{\partial}{\partial t} u(x, t)\right) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{matrix} \cos(u - v) \frac{\partial}{\partial t} (u - v) + \\ \cos(\cos(x + 2t) - \sin x \sinh(t)) \\ (-2 \cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\} \right] \right], \\ N\left(\frac{\partial}{\partial t} v(x, t)\right) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{matrix} \cos(u - v) \frac{\partial}{\partial t} (u - v) + \\ \cos(\cos(x + 2t) - \sin x \sinh(t)) \\ (-2 \cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\} \right] \right]. \end{cases} \tag{83}$$

Then,

$$\begin{cases} N\left(\frac{\partial}{\partial t} u_0(x, t)\right) = -S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{matrix} -\left\{ \begin{matrix} \cos(\cos(x + 2t) - \sin x \sinh(t)) \\ (-2 \cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\} \\ + \left\{ \begin{matrix} \cos(\cos(x + 2t) - \sin x \sinh(t)) \\ (-2 \cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\} \end{matrix} \right\} \right] \right] = 0, \\ N\left(\frac{\partial}{\partial t} v_0(x, t)\right) = S_x^{-1} S_t^{-1} \left[ \frac{k^2 p^2}{k^2 - p^2} \left[ S_x S_t \left\{ \begin{matrix} -\left\{ \begin{matrix} \cos(\cos(x + 2t) - \sin x \sinh(t)) \\ (-2 \cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\} \\ + \left\{ \begin{matrix} \cos(\cos(x + 2t) - \sin x \sinh(t)) \\ (-2 \cos(x + 2t) - \sin x \sinh(t)) \end{matrix} \right\} \end{matrix} \right\} \right] \right] = 0. \end{cases} \tag{84}$$

Therefore,

$$\begin{cases} \|N'(u_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N'(v_0(x, t))\| = \|0\| = 0 < \frac{1}{e}. \end{cases} \tag{85}$$

Similarly, we have

$$\begin{cases} \|N''(u_0(x, t))\| = \|N^{(3)}(u_0(x, t))\| = \|N^{(4)}(u_0(x, t))\| = \dots = \|N^{(k)}(u_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \\ \|N''(v_0(x, t))\| = \|N^{(3)}(v_0(x, t))\| = \|N^{(4)}(v_0(x, t))\| = \dots = \|N^{(k)}(v_0(x, t))\| = \|0\| = 0 < \frac{1}{e}, \end{cases} \tag{86}$$

for all  $k \geq 0$  by the principle of mathematical induction.

As the conditions of Theorem 13 are satisfied, the series solution obtained by the new iterative method is convergent on the domain of interest.

### 6. Conclusion

The double Sumudu transform method has been known to be a powerful tool for solving many fractional partial differential equations, integral equations, and so many other equations. In this paper, we have presented a new method, that is, the amalgamation of double Sumudu transform and a numerical method which is known as an iterative method to obtain the exact solution of one dimensional coupled nonlinear sine-Gordon equation. From the solutions of illustrative Examples 1 and 2, we can see that the decomposition

$$\begin{cases} u(x, t) = \sum_{i=0}^{\infty} u_i(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots + u_n(x, t) + \dots, \\ v(x, t) = \sum_{i=0}^{\infty} v_i(x, t) = v_0(x, t) + v_1(x, t) + v_2(x, t) + \dots + v_n(x, t) + \dots, \end{cases} \tag{87}$$

consists only of one term, i.e.,

$$\begin{cases} u(x, t) = u_0(x, t), \\ v(x, t) = sv_0(x, t). \end{cases} \tag{88}$$

From this study, we concluded that double Sumudu transform linked with an iterative method finds quite practical analytical results with less computational work.

### Data Availability

No data were used to support the study.

### Conflicts of Interest

The author declares that there is no conflict of interest regarding the preparation of this research paper.

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