Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 7670773, 9 pages
https://doi.org/10.1155/2022/7670773

Research Article

Novel Particular Solutions, Breathers, and Rogue Waves for an Integrable Nonlocal Derivative Nonlinear Schrödinger Equation

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Received 14 August 2021; Accepted 13 December 2021; Published 10 January 2022

Academic Editor: Wen-Xiu Ma

A determinant representation of the n-fold Darboux transformation for the integrable nonlocal derivative nonlinear Schödinger (DNLS) equation is presented. Using the proposed Darboux transformation, we construct some particular solutions from zero seed, which have not been reported so far for locally integrable systems. We also obtain explicit breathers from a nonzero seed with constant amplitude, deduce the corresponding extended Taylor expansion, and obtain several first-order rogue wave solutions. Our results reveal several interesting phenomena which differ from those emerging from the classical DNLS equation.

1. Introduction

It is well known that the derivative nonlinear Schrödinger (DNLS) equation [1]

$$iq_t(x,t)=q_{xx}(x,t)+i\varepsilon\left(q(x,t)^2q^*(x,t)\right)_x(\varepsilon=\pm1), \quad (1)$$

has many physical applications, e.g., in analyzing the propagation of circular polarized nonlinear Alfvén waves and radiofrequency waves in plasmas [1, 2]. Equation (1) is local; that is, the evolution only depends on the value of its local time and space [3]. In the recent years, some new integrable nonlocal equations have been introduced and several interesting results have been obtained [4–12]. These nonlocal equations are significantly different from the local one due to their particular spatiotemporal coupling, which may stimulate new physical applications, as they describe novel physical effects [3].

Since the nonlinearity-induced potentials satisfy the $\mathscr{PT}$-symmetric condition, the nonlocal NLS equation is often referred to as $\mathscr{PT}$-symmetric [4]. The dynamics arising from the nonlocal NLS equation leads to several phenomena, including dark solitons [13] and rogue waves [14, 15]. This also fostered interest in other nonlocal integrable equations [15–18], which are generally obtained by using parity ($\widehat{P}$, $\widehat{P}=-x$), time inversion ($\widehat{T}$, $\widehat{T}=-t$), and charge conjugation ($\widehat{C}$) symmetries. The symmetry $\widehat{P}-\widehat{T}-\widehat{C}$ has in turn an important role in quantum physics [19] and in many other fields of physics [20–22]. In addition, an important physical link between the nonlocally integrable reduction of the newly discovered AKNS system and physically interesting equations has been established [23]. Overall, there are several reasons motivating further studies of these nonlocal systems.

Recently, Zhou presented an integrable nonlocal DNLS equation [24]

$$iq_t(x,t)=q_{xx}(x,t)+\varepsilon\left(q(x,t)^2q^*(-x,t)\right)_x(\varepsilon=\pm1), \quad (2)$$

where the symbol * denotes complex conjugation. Equation (2) has a $\mathscr{PT}$-symmetric conserved density $q^*(-x,t)q(x,t)$, that is,

$$i(q^*(-x,t)q(x,t))_t = \left(q^*(-x,t)q_x(x,t) - (q^*(-x,t))_x q(x,t)\right)_x + \frac{3\varepsilon}{2}\left(q^2(x,t)q^{*2}(-x,t)\right)_x. \tag{3}$$

By the transformations $x \longrightarrow -ix, t \longrightarrow -t$, Equation (2) can be obtained from Equation (1) [3], and as a matter of fact, it is more accurate to call Equation (2) $\widehat{P}\widehat{C}$ symmetric [25]. In addition, Ablowitz and Musslimani have presented a real space-time reversal DNLS equation [5].

In recent years, there are many new results in obtaining nonlinear waves. Based on the characteristic lines and phase shift analysis, Yin and Tian studied the transitions and mechanisms of nonlinear waves in the $(2+1)$-dimensional Sawada-Kotera (2DSK) equation [26]. According to the characteristic lines of breath, various nonlinear waves are obtained, including quasi-anti-dark soliton, M-shaped soliton, W-shaped soliton, multi-peak soliton, and quasi-periodic wave. The dynamic properties of these nonlinear waves are analyzed in detail by means of characteristic lines. Wang et al. have constructed the three-component coupled Hirota hierarchy and have obtained its soliton solutions by using the $\bar{\partial}$-dressing method [27]. By using the Riemann-Hilbert method, Li et al. systematically investigated the general n-component nonlinear Schödinger equations. The multisoliton solutions of two-component nonlinear Schödinger equations have been detailedly analyzed by means of parameter modulation. Many interesting new phenomena are presented, including elastic collision, parallel propagation, soliton reflection, and time-periodic propagation [28].

In mathematical physics, N-soliton solutions are very helpful in exploring nonlinear wave phenomena. Breather, rogue wave, and lump solutions, etc. are all special reductions of N-soliton solutions [29]. Ma et al. studies the soliton solutions of many famous nonlinear equations and presents an algorithm to verify the Hirota N-soliton conditions [29–31]. The existence of N-soliton solutions of two kinds of generalized KdV equations is verified by the common factorization of Hirota functions of N wave vectors and the comparison of the degrees of polynomials containing common factors [30]. And then, a weight number is used in the above algorithm to verify the Hirota N-soliton condition for the B-type Kadomtsev-Petviashvili equation and three integrable equations in $(2+1)$ dimensions [29, 31]. Breathers and rogue waves have been the subject of large interest in the recent years. Breathers have been found in many physically related models such as Bose-Einstein condensates [32], higher-order integrable systems [33], and reaction-diffusion systems [34]. The observation of rogue waves and the study of their formation and their dynamics have been experimentally performed in diverse physical media, such as optical fibers, water wave tanks, and plasmas [35–37].

In this paper, we focus on studying breathers and rogue waves of the nonlocal DNLS Equation (2). We first construct an n-fold Darboux transformation for Equation (2) in Section 2, from which we obtain two particular solutions that are obtained from zero seed. In Section 3, breathers by nonzero seeds with constant amplitude are studied, and their typical dynamics are analyzed. By Taylor expansion on the breather, we obtain and analyze the rogue wave of Equation (2). Section 4 closes the paper with some concluding remarks.

2. Darboux Transformation and Particular Solutions

In this part, we first construct the Darboux transformation of Equation (2), which is then exploited to obtain two novel solutions that are not soliton solutions, of which we analyze the asymptotic behavior.

Our Darboux transformation is mostly determined by the seed solutions and eigenvalues corresponding to the Lax pair and by the relationship between the potential function q and r. According to the form of the first-order Darboux matrix, we then provide the n-order Darboux matrix and finally obtain the n-order Darboux transformation of Equation (2).

Let us sketch the main steps as follows: first, given an eigenvalue λ in the Lax equation of Equation (2) and being $q=0$ be a zero seed solution, we seek the eigenfunction corresponding to the seed solution of the Lax equation, then substitute it into the first-order Darboux transformation, and get a novel solution. Using the same steps, we select two different eigenvalues, so that we can get four eigenvalues according to the conjugate relation. We still take the seed solution $q=0$, and solving the Lax equation, we can obtain four eigenfunctions $f_i (i=1,\cdots,4)$. According to the relationship among the potential functions, we can get the other four eigenfunctions $g_i (i=1,\cdots,4)$ and bring them into the second-order Darboux transformation to obtain the second-order novel solutions.

According to the above method, we can get higher-order new solutions and study more interesting phenomena by analyzing their properties.

2.1. Darboux Transformation. The integrability of Equation (2) is guaranteed by the Lax pair [1]

$$\psi_x = U\psi = \begin{pmatrix} \lambda^2 & \lambda q \\ \lambda r & -\lambda^2 \end{pmatrix}\psi, \tag{4}$$

$$\psi_t = V\psi = \begin{pmatrix} -2i\lambda^4 + i\lambda^2 qr & -2i\lambda^3 q + \left(-iq_x + iq^2 r\right)\lambda \\ -2i\lambda^3 r + \left(ir_x + iqr^2\right)\lambda & 2i\lambda^4 - i\lambda^2 qr \end{pmatrix}\psi, \tag{5}$$

where λ is the spectral parameter, $q=q(x,t)$, $r=r(x,t)$, and $r=-\varepsilon\bar{q}^* = -\varepsilon q^*(-x,t)$, $\psi=(\psi_1(x,t),\psi_2(x,t))^T$. Refering to the construction of Darboux transformation of classical integrable equations [38], we could obtain a Darboux transformation of Equation (2).

Consider a canonical transformation

$$\psi^{[1]} = T\psi. \tag{6}$$

Equations (4) and (5) become

$$\psi_x^{[1]} = (T_x + TU)T^{-1}\psi^{[1]} \triangleq U^{[1]}\psi^{[1]},\\ \psi_t^{[1]} = (T_t + TV)T^{-1}\psi^{[1]} \triangleq V^{[1]}\psi^{[1]}. \tag{7}$$

Substituting potential functions q, r with the new ones $q^{[1]}$, $r^{[1]}$, we can determine T such that $U^{[1]}$ and $V^{[1]}$ have the same forms of U and V, respectively.

Let

$$T = \lambda I - P, \tag{8}$$

where I is unit matrix and $P = (p_{ij})_{2\times 2}$ with $p_{ij} = p_{ij}(x, t)$ $(i, j = 1, 2)$. The relation between the old potentials and new ones is given by

$$q^{[1]} = q + 2p_{12}, r^{[1]} = r - 2p_{21}. \tag{9}$$

According to the relation between q and r

$$r = -\varepsilon\bar{q}^* = -\varepsilon q^*(-x, t), \tag{10}$$

we have the following constraint

$$p_{21}(x, t) = \varepsilon p_{12}^*(-x, t). \tag{11}$$

For the sake of convenience, we use the notation $\bar{f}(x, t) = f(-x, t)$.

In order to determine T in Equation (8), one may set

$$P = H\Lambda H^{-1}, \tag{12}$$

with

$$H = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{13}$$

where $(f_1, f_2)^T = (f_1(x, t), f_2(x, t))^T$ is a solution of Equations (4) and (5) corresponding to the seed solution and the eigenvalue $\lambda = \lambda_1$. According to (10), we know that $(g_1, g_2)^T = (\bar{f}_2^*, \varepsilon\bar{f}_1^*)^T$ is a solution of Equations (4) and (5) when $\lambda = \lambda_1^* \triangleq \lambda_2$.

Then, we get

$$P = \frac{1}{\Delta}\begin{pmatrix} \varepsilon\lambda_1 f_1\bar{f}_1^* - \lambda_1^* f_2\bar{f}_2^* & (\lambda_1^* - \lambda_1)f_1\bar{f}_2^* \\ (\lambda_1 - \lambda_1^*)\varepsilon f_2\bar{f}_1^* & \varepsilon\lambda_1^* f_1\bar{f}_1^* - \lambda_1 f_2\bar{f}_2^* \end{pmatrix}, \tag{14}$$

with $\Delta = \varepsilon f_1\bar{f}_1^* - f_2\bar{f}_2^*$. From (14), we can directly verify the constraint (11). Therefore, from (9) and (14), a new solution of Equation (2) is obtained as follows:

$$q^{[1]} = q + \frac{2}{\Delta}(\lambda_1^* - \lambda_1)f_1\bar{f}_2^*. \tag{15}$$

The n-fold Darboux transformation of Equation (2) can be written in the following determinant representation:

$$q^{[n]} = q + 2\frac{P_{2n}}{W_{2n}}, \tag{16}$$

where

$$P_{2n} = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \cdots & \lambda_1^{n-1} f_1 & \lambda_1^n f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \cdots & \lambda_2^{n-1} g_1 & \lambda_2^n g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 f_4 & \cdots & \lambda_3^{n-1} f_3 & \lambda_3^n f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \cdots & \lambda_4^{n-1} g_3 & \lambda_4^n g_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \cdots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^n g_{2n-1} \end{vmatrix},$$

$$W_{2n} = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \cdots & \lambda_1^{n-1} f_1 & \lambda_1^{n-1} f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \cdots & \lambda_2^{n-1} g_1 & \lambda_2^{n-1} g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 f_4 & \cdots & \lambda_3^{n-1} f_3 & \lambda_3^{n-1} f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \cdots & \lambda_4^{n-1} g_3 & \lambda_4^{n-1} g_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \cdots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^{n-1} g_{2n} \end{vmatrix}. \tag{17}$$

By using (16), we can obtain multiple particular solutions, multibreathers, and higher-order rogue waves of Equation (2). The determinant representation of the DT provides a powerful tool to calculate this tedious expansion.

2.2. One-fold Particular Solution. In order to construct one-fold particular solutions of Equation (2) using the above Darboux transformation, one may start from the zero seed solution $q = 0$ in Equations (4) and (5). Then, the eigenfunctions corresponding to the seed solution are given by

$$f_1 = e^{\lambda_1^2 x - 2i\lambda_1^4 t}, \quad f_2 = e^{-\lambda_1^2 x + 2i\lambda_1^4 t}. \tag{18}$$

Setting $\lambda_1 = a + ib$ and substituting (18) into (15), one can easily get the one-fold particular solution

$$q^{[1]} = \frac{-4ibe^{2x\left(a^2-b^2\right)-4it\left(a^4-6a^2b^2+b^4\right)}}{\varepsilon e^{-4iab\left(4ita^2-4itb^2-x\right)} - e^{4iab\left(4ita^2-4itb^2-x\right)}}. \tag{19}$$

Solution (19) is a complex function whose modulus is given by

$$\left|q^{[1]}\right| = \frac{4|b|e^{2x\left(a^2-b^2\right)}}{\sqrt{2\cosh\left[32abt\left(a^2-b^2\right)\right] - 2\varepsilon\cos\left(8abx\right)}}, \tag{20}$$

with singularities at $\{(x, t) \mid 2\cosh[32abt(a^2 - b^2)] - 2\varepsilon\cos(8abx) = 0\}$. From (20), it can be found that when $a^2 > b^2$,

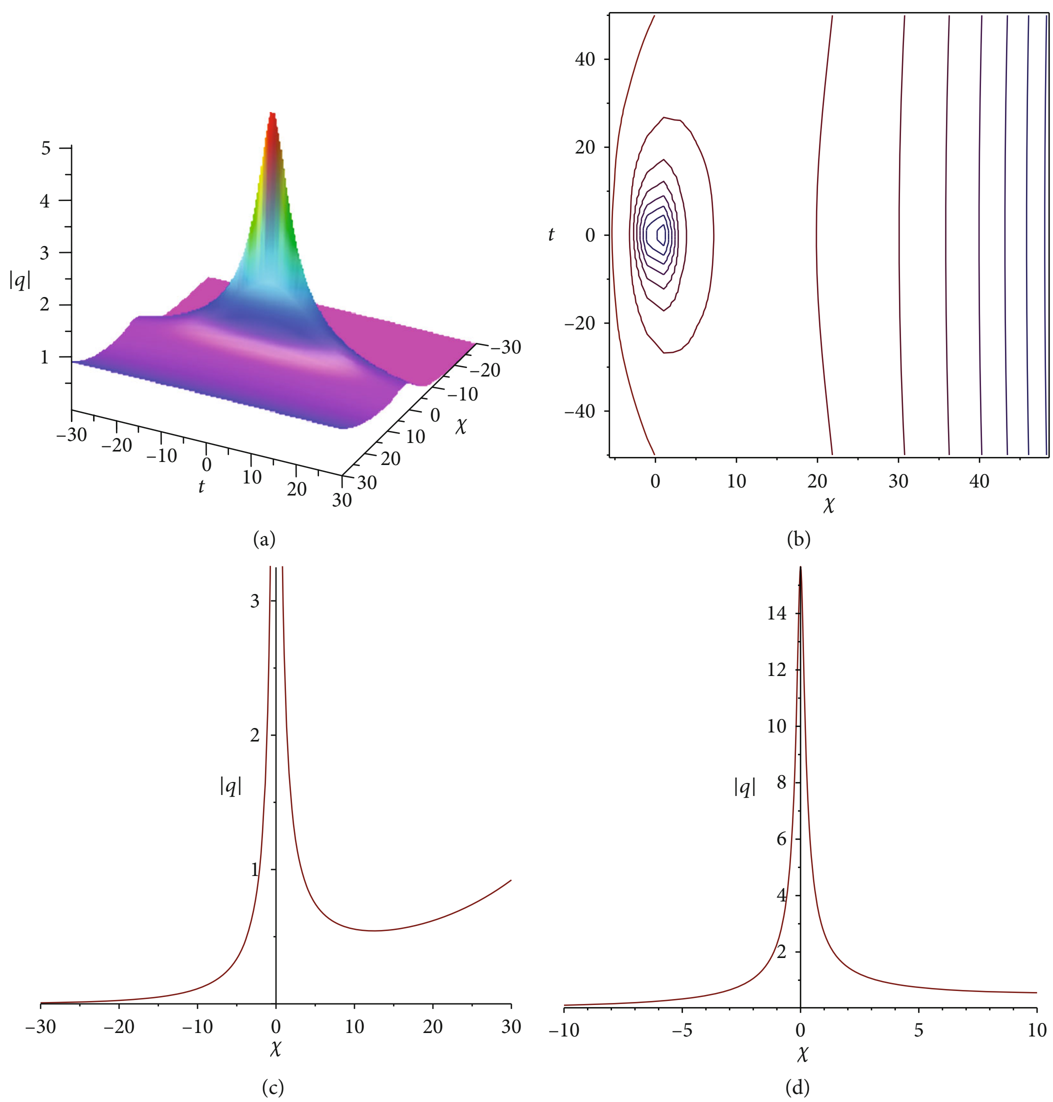


FIGURE 1: The one-fold particular solution evolution graph (a) and contour plot (b) for (20) with $\varepsilon = 1$, $a = 0.2$, $b = 0.01$. (c) The waveform at $t = 0$ and (d) the waveform at $t = 1$.

$|q^{[1]}| \longrightarrow +\infty$ as $x \longrightarrow +\infty$, and $|q^{[1]}| \longrightarrow 0$ as $x \longrightarrow -\infty$. We also see that $|q^{[1]}| \longrightarrow 0$ as $t \longrightarrow \pm\infty$. By setting $\varepsilon = 1$, $a = 0.2$, $b = 0.01$, Figures 1(a) and 1(b) illustrate the behaviour $|q^{[1]}|$ given in (20). By choosing $t = 0$ in (20) and using the previous parameters, one can get a one-fold particular solution with singularity. Figure 1(c) illustrates this case. For $t = 1$, we can see that $|q^{[1]}|$ reaches the maximum at $x = 0$. The maximum amplitude is 15.66400782.

2.3. Two-fold Particular Solution. In order to proceed further, we still choose zero seed solution $q = 0$ and use the 2-fold Darboux transformation, to obtain a two-fold particular solution of Equation (2) as follows:

$$q^{[2]} = q + 2\frac{P_4}{W_4}, \tag{21}$$

where

$$P_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1^2 f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2^2 g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3^2 f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4^2 g_3 \end{vmatrix}, W_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 \end{vmatrix}. \tag{22}$$

Let $\lambda_1 = \lambda_2^* = a + ib$ and $\lambda_3 = \lambda_4^* = c + id$. By solving Equations (4) and (5), we obtain the eigenfunctions

$$\begin{aligned} f_1 &= e^{(a+ib)^2 x - 2i(a+ib)^4 t}, f_2 = e^{-(a+ib)^2 x + 2i(a+ib)^4 t}, \\ f_3 &= e^{(c+id)^2 x - 2i(c+id)^4 t}, f_4 = e^{-(c+id)^2 x + 2i(c+id)^4 t}. \end{aligned} \tag{23}$$

According to (10), we get $g_1 = \bar{f}_2^*$, $g_2 = \varepsilon \bar{f}_1^*$, $g_3 = \bar{f}_4^*$, g_4

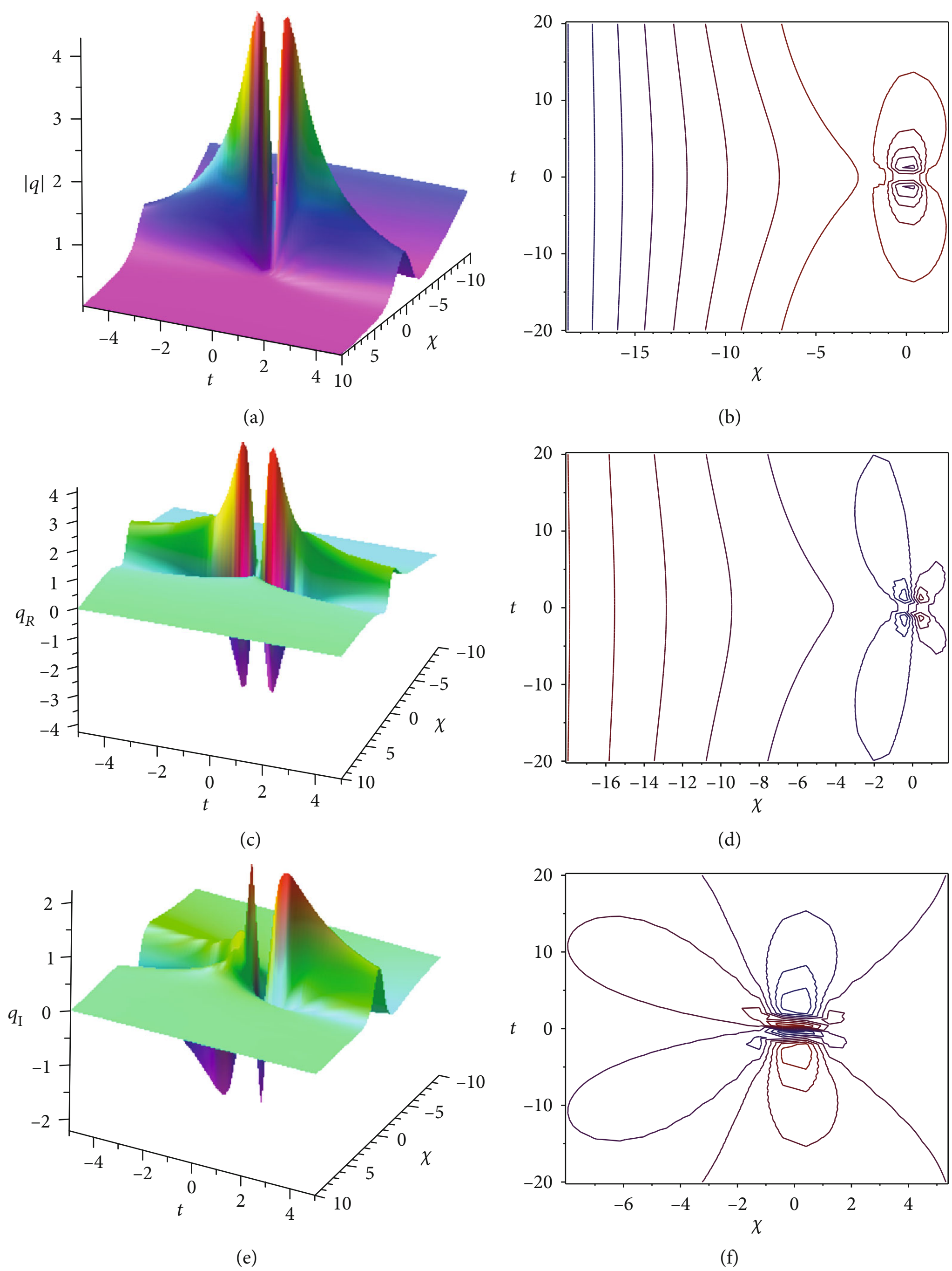


FIGURE 2: The two-fold particular solution evolution graph of (21) with parameters: $\varepsilon = 1, a = 0.1, b = 0.2, c = 0.01, d = 0.35$. (a) $|q^{[2]}|$. (c) Re $(q^{[2]})$. (e) Im $(q^{[2]})$. (b, d, f) Contour plots corresponding to (a, c, e), respectively.

$= \varepsilon \bar{f}_3^*$. Hence, a two-fold particular solution can be given explicitly by Equation (21). We take some special parameter values for the solution (21) and show the resulting behavior in Figure 2. Figure 2(a) reveals that there are two crests above the $x-t$ plane. We also found $|q^{[2]}| \longrightarrow +\infty$ as $x \longrightarrow -\infty$, and $|q^{[2]}| \longrightarrow 0$ as $x \longrightarrow +\infty$ as it can be seen from Figure 2(a). Figures 2(b) and 2(c) show beautiful butterfly contour graphs.

3. Breather and Rogue Wave Solutions

In this section, we construct breathers and rogue wave solutions of Equation (2) using the Darboux transformation illustrated in the previous one. At first, we construct breather solution from a nonzero seed. Then, we use the extended Taylor expansion method to reveal rogue wave solution of Equation (2) from breather ones.

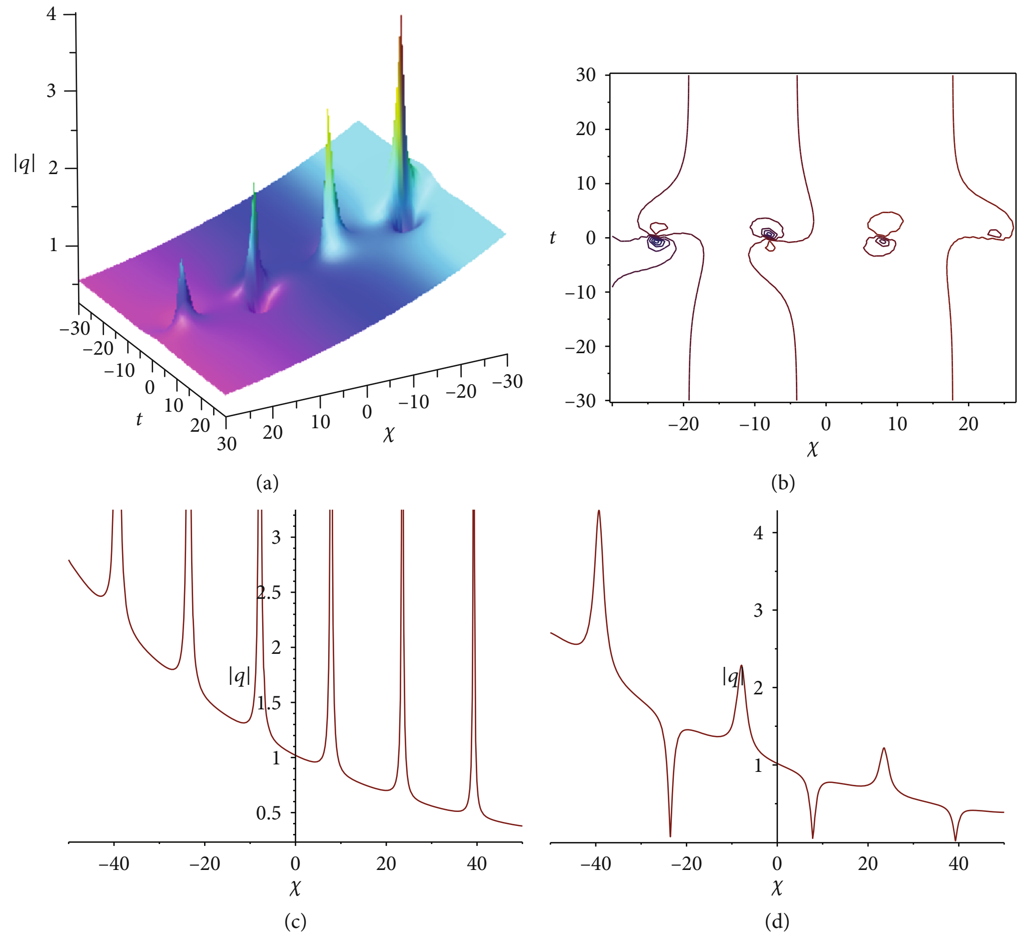


FIGURE 3: Evolution graph of the breather solution (32), (a) $|q^{[11]}_{\text{breather}}|$, (b) contour plot corresponding to (a) and (c, d) the waveform at $t=0$ and $t=1$, respectively.

3.1. Breather Solution. To obtain nontrivial periodic solutions, we choose a nonzero seed $q=\rho e^{kx+i\omega t}$, where ρ, k, and ω are all real constants and $\omega=-k(k+\varepsilon\rho^2)$. Substituting $q=\rho e^{kx+i\omega t}$ into Equations (4) and (5) and solving them, we get the eigenfunctions

$$f_1=\frac{\varepsilon}{2\lambda_1\rho}\left[C_1\left(k-2\lambda_1^2+\theta_1\right)e^{1/2(k-\theta_1)x+(1/2)i\left[\left(\rho^2\varepsilon+k+2\lambda_1^2\right)\theta_1-\left(k^2+\rho^2\varepsilon k\right)\right]t}+C_2\left(k-2\lambda_1^2-\theta_1\right)e^{1/2(k+\theta_1)x-(1/2)i\left[\left(\rho^2\varepsilon+k+2\lambda_1^2\right)\theta_1+\left(k^2+\rho^2\varepsilon k\right)\right]t}\right], \tag{24}$$

$$f_2=C_1e^{-(1/2)(k+\theta_1)x+(1/2)i\left[\left(k^2+\rho^2\varepsilon k\right)+\left(\rho^2\varepsilon+k+2\lambda_1^2\right)\theta_1\right]t}+C_2e^{-(1/2)(k-\theta_1)x+(1/2)i\left[\left(k^2+\rho^2\varepsilon k\right)-\left(\rho^2\varepsilon+k+2\lambda_1^2\right)\theta_1\right]t}, \tag{25}$$

where C_1, C_2 are constants and $\theta_1=\sqrt{k^2+4\lambda_1^4-4\varepsilon\lambda_1^2\rho^2-4k\lambda_1^2}$.

According to $g_1=\bar{f}_2^*$, $g_2=\varepsilon\bar{f}_1^*$, we get

$$g_1=C_1e^{(1/2)(k+\theta_2)x-(1/2)i\left[\left(k^2+\rho^2\varepsilon k\right)+\left(\rho^2\varepsilon+k+2\lambda_2^2\right)\theta_2\right]t}+C_2e^{(1/2)(k-\theta_2)x-(1/2)i\left[\left(k^2+\rho^2\varepsilon k\right)-\left(\rho^2\varepsilon+k+2\lambda_2^2\right)\theta_2\right]t},$$

$$g_2=\frac{1}{2\lambda_2\rho}\left[C_1\left(k-2\lambda_2^2+\theta_2\right)e^{-(1/2)(k-\theta_2)x-(1/2)i\left[\left(\rho^2\varepsilon+k+2\lambda_2^2\right)\theta_2-\left(k^2+\rho^2\varepsilon k\right)\right]t}+C_2\left(k-2\lambda_2^2-\theta_2\right)e^{-(1/2)(k+\theta_2)x+(1/2)i\left[\left(\rho^2\varepsilon+k+2\lambda_2^2\right)\theta_2+\left(k^2+\rho^2\varepsilon k\right)\right]t}\right], \tag{26}$$

where $\theta_2=\sqrt{k^2+4\lambda_2^4-4\varepsilon\lambda_2^2\rho^2-4k\lambda_2^2}$ and $\lambda_1=\lambda_2^*=a+ib$.

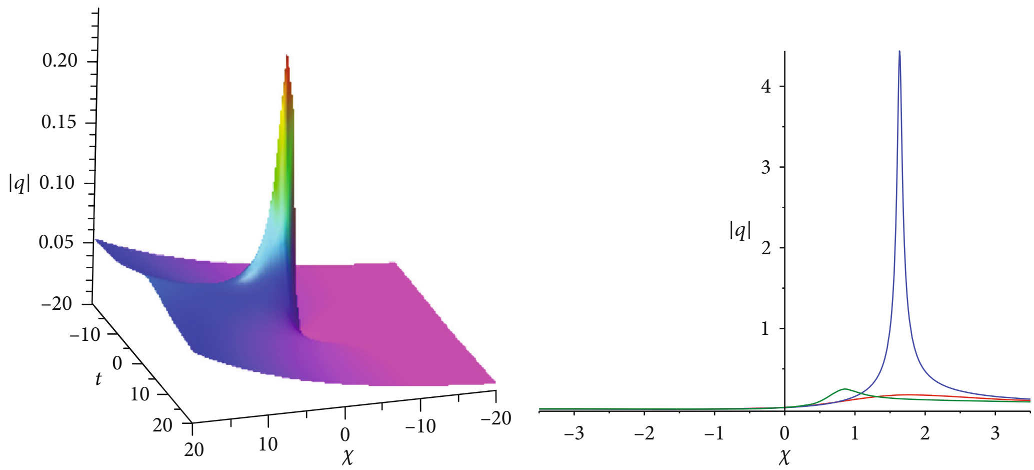
According to the principle of linear superposition, the new eigenfunctions associated to λ_1 of Equations (4) and (5) may be expressed by

$$F_1=f_1+g_1,\ F_2=f_2+g_2. \tag{27}$$

Next, we use Equations (24)–(27) to produce a new breather solution of Equation (2)

$$q^{[1]}=q+\frac{2}{\Delta}(\lambda_1^*-\lambda_1)F_1\bar{F}_2^*, \tag{28}$$

where $q=\rho e^{kx+i\omega t}$ and $\Delta=\varepsilon F_1\bar{F}_1^*-F_2\bar{F}_2^*$. By tedious calcula-



(a) (b)

FIGURE 4: The first-order rogue wave for $|q^{[1]}_{\text{rogue}}|$ with parameter selections: $k = 0.05$, $\rho = 0.001$, $a = 1$, $b = 0.01$. (a) $|q^{[1]}_{\text{rogue}}|$ and (b) its profiles at different times $t = -0.5$ (red), $t = 0$ (blue), and $t = 1$ (green).

tions, the breather solution is obtained as follows:

$$q^{[1]}_{\text{breather}} = e^{kx-i\left(k^2+\rho^2\varepsilon k\right)t} \cdot \left[\rho + \frac{2\lambda_2(\lambda_2-\lambda_1)\left[2\lambda_1\rho(1+\varepsilon)\cosh(d_2)A + 4\lambda_1^2\varepsilon\rho^2\cosh^2(d_2) + A^2\right]}{\lambda_1\varepsilon(\varepsilon-1)\left[AB + 4\lambda_1\lambda_2\rho^2\cosh(d_1)\cosh(d_2)\right]}\right]. \tag{29}$$

In fact $\varepsilon = \pm 1$, but we only take $\varepsilon = -1$ in Equation (29) and the breather solution is given by

$$q^{[1]}_{\text{breather}} = e^{kx-i\left(k^2-\rho^2 k\right)t}\left[\rho + \frac{\lambda_2(\lambda_2-\lambda_1)\left[A^2 - 4\lambda_1^2\rho^2\cosh^2(d_2)\right]}{\lambda_1\left[AB + 4\lambda_1\lambda_2\rho^2\cosh(d_1)\cosh(d_2)\right]}\right], \tag{30}$$

where

$$\begin{aligned} A &= \left(k - 2\lambda_1^2\right)\cosh(d_1) - \theta_1\sinh(d_1), B = \left(k - 2\lambda_2^2\right)\cosh(d_2) + \theta_2\sinh(d_2),\\ d_1 &= \frac{1}{2}\theta_1\left(x - it\left(k - \rho^2 + 2\lambda_1^2\right)\right), d_2 = \frac{1}{2}\theta_2\left(x - it\left(k - \rho^2 + 2\lambda_2^2\right)\right),\\ \theta_1 &= \sqrt{k^2 + 4\lambda_1^4 + 4\lambda_1^2\rho^2 - 4k\lambda_1^2}, \theta_2 = \sqrt{k^2 + 4\lambda_2^4 + 4\lambda_2^2\rho^2 - 4k\lambda_2^2},\\ \lambda_1 &= \lambda_2^* = a + ib. \end{aligned} \tag{31}$$

Specifically, choosing some specific parameter values $k = -0.02$, $\rho = 1$, $a = 0$, $b = 0.1$, we get

$$q^{[1]}_{\text{breather}} = e^{-0.02x-0.0204it} \cdot \left[1 + \frac{0.2i\left[\cos^2(0.1x + 0.104it) + \sin^2(0.1x + 0.104it)\right]}{\left[\cos^2(0.1x + 0.104it) - \sin^2(0.1x + 0.104it)\right]}\right], \tag{32}$$

Figure 3 illustrates this solution. From Figure 3(a), we see that the amplitude of (32) is increasing for $x \longrightarrow -\infty$. Moreover, we find that the breather solution (32) has periodic singularities at $t = 0$ and Figure 3(c) depicts this phenomenon. Figure 3(d) depicts the wave shape at $t = 1$.

3.2. Rogue Wave Solution. Motivated by recent findings [37, 39], we shall use the extended Taylor expansion method to construct the rogue wave solutions of Equation (2). At first, we need to find the parameters $a = \sqrt{2k - \rho^2}/2$, $b = \rho/2$ in the eigenvalues λ_1, λ_2 such that $\theta_1 \longrightarrow 0$, $\theta_2 \longrightarrow 0$ (i.e., $d_1 \longrightarrow 0$, $d_2 \longrightarrow 0$) in the breather solution (29), and, in turn, to put the breather into the indeterminate form 0/0, which is a critical point to transform a first-order breather into a first-order rogue wave. However, due to the complexity of the breather solution, we perform Taylor expansion at $d_1 = 0$, $d_2 = 0$ directly. The Taylor expansion of (30) at $d_1 = 0$, $d_2 = 0$ gives a first-order rogue wave solution of Equation (2)

$$q^{[1]}_{\text{rogue}} = e^{kx-i\left(k^2-\rho^2 k\right)t} \cdot \left[\rho + \frac{\lambda_2(\lambda_2-\lambda_1)\left[\left(\left(k-2\lambda_1^2\right)k_1 - \theta_1 k_2\right)^2 - 4\lambda_1^2\rho^2 k_3^2\right]}{\lambda_1\left[\left(k-2\lambda_1^2\right)k_1 - \theta_1 k_2\right]\left[\left(k-2\lambda_2^2\right)k_3 + \theta_2 k_4\right] + 4\lambda_1^2\lambda_2\rho^2 k_1 k_3}\right], \tag{33}$$

where

$$\begin{aligned} k_1 &= 1 + \frac{\theta_1^2}{8}\left[x - it\left(k - \rho^2 + 2\lambda_1^2\right)\right]^2,\\ k_2 &= \frac{\theta_1}{2}\left[x - it\left(k - \rho^2 + 2\lambda_1^2\right)\right] + \frac{\theta_1^3}{48}\left[x - it\left(k - \rho^2 + 2\lambda_1^2\right)\right]^3,\\ k_3 &= 1 + \frac{\theta_2^2}{8}\left[x - it\left(k - \rho^2 + 2\lambda_2^2\right)\right]^2,\\ k_4 &= \frac{\theta_2}{2}\left[x - it\left(k - \rho^2 + 2\lambda_2^2\right)\right] + \frac{\theta_2^3}{48}\left[x - it\left(k - \rho^2 + 2\lambda_2^2\right)\right]^3, \end{aligned} \tag{34}$$

and θ_j, λ_j $(j = 1, 2)$ satisfy (31). Next, we show the shape of $|q^{[1]}_{\text{rogue}}|$ in Figure 4(a) by selecting some special parameter values. From Figure 4(a), we can see that $|q_{\text{rogue}}^{[1]}| \longrightarrow +\infty$ as $x \longrightarrow +\infty$, and $|q_{\text{rogue}}^{[1]}| \longrightarrow 0$ as $x \longrightarrow -\infty$. The evolution of this rogue wave at different times is shown in Figure 4(b), where we can see that $|q_{\text{rogue}}^{[1]}|$ achieves its maximum at $(x = 1.63698271, t = 0)$. The maximum amplitude is 4.448628515. It is shown that rogue waves exist in integrable systems.

4. Summary

In summary, a determinant representation of n-fold Darboux transformation for the integrable nonlocal DNLS equation has been obtained. The determinant representation provides a powerful tool to calculate otherwise tedious expansion and it is very useful to calculate the higher-order Taylor expansion in an indeterminate form 0/0 from the double degeneration of eigenvalues. Armed with the Darboux transformation, we have got two particular solutions from zero seed, which are not soliton solutions, and then, their dynamics patterns are analyzed in some details. We show that the properties of these solutions crucially depend on the selection of the parameter values. By choosing different parameter values, we obtain several interesting and novel solutions, which are different from those of the previously studied locally integrable systems. Breather solution is given explicitly by a nonzero seed with constant amplitude. The variable x in the breather solution (30) may be referred to as the 'transverse variable,' and the breather periodic in the 'transverse variable' corresponds to the so-called 'Akhmediev' breather [40]. Finally, we have got a rogue wave solution of Equation (2) from the breather by an extended Taylor expansion method. We have analyzed the dynamical features of the rogue wave and have shown the evolution at different times. Our results pave the way for further studies on higher-order rogue waves of the nonlocal DNLS Equation (2) from the determinant representation (16) of the Darboux transformation. More generally, we hope that our results can enrich the study of nonlocal integrable systems.

Data Availability

The Maple data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is sponsored by the National Natural Science Foundation of China (11471004, 12071418), Applied Basic Research Program of Shanxi Province (20210302123082, 201901D211461), Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (2019L0868), and the Doctoral Research Project of Yuncheng University (YQ-2020019).

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