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Research Article

Generalization Contractive Mappings on Rectangular *b*-Metric Space

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In this paper, we introduce new coincidence fixed point theorems for generalized (ϕ, ψ)-contractive mappings fulfilling kind of an admissibility provision in a Hausdorff b-rectangular metric space with the support of C-functions. We applied our results to establish the existence of a solution for some integralitions. Finally, an example is presented to clarify our theorem.

1. Introduction

One of the main results in the development area of fixed point theorems is the Banach contraction principle [1]. It has been evolutionized and generalized in several directions. From those directions, we faced some new kinds of metric spaces in literature as the one established by Branciari [2] and gave the principle of a rectangular metric space in which the replacement of the triangle inequality with a weaker hypothesis called quadrilateral inequality and an analogue of the Banach contraction principle is shown. Then next, fixed point results in these spaces were studied by many authors. For more informations on fixed point theorems, in rectangular metric space, see [3–10]. In fact, a contraction principle in rectangulat *b*-metric space and its properties appeared by George [11]. Many definitions of various mathematical concepts and terms in rectangular *b*-metric space can be found in [12–19]. Separation of the Hausdorff space from rectangular *b*-metric space is not useful for our theorem, as Hausdorff space plays an important role in Theorem 16 and its corollaries. On another hand, Samet in [20] introduced the principle of (ϕ, ψ)-contractive mapping. Newly, two separate evaluations of α-admissible mapping were introduced in which the researcher Ansari in [21] used the notion of C-functions, whereas Budhia et al. in [22] used a rectangular metric space. By ideas from [21, 22], we prove several coincide fixed point results in rectangular b-metric space. That should be considered as development of [23], which are applied to find the existence and uniqueness of a solution for many problems in different mathematical branches. Moreover, one of the most attractive research subjects in fixed point theorem is the investigation of the existence and uniqueness of coincidence points of various operators in the setting of metric spaces (see [24–29]).

2. Mathematical Preliminaries

We recall some basic notions and needful results on the work in the literature.

Definition 1 (see [2]). Suppose X is a nonempty set. A function $d : X \times X \longrightarrow [0,\infty)$ is a rectangular metric (RM) on X if, for all $s_1, s_2 \in X$ and all distinct points $r_1, r_2 \in X$ such that $r_1, r_2 \notin \{s_1, s_2\}$, the following terms hold

(i) $d(s_1, s_2) = 0$, if and only if $s_1 = s_2$

(ii) $d(s_1, s_2) = d(s_2, s_1)$

(iii) $d(s_1, r_1) \leq d(s_1, r_2) + d(r_2, s_2) + d(s_2, r_1)$ (rectangular inequality)

Then, (X, d) is called a rectangular-metric space.

Definition 2 (see [11, 30]). Let X be a nonempty set and the mapping $d : X \times X \longrightarrow [0,\infty)$ satisfies

(i) $d(s_1, s_2) = 0$, if and only if $s_1 = s_2$ for all $s_1, s_2 \in X$

(ii) $d(s_1, s_2) = d(s_2, s_1)$ for all $s_1, s_2 \in X$

(iii) there exists a real number $b \geq 1$ such that

$$d(s_1, r_1) \leq b[d(s_1, r_2) + d(r_2, s_2) + d(s_2, r_1)], \text{(b-rectangularinequality)} \tag{1}$$

For all $s_1, s_2 \in X$ and all $(r_1 \neq r_2) \notin \{s_1, s_2\}$.

Then, d is called a bRM on X and (X, d) is called a rectangular b-metric space with coefficient b.

Remark 3 (see [31]). The type of rectangular b-metric space is greater than the type of metric space, where a b-metric space is a metric space when $b = 1$.

Example 1 (see [31]). Let $X = \mathbb{R}$ and $d : X \times X \longrightarrow \mathbb{R}^+$, such that $d(s_1, s_2) = |s_1 - s_2|^3$. Then, (X, d) is a bMS, with $b = 3$.

Remark 4 (see [11]). Every metric space is a rectangular metric space, and every rectangular metric space is a rectangular b-metric space, with $b = 1$. However, the opposite of implying above is not valid.

Example 2 (see [11]). Suppose, $X = \mathbb{N}$ and $d : X \times X \longrightarrow X$ as

$$\mathrm{d}(\mathrm{s}_1, \mathrm{s}_2) = \begin{cases} 0, & \text{if} \quad \mathrm{s}_1 = \mathrm{s}_2, \\ 4\eta, & \text{if} \quad \mathrm{s}_1, \mathrm{s}_2 \in \{1, 2\}, \mathrm{s}_1 \neq \mathrm{s}_2, \\ \eta, & \text{if} \quad \mathrm{s}_1 \mathrm{or} \mathrm{s}_2 \in \{1, 2\}, \mathrm{s}_1 \neq \mathrm{s}_2. \end{cases} \tag{2}$$

Consider $\eta \in (0,\infty)$. Hence, (X, d) is a rectangular b -metric space, with $b = 4/3$, but (X, d) is not rectangular metric space, take

$$d(1, 2) = 4\eta > 3\eta = d(1, 3) + d(3, 4) + d(4, 2). \tag{3}$$

In 1914, German mathematician Felix Hausdorff [32] defined a new distance idea called Hausdorff metric, denoted by $H(X, Y)$, as

$$H(X, Y) = \max \left\{\bar{\delta}(X, Y), \bar{\delta}(Y, X)\right\}, \tag{4}$$

where

$$\bar{\delta}(X, Y) = \max \{\delta(s, Y): s \in X\}. \tag{5}$$

The Hausdorff distance is the largest one of all the distances measured from one set to another.

Definition 5 (see [33]). A Hausdorff space H is a topological space where for any two distinct points $s_1, s_2 \in H$, there exist neighbourhoods of each are disjoint from each other.

Definition 6 (see [11]). Let (X, d) be a rectangular b-metric space, $\{n_i\}$ be a sequence in X, and $n \in X$. Then,

(i) a sequence $\{s_i\}$ is said to be convergent in (X, d) and converges to s, if for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $d(s_i, s) < \varepsilon$, for all $i > i_0$ and this truth is acted via $\lim_{i \longrightarrow \infty} \{s_i\} = s$ or $\{s_i\} \longrightarrow s$ as $i \longrightarrow \infty$

(ii) a sequence $\{s_i\}$ is said to be Cauchy sequence in (X, d) if for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $d(s_i, s_{i+\eta}) < \varepsilon$, for all $i > i_0, \eta > 0$ or equivalently, if $\lim_{i \longrightarrow \infty} d(s_i, s_{i+\eta}) = 0$, for all $\eta > 0$

(iii) (X, d) is said to be a complete rectangular b-metric space if every Cauchy sequence in X converges to some $s \in X$

The next lemmas are helpful in providing principle outcomes.

Lemma 7 (see [34]). *Let (X, d) be a rectangular b-metric space with $b \geq 1$ and let $\{s_i\}$ be a Cauchy sequence in X such that $s_i \neq s_j$ when it was $i \neq j$. Then, $\{s_i\}$ be able convergence at most one point.*

Lemma 8 (see [34]). *Let (X, d) be a rectangular b-metric space with $b \geq 1$.*

(i) Suppose that the sequences $\{s_i\}, \{r_i\} \in X$ where $s_i \longrightarrow s, r_i \longrightarrow r$ as $i \longrightarrow \infty$, such that $s_i \neq s, r_i \neq r$, for all $i \in \mathbb{N}$. Thus, we have

$$\frac{1}{b} d(s, r) \leq \lim_{i \longrightarrow \infty} \inf \; d(s_i, r_i) \leq \lim_{i \longrightarrow \infty} \sup \; d(s_i, r_i) \leq b d(s, r) \tag{6}$$

(ii) Suppose $s, m \in X$ and $\{s_i\}$ are a Cauchy sequence in X where $s_i \neq s_j$, for all $i, j \in \mathbb{N}, i \neq j$. where $s_i \longrightarrow s, s_j \longrightarrow m$ as $i, j \longrightarrow \infty, s \neq m$. Thus, we have

$$\frac{1}{b}d(s,m) \le \lim_{i\longrightarrow\infty} \inf\ d(s_i,m) \le \lim_{i\longrightarrow\infty} \sup\ d(s_i,m) \le bd(s,m),$$

$$\frac{1}{b}d(m,s) \le \lim_{i\longrightarrow\infty} \inf\ d(s_j,s) \le \lim_{i\longrightarrow\infty} \sup\ d(s_j,s) \le bd(m,s) \tag{7}$$

For all $s_1, s_2 \in [0,\infty)$.

Definition 9 (see [35, 36]). Let $F : X \longrightarrow X$ be a self-mapping on a metric space (X, d) and let $\alpha : X \times X \longrightarrow [0,\infty)$ be a function. F is called a α-admissible function if

$$\alpha(Fs_1, Fs_2) \ge 1 \text{ whenever } \alpha(s_1, s_2) \ge 1, \forall s_1, s_2 \in X. \tag{8}$$

Definition 10 (see [35, 36]). Let F be a self-mapping on a metric space (X, d). A map F is called a (ϕ, ψ)-contractive mapping if there exist two functions $\phi : X \times X \longrightarrow [0,\infty)$ and $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ such that

$$\phi(s_1, s_2)d(Fs_1, Fs_2) \le \psi(s_1, s_2) \forall s_1, s_2 \in X, \tag{9}$$

where ψ is a nondecreasing functions such that $\sum_{i=1}^{\infty}\psi^i(t) < +\infty, \forall t > 0$, where ψ^i is the i^{th} iteration of ψ.

For more informations for α-admissible and (ϕ, ψ)-contractive mappings, see [35–37].

Definition 11 (see [38]). Let T be a self-mapping on a metric space (X, d) and let $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ are two mappings. A function T is called α-admissible with regard to η if

$$\alpha(Ts_1, Ts_2) \ge \eta(Ts_1, Ts_2) \text{ where } \alpha(s_1, s_2) \ge \eta(s_1, s_2), \forall s_1, s_2 \in X. \tag{10}$$

Observe that, if $\alpha(s_1, s_2) = 1, \forall s_1, s_2 \in X$. Thus, this definition led to Definition 9. Likewise, if we pick $\alpha(s_1, s_2) = 1$, then we state that T is a η-subadmissible functions.

Definition 12 ([21]). A C-function $\vartheta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is a continuous functions such that

(i) $\vartheta(s_1, s_2) \le s_1$

(ii) $\vartheta(s_1, s_2) = s_1 \Rightarrow s_1 = 0$ or $s_2 = 0$

Definition 13 (see [39]). A nondecreasing continuous map $\phi : [0,\infty) \longrightarrow [0,\infty)$ is called an altering distance mapping whenever $\phi(\beta) = 0 \Leftrightarrow \beta = 0$.

Definition 14 (see [29]). Let (X, d) be a rectangular metric space and let α, η in Definition 10. X is said to be α-orderly with respect to η if for $\{s_i\} \in X$ with $\alpha(s_i, s_{i+1}) \ge \eta(s_i, s_{i+1})$, for all $i \ge N$ where $s_i \longrightarrow s$ as $i \longrightarrow \infty$; therefore, $\alpha(s_i, s) \ge \eta(s_i, s)$, for all $i \ge N$.

Remark 15. In altering the type of distance mapping, we denoted it by symbol Φ.

In the next section, we present a new coincide FPT by the generalization (ϕ, ψ)-contractive mappings on rectangular b-metric space fulfilling α-admissibility by the concept of C-functions.

A lot of authors used altering functions to prove the existence and uniqueness of the fixed point; see [3, 22, 34]. We will use generalizing some of them to prove our results.

3. Main Results

We will introduce new results of coincide fixed point in RMS. Let us start with the following.

Theorem 16. *Let (X, d) be a Hausdorff rectangular b-metric space, with $(b \ge 1)$. Assume that $T, F : X \longrightarrow X$ be an α-admissible function with respect to η such that $TX \subset FX$. Assume that (FX, d) is a complete rectangular metric space. Let $\Theta \in$ C-functions and $\phi, \psi \in \Phi$ such that, for all $s_1, s_2 \in X$,*

$$\alpha(Fs_1, Fs_2) \ge \eta(Fs_1, Fs_2) \Rightarrow \phi\left(\frac{1}{b}d(Ts_1, Ts_2)\right) \le \Theta[\phi(\vartheta(Fs_1, Fs_2)), \psi(\vartheta(Fs_1, Fs_2))], \tag{11}$$

where

$$\vartheta(Fs_1, Fs_2) = \sup\left\{\frac{1}{b}d(Fs_1, Fs_2), \frac{1}{b}d(Fs_1, Ts_1), \frac{1}{b}d(Fs_2, Ts_2), \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Ts_1, Ts_2)}\right\}. \tag{12}$$

Let that

(a) $\alpha(Fs_1, Fs_2) \ge \eta(Fs_1, Fs_2)$ and $\alpha(Fs_2, Fs_3) \ge \eta(Fs_2, Fs_3) \Rightarrow \alpha(Fs_1, Fs_3) \ge \eta(Fs_1, Fs_3)$, for all $s_1, s_2, s_3 \in X$

(b) either T is continuous or X is α-orderly with respect to η

Then, there exist $w \in X$ such that $T^k w = F^k w$, for some $k \in \mathbb{N}$; i.e., w is a periodic point. If all periodic point w satisfies

$$\alpha(Fw, Tw) \ge \eta(Fw, Tw), \tag{13}$$

we can decide that T and F have a fixed point. The fixed point is unique if for all $z_1, z_2 \in \Theta(T) = \{w \in X : Fw = Tw = w\}$, such that

$$\alpha(Fz_1, Fz_2) \ge \eta(Fz_1, Fz_2). \tag{14}$$

Proof. We shall prove the existence of coincide point of T and F. Assume that $s_0 \in X$ be an arbitrary point such that

$$\alpha(Fs_0, Ts_0) \geq \eta(Fs_0, Ts_0). \tag{15}$$

□

Consider the iteration

$$Ts_i = Fs_{i+1} = v_i, \tag{16}$$

such that $v_i \neq v_{i+1}$, for all $i \in \mathbb{N}$. Since T satisfied Definition 11 and by (16) using (15), we have

$$\alpha(v_i, v_{i+1}) \geq \eta(v_i, v_{i+1}), \forall i \in \mathbb{N}. \tag{17}$$

In the beginning, we will show that $d(v_i, v_{i+1}) \longrightarrow 0$, as $i \longrightarrow \infty$; i.e., $d(v_i, v_{i+1})$ is nonincreasing.

By (11), we get

$$\begin{aligned}\phi\left(\frac{1}{b}d(v_i, v_{i+1})\right) &= \phi(d(Ts_i, Ts_{i+1}))\\ &\leq \Theta[\phi(\vartheta(Fs_i, Fs_{i+1})), \psi(\vartheta(Fs_i, Fs_{i+1}))],\end{aligned} \tag{18}$$

where

$$\begin{aligned}\vartheta(Fs_i, Fs_{i+1}) &= \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_i, Fs_{i+1}), \frac{1}{b}d(Fs_i, Ts_i), \frac{1}{b}d(Fs_{i+1}, Ts_{i+1}),\\ \frac{d(Fs_i, Ts_i)d(Fs_{i+1}, Ts_{i+1})}{b + bd(Fs_i, Fs_{i+1})}, \frac{d(Fs_i, Ts_i)d(Fs_{i+1}, Ts_{i+1})}{b + bd(Ts_i, Ts_{i+1})}\end{array}\right\}\\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_i, Fs_{i+1}), \frac{1}{b}d(Fs_i, Fs_{i+1}), \frac{1}{b}d(Fs_{i+1}, Fs_{i+2}),\\ \frac{d(Fs_i, Fs_{i+1})d(Fs_{i+1}, Fs_{i+2})}{b + bd(Fs_i, Fs_{i+1})}, \frac{d(Fs_i, Fs_{i+1})d(Fs_{i+1}, Fs_{i+2})}{b + bd(Fs_{i+1}, Fs_{i+2})}\end{array}\right\}\\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d(v_{i-1}, v_i), \frac{1}{b}d(v_{i-1}, v_i), \frac{1}{b}d(v_i, v_{i+1}),\\ \frac{d(v_{i-1}, v_i)d(v_i, v_{i+1})}{b + bd(v_{i-1}, v_i)}, \frac{d(v_{i-1}, v_i)d(v_i, v_{i+1})}{b + bd(v_i, v_{i+1})}\end{array}\right\}\\ &= \sup\left\{\frac{1}{b}d(v_{i-1}, v_i), \frac{1}{b}d(v_i, v_{i+1})\right\}.\end{aligned} \tag{19}$$

We have two cases.

Case 1. If $\vartheta(Fs_i, Fs_{i+1}) = (1/b)d(v_{i-1}, v_i)$ for some $i \in \mathbb{N}$. Thus, inequality (18) will become

$$\begin{aligned}\phi\left(\frac{1}{b}d(v_i, v_{i+1})\right) &\leq \Theta\left[\phi\left(\frac{1}{b}d(v_{i-1}, v_i)\right), \psi\left(\frac{1}{b}d(v_{i-1}, v_i)\right)\right],\\ &\leq \phi\left(\frac{1}{b}d(v_{i-1}, v_i)\right)\end{aligned} \tag{20}$$

Since $\phi \in \Phi$ and ϕ is nondecreasing function,

$$\frac{1}{b}d(v_i, v_{i+1}) \leq \frac{1}{b}d(v_{i-1}, v_i) \tag{21}$$

was obtained; $\{d(v_i, v_{i+1})\}$ is a nonincreasing sequence which $v_1 \geq 0$ and satisfies

$$\begin{aligned}&\lim_{i\longrightarrow\infty} d(v_i, v_{i+1}) = v_1,\\ &\lim_{i\longrightarrow\infty} \vartheta(v_{i-1}, v_i) = v_1.\end{aligned} \tag{22}$$

Also, since ϕ, ψ are continuous functions,

$$\begin{aligned}\lim_{i\longrightarrow\infty} \phi(d(v_i, v_{i+1})) &\leq \lim_{i\longrightarrow\infty} \Theta[\phi(\vartheta(v_{i-1}, v_i)), \psi(\vartheta(v_{i-1}, v_i))]\\ &= \Theta\left[\lim_{i\longrightarrow\infty} \phi(\vartheta(v_{i-1}, v_i)), \lim_{i\longrightarrow\infty} \psi(\vartheta(v_{i-1}, v_i))\right].\end{aligned} \tag{23}$$

Thus,

$$\phi(v_1) \leq \Theta[\phi(v_1), \psi(v_1)] \leq \phi(v_1). \tag{24}$$

By Definition 13, we get $v_1 = 0$; hence,

$$\lim_{i\longrightarrow\infty} d(v_i, v_{i+1}) = 0. \tag{25}$$

Case 2. If $\vartheta(Fs_i, Fs_{i+1}) = (1/b)d(v_i, v_{i+1})$ for some $i \in \mathbb{N}$. Thus, inequality (18) becomes

$$\begin{aligned}\phi\left(\frac{1}{b}d(v_i, v_{i+1})\right) &\leq \Theta\left[\phi\left(\frac{1}{b}d(v_{i+1}, v_i)\right), \psi\left(\frac{1}{b}d(v_{i+1}, v_i)\right)\right]\\ &\leq \phi\left(\frac{1}{b}d(v_{i+1}, v_i)\right).\end{aligned} \tag{26}$$

By Definition 13 we get either $\phi((1/b)d(v_{i+1}, v_i)) = 0$ or $\psi((1/b)d(v_{i+1}, v_i)) = 0$ and then $(1/b)d(v_{i+1}, v_i) = 0$, but this is a contradiction with $v_{i+1} \neq v_i$.

In the next step, we will show that $(1/b)d(v_i, v_{i+2}) \longrightarrow 0$, as $i \longrightarrow \infty$.

By (11), we have

$$\begin{aligned}\phi\left(\frac{1}{b}d(v_i, v_{i+2})\right) &= \phi\left(\frac{1}{b}d(Ts_i, Ts_{i+2})\right)\\ &\leq \Theta[\phi(\vartheta(Fs_i, Fs_{i+2})), \psi(\vartheta(Fs_i, Fs_{i+2}))]\\ &\leq \phi(\vartheta(Fs_i, Fs_{i+2})).\end{aligned} \tag{27}$$

Well, it could be

$$\phi\left(\frac{1}{b}d(v_i, v_{i+2})\right) \leq \phi(\vartheta(Fs_i, Fs_{i+2})) \tag{28}$$

since ϕ is altering distance. We find that

$$\begin{aligned}\frac{1}{b}d(v_i, v_{i+2}) &\le \vartheta(Fs_i, Fs_{i+2})\\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_i, Fs_{i+2}), \frac{1}{b}d(Fs_i, Ts_i), \frac{1}{b}d(Fs_{i+2}, Ts_{i+2}),\\ \frac{d(Fs_i, Ts_i)d(Fs_{i+2}, Ts_{i+2})}{b+bd(Fs_i, Fs_{i+2})}, \frac{d(Fs_i, Ts_i)d(Fs_{i+2}, Ts_{i+2})}{b+bd(Ts_i, Ts_{i+2})}\end{array}\right\}\\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d(v_{i-1}, v_{i+1}), \frac{1}{b}d(v_{i-1}, v_i), \frac{1}{b}d(v_{i+1}, v_{i+2}),\\ \frac{d(v_{i-1}, v_i)d(v_{i+1}, v_{i+2})}{b+bd(v_{i-1}, v_{i+1})}, \frac{d(v_{i-1}, v_i)d(v_{i+1}, v_{i+2})}{b+bd(v_i, v_{i+2})}\end{array}\right\}\\ &\le \sup\left\{\begin{array}{c}\frac{1}{b}d(v_{i-1}, v_{i+1}), \frac{1}{b}d(v_{i-1}, v_i), \frac{1}{b}d(v_{i+1}, v_{i+2}),\\ d(v_{i-1}, v_i)d(v_{i+1}, v_{i+2}), d(v_{i-1}, v_i)d(v_{i+1}, v_{i+2})\end{array}\right\}\\ &\le \sup\left\{\frac{1}{b}[d(v_{i-1}, v_{i+1}) + d(v_{i-1}, v_i) + d(v_{i+1}, v_{i+2})], d(v_{i-1}, v_i)d(v_{i+1}, v_{i+2})\right\}\\ &= \sup\{d(v_i, v_{i+2}), d(v_{i-1}, v_i)d(v_{i+1}, v_{i+2})\}.\end{aligned} \tag{29}$$

We obtain $\vartheta(Fs_i, Fs_{i+2}) = d(v_i, v_{i+2})$, when $i \longrightarrow \infty$. Therefore, by (27)

$$\begin{aligned}\phi\left(\lim_{i\longrightarrow\infty} d(v_i, v_{i+2})\right) &\le \Theta\left[\phi\left(\lim_{i\longrightarrow\infty} \vartheta(Fs_i, Fs_{i+2})\right), \psi\left(\lim_{i\longrightarrow\infty} \vartheta(Fs_i, Fs_{i+2})\right)\right]\\ &\le \phi\left(\lim_{i\longrightarrow\infty} \vartheta(Fs_i, Fs_{i+2}) = \lim_{i\longrightarrow\infty} \phi(d(v_i, v_{i+2}))\right).\end{aligned} \tag{30}$$

Consequently, by Definition 13, we get

$$\lim_{i\longrightarrow\infty} d(v_i, v_{i+2}) \longrightarrow 0, \text{ as } i \longrightarrow \infty. \tag{31}$$

The next lemma is useful for the rest and its proof is classical. We omit it.

Lemma 17. *Let (X, d) be a rectangular b-metric space with $b \ge 1$ and let $\{v_i\}$ be a sequence in X such that*

$$\lim_{i\longrightarrow\infty} (v_i, v_{i+1}) = \lim_{i\longrightarrow\infty} (v_i, v_{i+2}) = 0, \tag{32}$$

where $v_i \neq m_j$, for all $i \neq j$. If $\{v_i\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two subsequences $v_{i(k)}, v_{j(k)} \subset \{v_i\}$, where $i(k) > j(k) > k, k \in \mathbb{N}$. Also,

$$\begin{aligned}d\left(v_{i(k)}, v_{j(k)}\right) &\ge b\varepsilon,\\ d\left(v_{i(k)}, v_{j(k)-1}\right) &\le b\varepsilon,\end{aligned} \tag{33}$$

such that for the next sequences

$$d\left(v_{i(k)}, v_{j(k)}\right), d\left(v_{i(k)-1}, v_{j(k)}\right), d\left(v_{i(k)}, v_{j(k)-1}\right), d\left(v_{i(k)+1}, v_{j(k)+1}\right), \tag{34}$$

it satisfies

$$\begin{aligned}\varepsilon \le \lim_{i\longrightarrow\infty} \inf d\left(v_{i(k)}, v_{j(k)}\right) &\le \lim_{i\longrightarrow\infty} \sup d\left(v_{i(k)}, v_{j(k)}\right) \le b\varepsilon,\\ \varepsilon \le \lim_{i\longrightarrow\infty} \inf d\left(v_{i(k)-1}, v_{j(k)}\right) &\le \lim_{i\longrightarrow\infty} \sup d\left(v_{i(k)-1}, v_{j(k)}\right) \le b^3\varepsilon,\\ \varepsilon \le \lim_{i\longrightarrow\infty} \inf d\left(v_{i(k)}, v_{j(k)}\right) &\le \lim_{i\longrightarrow\infty} \sup d\left(v_{i(k)}, v_{j(k)}\right) \le b^3\varepsilon,\\ \varepsilon \le \lim_{i\longrightarrow\infty} \inf d\left(v_{i(k)-1}, v_{j(k)-1}\right) &\le \lim_{i\longrightarrow\infty} \sup d\left(v_{i(k)-1}, v_{j(k)-1}\right) \le b^5\varepsilon.\end{aligned} \tag{35}$$

Now, replace s_1 by x_{i_k} and s_2 by x_{j_k} in (11). Then,

$$\begin{aligned}\phi\left(\frac{1}{b}d\left(v_{j_k}, v_{i_k}\right)\right) &= \phi\left(\frac{1}{b}d\left(Ts_{j_k}, Ts_{i_k}\right)\right)\\ &\le \Theta\left[\phi\left(\vartheta\left(Fs_{j_k}, Fs_{i_k}\right)\right), \psi\left(\vartheta\left(Fs_{j_k}, Fs_{i_k}\right)\right)\right],\end{aligned} \tag{36}$$

where

$$\begin{aligned}\vartheta\left(Fs_{j_k}, Fs_{i_k}\right) &= \sup\left\{\begin{array}{c}\frac{1}{b}d\left(Fs_{j_k}, Fs_{i_k}\right), \frac{1}{b}d\left(Fs_{j_k}, Ts_{j_k}\right), \frac{1}{b}d\left(Fs_{i_k}, Ts_{i_k}\right),\\ \frac{d\left(Fs_{j_k}, Ts_{j_k}\right)d\left(Fs_{i_k}, Ts_{i_k}\right)}{b+bd\left(Fs_{j_k}, Fs_{i_k}\right)}, \frac{d\left(Fs_{j_k}, Ts_{j_k}\right)d\left(Fs_{i_k}, Ts_{i_k}\right)}{b+bd\left(Ts_{j_k}, Ts_{i_k}\right)}\end{array}\right\}\\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d\left(v_{j_{k-1}}, v_{i_{k-1}}\right), \frac{1}{b}d\left(v_{j_{k-1}}, v_{j_k}\right), \frac{1}{b}d\left(v_{i_{k-1}}, v_{i_k}\right),\\ \frac{d\left(v_{j_{k-1}}, v_{j_k}\right)d\left(v_{i_{k-1}}, v_{i_k}\right)}{b+bd\left(v_{j_{k-1}}, v_{i_{k-1}}\right)}, \frac{d\left(v_{j_{k-1}}, v_{j_k}\right)d\left(v_{i_{k-1}}, v_{i_k}\right)}{b+bd\left(v_{j_k}, v_{j_k}\right)}\end{array}\right\}\\ &= \sup\left\{b^4\varepsilon, 0, 0, 0, 0\right\}.\end{aligned} \tag{37}$$

Taking $k \longrightarrow \infty$ in (36), we obtain

$$0 \le \phi\left(b^4\varepsilon\right) \le \Theta\left[\phi\left(b^4\varepsilon\right), \psi\left(b^4\varepsilon\right)\right] \le \phi\left(b^4\varepsilon\right). \tag{38}$$

This implies that $\phi(b^4\varepsilon) = 0$ or $\psi(b^4\varepsilon) = 0$; thus, $\varepsilon = 0$, but this is a contradiction with the fact $\varepsilon > 0$. Hence, $\{v_i\}$ is a Cauchy sequence in a rectangular b-metric space. Since (FX, d) is complete in a rectangular b-metric space, then there exist $s \in FX$ such that $v_i \longrightarrow s$ as $i \longrightarrow \infty$.

For the case that T is continuous and by relation (16) we have

$$\lim_{i\longrightarrow\infty} Tv_i = \lim_{i\longrightarrow\infty} Fv_{i+1} \longrightarrow Ts. \tag{39}$$

X is Hausdorff; then $Ts = Fs$. Hence, T and F have a common fixed point. On the other case, assume that X is α-orderly with respect to η, then from $\alpha(Fv_i, F\omega) \ge \eta(Fv_i, Fs)$, for all $i \in \mathbb{N}$ we have

$$\phi\left(\frac{1}{b}d(Tv_i, Ts)\right) \le \Theta[\phi(\vartheta(Fv_i, Fs)), \psi(\vartheta(Fv_i, Fs))], \tag{40}$$

where

$$\begin{aligned}\vartheta(Fv_i, Fs) &= \sup\left\{\frac{1}{b}d(Fv_i, Fs), \frac{1}{b}d(Fv_i, Tv_i), \frac{1}{b}d(Fs, Ts), \frac{d(Fv_i, Tv_i)d(Fs, Ts)}{b + bd(Fv_i, Fs)}, \frac{d(Fv_i, Tv_i)d(Fs, Ts)}{b + bd(Tv_i, Ts)}\right\} \\ &= \sup\left\{\frac{1}{b}d(Fs, Fs), \frac{1}{b}d(Fs, Ts), \frac{1}{b}d(Fs, Ts)\right\}.\end{aligned} \tag{41}$$

Since $\{v_i\} \longrightarrow s$ where $i \longrightarrow \infty$.

$$\lim_{i\longrightarrow\infty} \vartheta(Fv_i, Fs) = d(Fs, Ts). \tag{42}$$

Then, $d(Fs, Ts) = 0$ and $Fs = Ts$. Hence, T and F have a periodic coincide fixed point.

Now, we will prove that T and F have a coincide fixed point. Assume that $v_1 \in X$ is the coincide of T and F such that $v_1 = T^k s = F^k s$. When $k = 1$, then v_1, is a coincide of T and F. We shall show $v_2 = T^{k-1}s = F^{k-1}s$ is the coincide of T and F in case $k > 1$. Assume that $T^{k-1}s \neq T^k s$ and $F^{k-1}s \neq F^k s$, for all $k > 1$, such that $\alpha(Fs, Ts) \geq \eta(Fs, Ts)$ for a periodic point s. Therefore, from (11) and (12),

$$\begin{aligned}\phi\left(\frac{1}{b}d\left(T^{k-1}s, T^k s\right)\right) &\leq \Theta\left[\phi\left(\vartheta\left(FT^{k-2}s, FT^{k-1}s\right)\right), \psi\left(\vartheta\left(FT^{k-2}s, FT^{k-1}s\right)\right)\right] \\ &\leq \Theta\left[\phi\left(\vartheta\left(FF^{k-2}s, FF^{k-1}s\right)\right), \psi\left(\vartheta\left(FF^{k-2}s, FF^{k-1}s\right)\right)\right] \\ &= \Theta\left[\phi\left(\vartheta\left(F^{k-1}s, F^k s\right)\right), \psi\left(\vartheta\left(F^{k-1}s, F^k s\right)\right)\right],\end{aligned} \tag{43}$$

where

$$\begin{aligned}\vartheta\left(F^{k-1}s, F^k s\right) &= \sup\left\{\begin{array}{c}\frac{1}{b}d\left(F^{k-1}s, F^k s\right), \frac{1}{b}d\left(F^{k-1}s, TF^{k-2}s\right), \frac{1}{b}d\left(F^k s, TF^{i-1}s\right), \\ \frac{d\left(F^{k-1}s, TF^{k-2}s\right)d\left(F^k s, TF^{k-1}s\right)}{b + bd\left(F^{k-1}s, ,F^k s\right)}, \frac{d\left(F^{k-1}s, TF^{k-2}s\right)d\left(F^i s, TF^{k-1}s\right)}{b + bd\left(TF^{k-1}s, TF^{k-2}s\right)}\end{array}\right\} \\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d\left(F^{k-1}s, F^k s\right), \frac{1}{b}d\left(F^{k-1}s, T^{k-1}s\right), \frac{1}{b}d\left(F^k s, T^i s\right), \\ \frac{d\left(F^{k-1}s, F^k s\right)d\left(F^k s, T^k s\right)}{b + bd\left(F^{k-1}s, ,F^k s\right)}, \frac{d\left(F^{k-1}s, F^k s\right)d\left(F^k s, T^k s\right)}{b + bd\left(T^k s, T^{k-1}s\right)}\end{array}\right\} = \frac{1}{b}d\left(F^{k-1}s, F^k s\right).\end{aligned} \tag{44}$$

Take $\vartheta(F^{k-1}s, F^k s) = (1/b)d(F^{k-1}s, F^k s)$. Then,

$$\begin{aligned}\phi\left(\frac{1}{b}d\left(T^{k-1}s, T^k s\right)\right) &\leq \Theta\left[\phi\left(\frac{1}{b}d\left(F^{k-1}s, F^k s\right)\right), \psi\left(\frac{1}{b}d\left(F^{k-1}s, F^k s\right)\right)\right] \\ &\leq \Theta\left[\phi\left(\frac{1}{b}d\left(F^{k-1}s, F^k s\right)\right)\right].\end{aligned} \tag{45}$$

Either $\phi((1/b)d(F^{i-1}s, F^i s)) = 0$ or $\psi((1/b)d(F^{i-1}s, F^i s)) = 0$, i.e., $d(F^{i-1}s, F^i s) = 0$; this leads to $F^{i-1}s = F^i s$ which is a contradiction. Hence, our assumption that $v_2 = T^{i-1}s$ is a coincide fixed point of T and F is not true. Accordingly, T and F have a coincide fixed point.

To make sure the uniqueness of the coincide fixed point, let us assume that $v_1, v_2 \in X$ such that $w_1 \neq w_2$ are two coincide fixed points of T and F. By the inequality $\alpha(Fv_1, Fv_2) \geq \eta(Fv_1, Fv_2)$ and (11) and (12), we get

$$\begin{aligned}\phi\left(\frac{1}{b}d(Fv_1, Fv_2)\right) &= \phi\left(\frac{1}{b}d(Tv_1, Tv_2)\right) \\ &\leq \Theta[\phi(\vartheta(Fv_1, Fv_2)), \psi(\vartheta(Fv_1, Fv_2))],\end{aligned} \tag{46}$$

where

$$\begin{aligned}\vartheta(Fv_1, Fv_2) &= \sup\left\{\begin{array}{c}d(Fv_1, Fv_2), d(Fv_1, Tv_1), d(Fv_2, Tv_2), \\ \frac{d(Fv_1, Tv_1)d(Fv_2, Tv_2)}{b + bd(Fv_1, Fv_2)}, \frac{d(Fv_1, Fv_2)d(Tv_1, Tv_2)}{b + bd(Tv_1, Tv_2)}\end{array}\right\} \\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d(v_1, v_2), \frac{1}{b}d(v_1, v_1), \frac{1}{b}d(v_2, v_2), \\ \frac{d(v_1, v_1)d(v_2, v_2)}{b + bd(v_1, v_2)}, \frac{d(v_1, v_2)d(v_1, v_2)}{b + bd(v_1, v_2)}\end{array}\right\}.\end{aligned} \tag{47}$$

Then, $\vartheta(Fv_1, Fv_2) = (1/b)d(v_1, v_2)$. Applying it in (46), we get

$$\begin{aligned}\phi\left(\frac{1}{b}d(Fv_1, Fv_2)\right) &= \phi\left(\frac{1}{b}d(Tv_1, Tv_2)\right)\\ &\le \Theta\left[\phi\left(\frac{1}{b}d(v_1, v_2)\right), \psi\left(\frac{1}{b}d(v_1, v_2)\right)\right]\\ &\le \phi\left(\frac{1}{b}d(v_1, v_2)\right).\end{aligned} \tag{48}$$

Thus, either $\phi((1/b)d(v_1, v_2)) = 0$ or $\psi((1/b)d(v_1, v_2)) = 0$, which implies that $d(v_1, v_2) = 0$. Hence, $v_1 = v_2$. This proves the uniqueness coincide fixed point of T and F on X.

Corollary 18. *Let (X, d) be a complete Hausdorff rectangular b-metric space, with $b \ge 1$. Let $T, F : X \times X$ be a self-mapping such that $TX \subset FX$ satisfies*

$$d(Ts_1, Ts_2) \le \eta \sup\left\{\frac{1}{b}d(Fs_1, Fs_2), \frac{1}{b}d(Fs_1, Ts_1), \frac{1}{b}d(Fs_2, Ts_2), \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Ts_1, Ts_2)}\right\} \tag{49}$$

For all $s_1, s_2 \in X$, such that $0 \le \eta < 1$. Then, T and F have a unique coincide fixed point in X.

Corollary 19. *Let (X, d) be a complete Hausdorff rectangular b-metric space, with $b \ge 1$. Let $T, F : X \times X$ be a self-mappings such that $TX \subset FX$ satisfies*

$$d(Ts_1, Ts_2) \le \eta\left\{\begin{array}{c}\frac{1}{b}[d(Fs_1, Fs_2) + d(Fs_1, Ts_1) + d(Fs_2, Ts_2)] + \\ \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Fs_1, Fs_2)} + \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Ts_1, Ts_2)}\end{array}\right\}. \tag{50}$$

For all $s_1, s_2 \in X$ such that $0 \le \eta < 1$. Then, T and F have a unique coincide fixed point in X.

Corollary 20. *Let (X, d) be a complete bRMS Hausdorff, with $b \ge 1$. Let $T, F : X \times X$ be the self-mappings such that $TX \subset FX$ satisfies*

$$d(Ts_1, Ts_2) \le \Theta[\vartheta(Fs_1, Fs_2), \psi(\vartheta(Fs_1, Fs_2))], \tag{51}$$

where

$$\vartheta(Fs_1, Fs_2) \le \eta\left\{\frac{1}{b}d(Fs_1, Fs_2), \frac{1}{b}d(Fs_1, Ts_1), \frac{1}{b}d(Fs_2, Ts_2), \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Ts_1, Ts_2)}\right\}. \tag{52}$$

For all $s_1, s_2 \in X$ such that $0 \le \eta < 1$. Then, T and F have a unique coincide fixed point in X.

Theorem 21. *Let (X, d) be a complete Haudorff rectangular b -metric space, with $b \ge 1$. Let $T, F : X \longrightarrow X$ be the self-mappings such that $TX \subset FX$ satisfies*

$$d(Ts_1, Ts_2) \le \Theta[\phi(\vartheta(Fs_1, Fs_2)), \psi(\vartheta(Fs_1, Fs_2))]. \tag{53}$$

For all $s_1, s_2 \in X$ and $\phi, \psi \in \Phi$, such that

$$\vartheta(Fs_1, Fs_2) \le \sup\left\{\frac{1}{b}d(Fs_1, Fs_2), \frac{1}{b}d(Fs_2, Ts_2)\frac{b + bd(Fs_1, Ts_1)}{b + bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b + bd(Ts_1, Ts_2)}\right\}. \tag{54}$$

Then, T and F have a unique coincide fixed point.

Proof. Assume that $s_0 \in X$ be an arbitrary point. Since $TX \subset FX$, consider the sequence $\{s_i\} \subset X$ such that $Fs_i = Ts_{i-1}$ for all $i \ge 1$. Let that $Fs_i \ne Fs_{i+1} = Ts_i$ for all $i \ge 0$. We have from (11)

$$\begin{aligned}\phi\left(\frac{1}{b}d(Fs_i, Fs_{i+1})\right) &= \phi\left(\frac{1}{b}d(Ts_{i-1}, Ts_i)\right)\\ &\le \Theta[\phi(\vartheta(Fs_{i-1}, Fs_i)), \psi(\vartheta(Fs_{i-1}, Fs_i))],\end{aligned} \tag{55}$$

where

$$\begin{aligned}\vartheta(Fs_{i-1}, Fs_i) &\leq \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_{i-1}, Fs_i), \frac{1}{b}d(Fs_i, Ts_i)\frac{b+bd(Fs_{i-1}, Ts_{i-1})}{b+bd(Fs_{i-1}, Fs_i)},\\ \frac{d(Fs_{i-1}, Ts_{i-1})d(Fs_i, Ts_i)}{b+bd(Fs_{i-1}, Fs_i)}, \frac{d(Fs_{i-1}, Ts_{i-1})d(Fs_i, Ts_i)}{b+bd(Ts_{i-1}, Ts_i)}\end{array}\right\}\\ &= \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_{i-1}, Fs_i), \frac{1}{b}d(Fs_i, Fs_{i+1})\frac{b+bd(Fs_{i-1}, Fs_i)}{b+bd(Fs_{i-1}, Fs_i)},\\ \frac{d(Fs_{i-1}, Fs_i)d(Fs_i, Fs_{i+1})}{b+bd(Fs_{i-1}, Fs_i)}, \frac{d(Fs_{i-1}, Fs_i)d(Fs_i, Fs_{i+1})}{b+bd(Fs_i, Fs_{i+1})}\end{array}\right\}\\ &= \sup\left\{\frac{1}{b}d(Fs_{i-1}, Fs_i), \frac{1}{b}d(Fs_i, Fs_{i+1})\right\}.\end{aligned} \tag{56}$$

The remainder of the proof is identical as the proof of Theorem 16. □

4. Applications

Definition 22. Let Γ be the class of functions $\tau : [0,\infty) \longrightarrow [0,\infty)$ such the following are satisfying

(i) τ is Lebesgue integral function for all compact subset of $[0, \infty)$

(ii) $\int_0^\varepsilon \tau(x)dx > 0$ for each $\varepsilon > 0$

Theorem 23. *Let (X, d) be a complete Hausdorff rectangular b-metric space, with $b \geq 1$. Let $T, F : X \longrightarrow X$ be a self-mappings. Assume that (FX, d) is a complete rectangular b -metric space and that the next condition holds*

$$\int_0^{(1/b)d(Ts_1,Ts_2)} \tau(s)d(s) \leq \int_0^{\vartheta(Fs_1,Fs_2)} \tau(s)d(s) - \int_0^{\vartheta(Fs_1,Fs_2)} \chi(s)d(s). \tag{57}$$

$\forall s_1, s_2 \in X$ and $\tau, \chi \in \Gamma$, such that T and F satisfy inequality (11), where

$$\vartheta(Fs_1, Fs_2) \leq \sup\left\{\frac{1}{b}d(Fs_1, Fs_2), \frac{1}{b}d(Fs_1, Ts_1), \frac{1}{b}d(Fs_2, Ts_2), \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b+bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b+bd(Ts_1, Ts_2)}\right\}. \tag{58}$$

Then, T and F have a unique coincide fixed point.

Proof. Let that $\phi(s) = \int_0^s \tau(u)du$ and $\psi(s) = \int_0^s \chi(u)du$. Then, ϕ and $\psi \in \Phi$. Hence, by Theorem 16, T and F have a unique coincide fixed point. □

Theorem 24. *Let (X, d) be a complete bRMS Hausdorff with $b \geq 1$. Let $T, F : X \longrightarrow X$ be a self-mappings, such that*

$$\int_0^{(1/b)d(Ts_1,Ts_2)} \tau(r)d(r) \leq \eta\int_0^{\vartheta(Fs_1,Fs_2)} \tau(r)d(r). \tag{59}$$

For all $s_1, s_2 \in X$ and $\tau \in \Gamma$ and $0 \leq \eta < 1$, such that T and F satisfy inequality (11), where

$$\vartheta(Fs_1, Fs_2) \leq \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_1, Fs_2), \frac{1}{b}d(Fs_1, Ts_1), \frac{b+bd(Fs_1, Ts_1)}{b+bd(Fs_1, Fs_2)}d(Fs_2, Ts_2),\\ \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b+bd(Fs_1, Fs_2)}, \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b+bd(Ts_1, Ts_2)}\end{array}\right\}. \tag{60}$$

Then, T and F have a unique coincide fixed point.

Proof. Suppose $g(s) = \tau(s) - \eta\tau(s)$. Hence, by Theorem 23, T and F have a unique fixed point. □

Theorem 25. *Let (X, d) be a complete Hausdorff rectangular b-metric space, with $b \geq 1$. Assume that $T, F : X \longrightarrow X$ be a self-mappings. Let*

$$\int_0^{(1/b)d(Ts_1,Ts_2)} \tau(s)d(s) \leq \int_0^{\vartheta(Fs_1,Fs_2)} \tau(s)d(s) - \int_0^{\vartheta(Fs_1,Fs_2)} \chi(s)d(s). \tag{61}$$

For all $s_1, s_2 \in X$ and $\tau, \chi \in \Gamma$, such that T and F satisfy inequality (11), where

$$\vartheta(Fs_1, Fs_2) \leq \sup\left\{\begin{array}{c}\frac{1}{b}d(Fs_1, Fs_2) + \frac{1}{b}d(Fs_1, Ts_1) + \frac{1}{b}d(Fs_2, Ts_2) +\\ \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b+bd(Fs_1, Fs_2)} + \frac{d(Fs_1, Ts_1)d(Fs_2, Ts_2)}{b+bd(Ts_1, Ts_2)}\end{array}\right\}. \tag{62}$$

Then T and F have a unique coincide fixed point.

Proof. Suppose $\phi(s) = \int_0^s \zeta(u)du$ and $\psi(s) = \int_0^s f(u)du$. Then, ϕ and $\psi \in \Phi$. Hence, by Theorem 21, T and F have a unique coincide fixed point. □

Example 3. Suppose $X = [0, 1]$ and define $T, F : X \longrightarrow X$ such that

$$Ts_1 = \begin{cases} s_1 + \frac{1}{2}, & s_1 \in \left[0, \frac{1}{2}\right), \\ \frac{1}{2}, & s_1 \in \left[\frac{1}{2}, 1\right]. \end{cases} \quad (63)$$

Consider $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ since $\alpha(Fs_1, Fs_2) = 3$ and $\eta(Fs_1, Fs_2) = 2$, for all $s_1, s_2 \in X$. Assume $d : X \times X \longrightarrow [0, 1)$ be rectangular b-metric space, with $b = 3$, where

$$\frac{1}{b} d(Fs_1, Fs_2) = \begin{cases} \frac{1}{5}, & s_1, s_2 \in \left[0, \frac{1}{2}\right], \\ \frac{1}{20}, & s_1, s_2 \in \left[\frac{1}{2}, 0\right], \\ \frac{1}{9} & s_1 \in \left[0, \frac{1}{2}\right], s_2 \in \left[0, \frac{1}{2}\right]. \end{cases} \quad (64)$$

It is very well maybe the next step is to:

(i) $d(Ts_1, Ts_2) = 1/20$ and $\vartheta(Fs_1, Fs_2) = 1/5$, if $s_1, s_2 \in [0, 1/2]$

(ii) $d(Ts_1, Ts_2) = 0$ and $\vartheta(Fs_1, Fs_2) = 1/20$, if $s_1, s_2 \in [1/2, 1]$

(iii) $d(Fs_1, Fs_2) = 1/20$ and $\vartheta(Fs_1, Fs_2) = 1/9$, if $s_1 \in [0, 1 /2]$, $s_2 \in [0, 1]$

Consider the function $\Theta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ and ϕ, $\psi : [0,\infty) \longrightarrow [0,\infty)$ defined as $\Theta(s_1, s_2) = s_1 - s_2$, $\phi(s_2) = 4 s_2/5$ and $\psi(s_2) = s_2/3$. Then, Theorem 16 has been fulfilled. Hence, $v_1 = 1/2$ is a unique coincide fixed point of T and F.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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