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Research Article

Breather Wave and Traveling Wave Solutions for A $(2+1)$-Dimensional KdV4 Equation

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In this paper, an integrable $(2+1)$-dimensional KdV4 equation is considered. By considering variable transformation and Bell polynomials, an effective and straightforward way is presented to derive its bilinear form. The homoclinic breather test method is employed to construct the breather wave solutions of the equation. Then, the dynamic behaviors of breather waves are discussed with graphic analysis. Finally, the (G'/G^2) expansion method is employed to obtain traveling wave solutions of the $(2+1)$-dimensional integrable KdV4 equation, including trigonometric solutions and exponential solutions.

1. Introduction

In the research of nonlinear science, more and more attention has been paid to the nonlinear evolution equations [1–3], which can depict many important phenomena in physics and other related fields. In order to describe these nonlinear phenomena, it is very necessary to seek exact solutions for nonlinear evolution equations in mathematical physics [4–6]. Over the last few decades, there exist a lot of methods to deal with nonlinear models, including Hirota bilinear method [7], the (G'/G)-expansion method [8], and the (G'/G^2)-expansion method [9, 10]. Particularly, Hirota bilinear method is one of the most direct and effective methods to search for the solitary wave solutions of nonlinear evolution equations. Recently, Yuan derived exact solutions of a $(2+1)$-dimensional extended shallow water wave equation by using Hirota bilinear method in [11]. Tao presented abundant soliton wave solutions for the $(3+1)$-dimensional variable-coefficient nonlinear wave equation in [12] by considering the Hirota bilinear operators. Meanwhile, breather waves and rouge waves also have attracted growing attention on both experimental observations and theoretical predictions [13, 14]. These giant wave phenomena have been found in different fields such as the plasmas, deep ocean, nonlinear optic, biophysics, and even finance. Particularly, based on Hirota bilinear method, there are a number of works to study breather waves [15–17] and rouge waves [18, 19].

For KdV series equations, many meaningful results have been presented. Dai et al. discussed interactions between exotic multivalued solitons of the $(2+1)$-dimensional Korteweg–De Vries equation describing shallow water wave in [20]. In [21], Abdul-Majid Wazwaz derived a $(2+1)$-dimensional Korteweg De Vries 4 (KdV4) equation by using the recursion operator of the KdV equation as follows:

$$v_{xy} + v_{xxxt} + v_{xxxx} + 3\left(v_x^2\right)_x + 4v_x v_{xt} + 2v_{xx} v_t = 0. \tag{1}$$

Multiple soliton solutions, traveling wave solutions, and other periodic solutions for the $(2+1)$-dimensional KdV4 equation were derived in [21]. Inspired by the ideas in above literature, we would like to consider the breather wave solutions, trigonometric solutions, and exponential solutions to the KdV4 equation.

The rest of this paper is organized as follows. In Section 2, the bilinear form of KdV4 equation is derived via using variable transformation and Bell's polynomials. In Section 3, the homoclinic breather limit method is employed to construct the breather wave solutions of KdV4. Then, the (G'/G^2) expansion method is applied to obtain traveling wave solutions of the $(2+1)$-dimensional integrable KdV4 equation, including trigonometric solutions and exponential solutions. Finally, some remarks are given.

2. Bilinear Forms of the (2 + 1)-Dimensional KdV4 Equation

Through calculation, we find it is impossible to the obtain the Hirota bilinear form of KdV4. So, we introduce the dependent variable transformation $\xi = x - kt$ in (1); then, the $(2+1)$-dimensional KdV4 equation (1) can be transformed into

$$v_{\xi y} + kv_{\xi\xi\xi\xi} + v_{\xi\xi\xi\xi} + 3\left(v_{\xi}^2\right)_{\xi} - 4kv_{\xi}v_{\xi\xi} - 2kv_{\xi\xi}v_{\xi} = 0, \tag{2}$$

where k is a real constant. Integrating the obtained equation with respect to ξ once, one obtains

$$v_y + (1-k)v_{\xi\xi\xi} + (3-3k)v_{\xi}^2 = 0. \tag{3}$$

Let us introduce a potential transformation

$$v = cq_{\xi}. \tag{4}$$

Substituting (4) into (3), we have

$$E(q) = cP_{\xi y} + c(1-k)P_{\xi\xi\xi\xi} = 0. \tag{5}$$

Based on the results about Bell polynomials in [22], equation (5) yields the following bilinear formalism:

$$\left[D_{\xi}D_y + (1-k)D_{\xi}^4\right]f \cdot f = 0, \tag{6}$$

with the aid of the following transformation:

$$q = 2\log(f) \Longleftrightarrow v = cq_{\xi} = 2\log(f)_{\xi}. \tag{7}$$

3. Breather Wave Solutions of the (2 + 1)-Dimensional KdV4 Equation

In this section, we will construct the breather wave solutions of the KdV4 equation (1) by using the homoclinic breather test method [23]. It is not hard to check that equation (1) does not exist an equilibrium solution. So, we suppose

$$v = 2\log(f)_{\xi}, \tag{8}$$

where $f = f(\xi, y)$ is a real function to be known later. According to the extended homoclinic test method, we seek for the breather wave solution of equation (6) in the following form:

$$\begin{aligned} f = {} & \exp(-p_1(\xi + \alpha y)) + \epsilon_1 \cos(p(\xi + \beta y)) \\ & + \epsilon_2 \exp(p_1(\xi + \alpha y)), \end{aligned} \tag{9}$$

where $p_1, p, \alpha, \beta, \epsilon_1$, and ϵ_2 are real constants to be determined later. Substituting equation (9) into equation (6) leads to an algebraic equation and equating each coefficient for the powers of $\exp(\pm p_1(\xi + \alpha y))$, $\sin(p(\xi + \beta y))$, and $\cos(p(\xi + \beta y))$ to zero, we obtain some algebraic equations. Taking $p = p_1$, we have

$$\begin{cases} -8p^4k\epsilon_1^2 - 32p^4k\epsilon_2 + 8p^4\epsilon_1^2 - 2p^2\epsilon_1^2\beta + 32p^4\epsilon_2 + 8p^2\epsilon_2\alpha = 0, \\ 8p^4k\epsilon_1 - 8p^4\epsilon_1 + 2p^2\epsilon_1\alpha - 2p^2\epsilon_1\beta = 0, \\ 8p^4k\epsilon_1\epsilon_2 - 8p^4\epsilon_1\epsilon_2 + 2p^2\epsilon_1\epsilon_2\alpha - 2p^2\epsilon_1\epsilon_2\beta = 0, \\ 2p^2\epsilon_1\epsilon_2\alpha + 2p^2\epsilon_1\epsilon_2\beta = 0. \end{cases} \tag{10}$$

Solving the obtained equations in (10) with the help of Maple, we have

$$\begin{aligned} \epsilon_1 &= \pm 2\sqrt{-\epsilon_2}, \\ \alpha &= 2(1-k)p^2, \\ \beta &= 2(k-1)p^2, \end{aligned} \tag{11}$$

in which k, p, and ϵ_2 are arbitrary real numbers with $p \neq 0$ and $\epsilon_2 \leq 0$. In addition, equation (9) can be written as

$$\begin{aligned} f &= \exp\left(-p\left(\xi + 2(1-k)p^2y\right)\right) + \epsilon_1\cos\left(p\left(\xi + 2(k-1)p^2y\right)\right) + \epsilon_2\exp\left(p\left(\xi + 2(1-k)p^2y\right)\right) \\ &= -2\sqrt{-\epsilon_2}\sinh\left[p\left(\xi + 2(1-k)p^2y\right) + \ln\sqrt{-\epsilon_2}\right] \pm 2\sqrt{-\epsilon_2}\cos\left(p\left(\xi + 2(k-1)p^2y\right)\right). \end{aligned} \tag{12}$$

Then, substituting the obtained results (12) and $\xi = x - kt$ into equation (6) yields the solutions of equation (1) as follows:

$$v_1 = \frac{2p\cosh(\Omega_1) - 2p\sin(\Omega_2)}{\sinh(\Omega_1) + \cos(\Omega_2)}, \tag{13}$$

$$v_2 = \frac{2p\cosh(\Omega_1) + 2p\sin(\Omega_2)}{\sinh(\Omega_1) - \cos(\Omega_2)}, \tag{14}$$

where

$$\begin{aligned} \Omega_1 &= p\left(x + 2(1-k)p^2 y - kt\right) + \ln\sqrt{-\epsilon_2}, \\ \Omega_2 &= p\left(x + 2(k-1)p^2 y - kt\right). \end{aligned} \tag{15}$$

The wave v_1 given via solution (13) would be closing to a point $2p$ as $t \longrightarrow +\infty$, and it would be closing to a point $-2p$ as $t \longrightarrow -\infty$. While the wave v_2 given via solution (14) would also be closing to a point $2p$ as $t \longrightarrow +\infty$, and it would be closing to a point $-2p$ as $t \longrightarrow -\infty$. This is the reason why we could not find an equilibrium solution at the beginning. v_1 and v_2 are the breather waves which can propagate with periodic oscillation. It indicates that the homoclinic breather wave can be spanned by the interaction between homoclinic wave and breather wave in a direction. Taking $\epsilon_2 = -1$, $\ln(\sqrt{-\epsilon_2}) = 0$ in equations (13) and (14); so, v can be rewritten as follows:

$$v_1 = \frac{2p\cosh\left(p\left(x + 2(1-k)p^2 y - kt\right)\right) - 2p\sin\left(p\left(x + 2(k-1)p^2 y - kt\right)\right)}{\sinh\left(p\left(x + 2(1-k)p^2 y - kt\right)\right) + \cos\left(p\left(x + 2(k-1)p^2 y - kt\right)\right)}, \tag{16}$$

$$v_2 = \frac{2p\cosh\left(p\left(x + 2(1-k)p^2 y - kt\right)\right) + 2p\sin\left(p\left(x + 2(k-1)p^2 y - kt\right)\right)}{\sinh\left(p\left(x + 2(1-k)p^2 y - kt\right)\right) - \cos\left(p\left(x + 2(k-1)p^2 y - kt\right)\right)}, \tag{17}$$

when $p \longrightarrow 0$,

$$\frac{2p\cosh\left(p\left(x + 2(1-k)p^2 y - kt\right)\right) \pm 2p\sin\left(p\left(x + 2(k-1)p^2 y - kt\right)\right)}{\sinh\left(p\left(x + 2(1-k)p^2 y - kt\right)\right) \pm \cos\left(p\left(x + 2(k-1)p^2 y - kt\right)\right)} \longrightarrow 0. \tag{18}$$

So, we could not obtain the rouge wave solutions of KdV4 by Taylor expansion in (14) at $p = 0$. By choosing the suitable parameters, we present the breather wave solutions of (16) and (17) in Figures 1 and 2, respectively. The evolution of v_1 with x and y at $t = 0$ is demonstrated in Figure 1. Figure 1(a) clearly shows the interactions of different waves. Figure 1(b) shows the overhead view of Figure 1(a). Figure 1(c) demonstrates the wave along the x axis with $y = 0$. The evolution of breather wave v_2 with x and y at $t = 0$ is demonstrated in Figure 1. Figure 2(a) clearly shows the interactions of different waves. Figure 2(b) shows the overhead view of Figure 2(a). Figure 2(c) demonstrates the wave along the x axis with $y = 0$. It is clear that v_1 and v_2 are much similar.

4. Traveling Wave Solutions of the $(2+1)$-Dimensional KdV4 Equation

In this section, we will construct the traveling wave solutions of the KdV4 equation (1) by using the (G'/G^2) expansion method [10]. Considering traveling wave transformation $\eta = lx + my - ct$, (1) is converted into the following ODE in the variable $V = V(\eta)$:

$$lmV'' - l^3 cV^{(4)} + l^4 V^{(4)} + 3l^3 V'V'' - 6l^2 cV'V'' = 0. \tag{19}$$

Eliminating l and then integrating (19) with respect to η once, by choosing the constant of integration to be zero, we obtain the following ODE:

$$mV' - l^2 cV''' + l^3 V''' + 3l^2 (V')^2 - 3lc(V')^2 = 0, \tag{20}$$

for which the homogeneous balance principle is applied. The highest order derivative V''' and the nonlinear term of the highest order $(V')^2$ are balanced as follows:

$$\deg\left[V'''\right] = N + 3 = \deg\left[(V')^2\right] = 2N + 2, \tag{21}$$

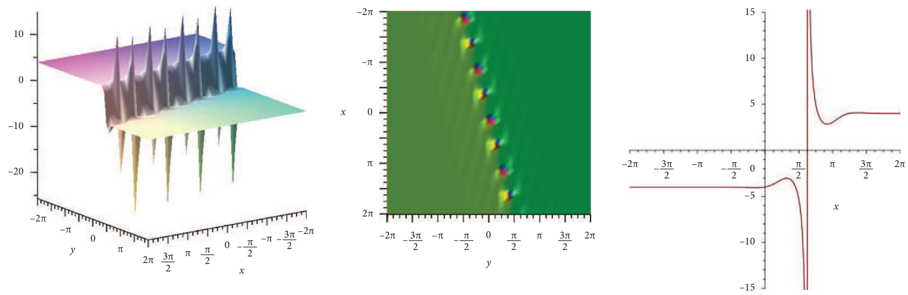
which leads to $N = 1$. Therefore, the form of exact solutions of the ODE in (20) using G'/G^2 expansion method can be expressed as

$$V(\eta) = a_{-1}\left(\frac{G'}{G^2}\right)^{-1} + a_0 + a_1\left(\frac{G'}{G^2}\right), \tag{22}$$

where a_{-1}, a_0, and a_1 are undetermined constants with

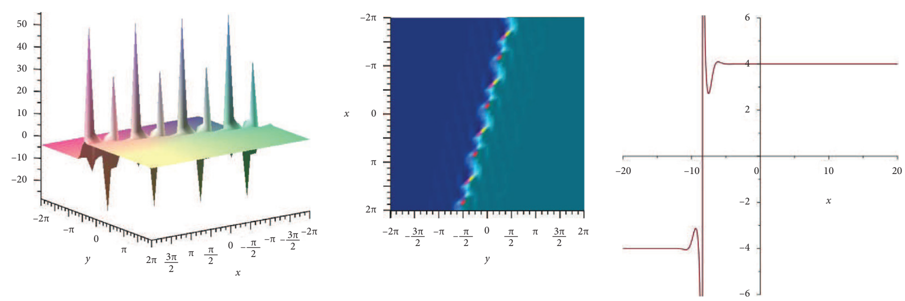
$$\left(\frac{G'}{G^2}\right)' = \mu + \lambda\left(\frac{G'}{G^2}\right)^2, \tag{23}$$

in which $\lambda \neq 1$ and $\mu \neq 0$ are arbitrary real numbers. Substituting (22) into (20) along with (23), then collecting all



(a) (b) (c)

Figure 1: Breather wave (16) for equation (1) with $\epsilon_1 = 2, \epsilon_2 = -1, p = 2, \alpha = -4, \beta = 4$, and $k = (3/2)$ at time $t = 0$. (a) Perspective view of the real part of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the x axis with $y = 0$.



(a) (b) (c)

Figure 2: Breather wave (17) for equation (1) with $\epsilon_1 = -4, \epsilon_2 = -4, p = 2, \alpha = 4, \beta = -4$, and $k = (1/2)$ at time $t = 0$. (a) Perspective view of the real part of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the x axis with $y = 0$.

the coefficients with the same power of $(G'/G^2)^j$, $(j = 0, \pm 1, \pm 2, \ldots)$, and finally setting these resulting coefficients to be zero, we consequently obtain the following system of algebraic equation in $a_{-1}, a_0, a_1, l, m, c, \lambda$, and μ:

$$\begin{aligned}
&\left(\frac{G'}{G^2}\right)^4 : 6l^3\lambda^3 a_1 - 6l^2 c\lambda^3 a_1 + 3l^2\lambda^2 a_1^2 - 3lc\lambda^2 a_1^2,\\
&\left(\frac{G'}{G^2}\right)^2 : 8l^3\lambda^2\mu a_1 - 8l^2 c\lambda^2\mu a_1 + 6cl\lambda^2 a_1 a_{-1} - 6l^2\lambda^2 a_1 a_{-1} - 6lc\lambda\mu a_1^2 + 6l^2\lambda\mu a_1^2 + \lambda m a_1,\\
&\left(\frac{G'}{G^2}\right)^0 : 2l^2 c\lambda^2\mu a_{-1} - 2l^3\lambda^2\mu a_{-1} - 2l^2 c\lambda\mu^2 a_1 + 2l^3\lambda\mu^2 a_1 - 3lc\lambda^2 a_{-1}^2 + 3l^2\lambda^2 a_{-1}^2 + 12lc\lambda\mu a_1 a_{-1}\\
&\quad - 12l^2\lambda\mu a_1 a_{-1} - 3lc\mu^2 a_1^2 + 3l^2\mu^2 a_1^2 - \lambda m a_{-1} + \mu m a_1,\\
&\left(\frac{G'}{G^2}\right)^{-2} : \mu a_{-1}\left(8l^2 c\lambda\mu - 8l^3\lambda\mu - 6lc\lambda a_{-1} + 6l^2\lambda a_{-1} + 6lc\mu a_1 - 6l^2\mu a_1 - m\right),\\
&\left(\frac{G'}{G^2}\right)^{-4} : 3l\mu^2 a_{-1}\left(2lc\mu - 2l^2\mu - ca_{-1} + la_{-1}\right).
\end{aligned} \tag{24}$$

Solving the obtained algebraic system (24) by using Maple, we obtain the following three cases.

$$\begin{aligned}\text{Case 1: } a_0 &= a_0,\\ a_1 &= -2l\lambda,\\ a_{-1} &= 2lm\mu,\\ c &= \frac{16\lambda\mu l^3 - m}{16\lambda\mu l^2},\\ l &= l,\\ m &= m,\end{aligned} \tag{25}$$

where a_0, l, m, λ, and μ are arbitrary constants.

$$\begin{aligned}\text{Case 2: } a_0 &= a_0,\\ a_1 &= 0,\\ a_{-1} &= 2l\mu,\\ c &= \frac{\left(4\lambda\mu l^3 - m\right)}{4\lambda\mu l^2},\\ l &= l,\\ m &= m,\end{aligned} \tag{26}$$

where a_0, l, m, λ, and μ are arbitrary constants.

$$\begin{aligned}\text{Case 3: } a_0 &= a_0,\\ a_1 &= -2l\lambda,\\ a_{-1} &= 0,\\ c &= \frac{\left(4\lambda\mu l^3 - m\right)}{4\lambda\mu l^2},\\ l &= l,\\ m &= m,\end{aligned} \tag{27}$$

where a_0, l, m, λ, and μ are arbitrary constants. When we substitute the above three cases of the obtained parameters along with the functions (G'/G^2) specified in reference [7], into the solution form (22), we can write three results of solutions of (1) as follows.

Result 1. For case 1 in (25), we have $\eta = lx + my - ((16\lambda\mu l^3 - m)/16\lambda\mu l^2)t$. When $\lambda\mu > 0$, the trigonometric function solution corresponding to the parameter values can be written as

$$v_3 = 2lm\sqrt{\lambda\mu}\left(\frac{C\cos(\sqrt{\lambda\mu}\,\eta) + D\sin(\sqrt{\lambda\mu}\,\eta)}{D\cos(\sqrt{\lambda\mu}\,\eta) - C\sin(\sqrt{\lambda\mu}\,\eta)}\right)^{-1} + a_0 - 2l\sqrt{\lambda\mu}\left(\frac{C\cos(\sqrt{\lambda\mu\eta}) + D\sin(\sqrt{\lambda\mu\eta})}{D\cos(\sqrt{\lambda\mu\eta}) - C\sin(\sqrt{\lambda\mu\eta})}\right). \tag{28}$$

The fashions of solutions (28) are displayed in Figure 3 by choosing suitable parameters.

When $\lambda\mu < 0$, the exponential function solution corresponding to the parameter values can be written as

$$v_4 = \frac{lm\mu}{\lambda}\left(2\sqrt{|\lambda\mu|} - \frac{4C\sqrt{|\lambda\mu|}\,e^{2\sqrt{|\lambda\mu|}\,\eta}}{Ce^{2\sqrt{|\lambda\mu|}\,\eta} - D}\right)^{-1} + a_0 - l\left(2\sqrt{|\lambda\mu|} - \frac{4C\sqrt{|\lambda\mu|}\,e^{2\sqrt{|\lambda\mu|}\,\eta}}{Ce^{2\sqrt{|\lambda\mu|}\,\eta} - D}\right). \tag{29}$$

The fashions of solutions (29) are displayed in Figure 4 by choosing suitable parameters.

Result 2. For case 2 in (26), we have $\eta = lx + my - ((4\lambda\mu l^3 - m)/4\lambda\mu l^2)t$. When $\lambda\mu > 0$, the trigonometric function solution corresponding to the parameter values can be written as

$$v_5 = 2l\sqrt{\lambda\mu}\left(\frac{C\cos(\sqrt{\lambda\mu}\,\eta) + D\sin(\sqrt{\lambda\mu}\,\eta)}{D\cos(\sqrt{\lambda\mu}\,\eta) - C\sin(\sqrt{\lambda\mu}\,\eta)}\right)^{-1} + a_0, \tag{30}$$

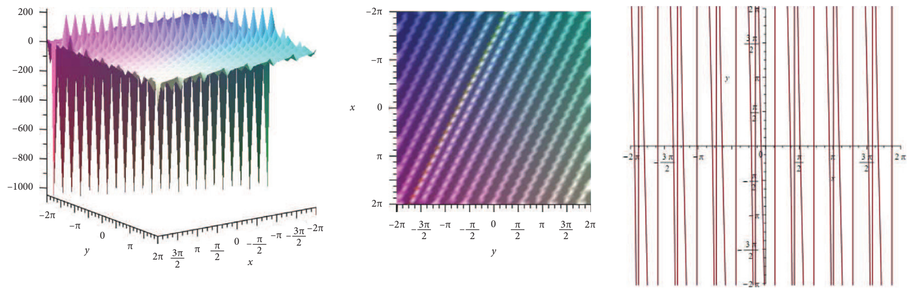
and when $\lambda\mu < 0$, the exponential function solution corresponding to the parameter values can be written as

$$v_6 = \frac{l\mu}{\lambda}\left(2\sqrt{|\lambda\mu|} - \frac{4C\sqrt{|\lambda\mu|}\,e^{2\sqrt{|\lambda\mu|}\,\eta}}{Ce^{2\sqrt{|\lambda\mu|}\,\eta} - D}\right)^{-1} + a_0. \tag{31}$$

Result 3. For case 3 in (27), we have $\eta = lx + my - ((4\lambda\mu l^3 - m)/4\lambda\mu l^2)t$. When $\lambda\mu > 0$, the trigonometric function solution corresponding to the parameter values can be written as

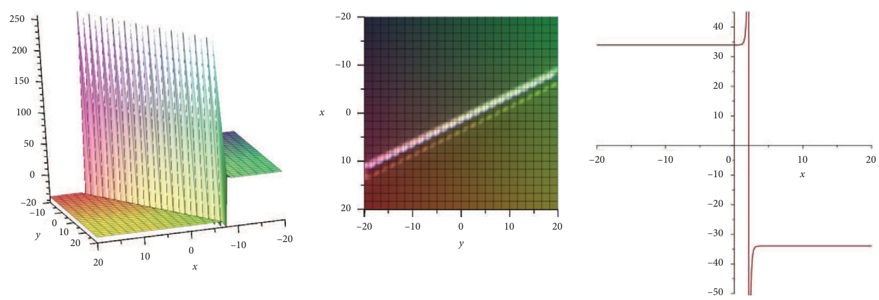
$$v_7 = a_0 - 2l\sqrt{\lambda\mu}\left(\frac{C\cos(\sqrt{\lambda\mu}\,\eta) + D\sin(\sqrt{\lambda\mu}\,\eta)}{D\cos(\sqrt{\lambda\mu}\,\eta) - C\sin(\sqrt{\lambda\mu}\,\eta)}\right)^{-1}. \tag{32}$$

When $\lambda\mu < 0$, the exponential function solution corresponding to the parameter values can be written as



(a) (b) (c)

Figure 3: Trigonometric function solution (28) for equation (1) with $l = 1, m = 2, c = 1, \lambda = 1, \mu = 3, a_0 = 2, C = 2$, and $D = 3$ at time $t = 1$. (a) Perspective view of the real part of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the x axis with $y = 1$.



(a) (b) (c)

Figure 4: Exponential function solution (29) for equation (1) with $l = 2, m = 1, \lambda = -1, \mu = 2, a_0 = 0, C = 1$, and $D = 2$ at time $t = 2$. (a) Perspective view of the real part of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the x axis with $y = 0$.

$$v_8 = a_0 - l\left(2\sqrt{|\lambda\mu|} - \frac{4C\sqrt{|\lambda\mu|}\, e^{2\sqrt{|\lambda\mu|}\,\eta}}{Ce^{2\sqrt{|\lambda\mu|}\,\eta} - D}\right). \quad (33)$$

It is clear that (28) is the superimposition of (30) and (32), while (29) is the superimposition of (31) and (33).

5. Remarks

In this paper, we introduced a dependent variable transformation to obtain the bilinear form of KdV4 equation. It is very interesting, although we applied homoclinic breather limit method to construct the breather wave solutions of the equation, we cannot obtain the rouge waves of KdV4 equation through Taylor expansion via breather waves. Then, the (G'/G^2) expansion method was employed to obtain traveling wave solutions of the $(2+1)$-dimensional integrable KdV4 equation, including trigonometric solutions and exponential solutions. It is necessary to point out that the solutions of Riccati equation (23) were derived by Ma in [24]. These solutions which we obtained in this paper are new, and they are different from the ones in [21]. Moreover, the method could also be employed efficiently for a broad range of nonlinear evolution equations.

Data Availability

All data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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