Research Article

Regularity for a Nonlinear Discontinuous Subelliptic System with Drift on the Heisenberg Group

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Received 2 September 2021; Accepted 6 October 2021; Published 24 January 2022

Abstract

In this paper, we prove the partial Hölder regularity of weak solutions and the partial Morrey regularity to horizontal gradients of weak solutions to a nonlinear discontinuous subelliptic system with drift on the Heisenberg group by the $A$-harmonic approximation, where the coefficients in the nonlinear subelliptic system are discontinuous and satisfy the VMO condition for ellipticity and growth condition with the growth index $1<p<2$ for the Heisenberg gradient variable, and the nonhomogeneous terms satisfy the controllable growth condition and the natural growth condition, respectively.

1. Introduction

Kohn in [1] proved $L^2$ estimates for the operator

$$Lu = \sum_{j=1}^{k} X_j^2 u + X_0 u + cu$$

constructed by Hörmander’s vector fields $\{X_1, X_2, \cdots, X_q, X_0\}$ (see [2]) based on the energy estimate and a subelliptic estimate. Moreover, some authors also inspected the regularity of solutions to linear degenerate elliptic equations with drift term by establishing singular integral estimates. For example, Folland and Stein in [3] established $L^p$ estimates with $a_{ij}(x)$ and $a_0(x)$ belonging to VMO spaces related to $\{X_1, X_2, \cdots, X_q, X_0\}$ and Schauder estimates with $a_{ij}(x)$ and $a_0(x)$ being in Hölder spaces for strong solutions. It is important in [4] that the difference between equations without $X_0$ and with $X_0$ was pointed out. When $X_1, X_2, \cdots, X_q$ in (3) is basis vector fields and $X_0$ is the drift vector field on homogeneous groups, many scholars have obtained regularities to the operator $\mathcal{L}$ with coefficients $a_{ij}$ and $a_0$ satisfying appropriate conditions, such as [5–8]. In addition, Austin and Tyson in [9] achieved the $C^{\infty}$-smoothness for the operator on the Heisenberg group $\mathbb{H}^n$ by using the geometric analysis method.

The equations studied in the above-cited papers are linear. In this paper, we consider the regularity to the...
weak solution of discontinuous subelliptic systems with drift term $Tu$ on $H^n$

$$-\sum_{i=1}^{2n} X_iA^k_j(x, u, \nabla_H u) - Tu = B^k(x, u, \nabla_H u), \quad x \in \Omega, k = 1, 2, \cdots, N, \quad (5)$$

where $\Omega$ is the bounded domain in $H^n$, $A^k_j$ belongs to the vanishing mean oscillation space (which is abbreviated as VMO) and satisfies the ellipticity on $\mathbb{R}^{2mN}$ and polynomial growth conditions with the growth index $1 < p < 2$ for $\nabla_H u$, and also $A^k_j$ is continuous for $u$ and differentiable for $\nabla_H u$ with continuous derivatives,

$$\nabla_H u = (X_1 u, X_2 u, \cdots, X_{2n} u), \quad (6)$$

$X_i$ $(i = 1, 2, \cdots, 2n)$ is the horizontal vector field and $T$ is the vertical vector $\theta$ with nondecreasing continuous modulus $\theta$ in continuous derivatives, that is, there exists a bounded, concave, growth index $1 < p < 2$ and also $A^k_j$ is continuous for $u$ and differentiable for $\nabla_H u$ with continuous derivatives,

$$\nabla_H u = (X_1 u, X_2 u, \cdots, X_{2n} u),$$

where $\Omega$ is the bounded domain in $H^n$, $A^k_j$ belongs to the vanishing mean oscillation space (which is abbreviated as VMO) and satisfies the ellipticity on $\mathbb{R}^{2mN}$ and polynomial growth conditions with the growth index $1 < p < 2$ for $\nabla_H u$, and also $A^k_j$ is continuous for $u$ and differentiable for $\nabla_H u$ with continuous derivatives,

$$\nabla_H u = (X_1 u, X_2 u, \cdots, X_{2n} u), \quad (6)$$

$X_i$ $(i = 1, 2, \cdots, 2n)$ is the horizontal vector field and $T$ is the vertical vector field in $H^n$. For more information about $H^n$, see Section 2. The nonhomogeneous term $B^k$ satisfies the controllable growth condition or natural growth condition. We will use the $A$-harmonic approximation method to conclude the partial Hölder regularity to the weak solutions and the partial Morrey regularity to the horizontal gradients of the weak solutions.

More regularity for the elliptic system without drift term, one can refer to [10–12] (Euclidean space) and [13–15] (Heisenberg group).

Now, for any $x \in \Omega$, $u, u_0 \in \mathcal{C}^N, P, P_0 \in \mathbb{R}^{2mN}$, and the growth index $1 < p < 2$, we list the hypotheses that the system satisfies,

(H1). Let $A^k_j$ satisfy the following ellipticity and polynomial growth conditions (growth index $1 < p < 2$):

$$\left\langle D_P A^k_j(x, u, P) P_0, P_0 \right\rangle \geq \lambda (1 + |P|)^{p-2} |P_0|^2, \quad (7)$$

$$\left| A^k_j(x, u, P) \right| + (1 + |P|) \left| D_P A^k_j(x, u, P) \right| \leq A(1 + |P|)^{p-1}, \quad (8)$$

where $D_P A^k_j$ denote the usual derivative of $A^k_j$ with respect to the variable $P$, $0 < \lambda \leq 1 \leq A < \infty$.

(H2). Assume that $A^k_j(x, u, P)/(1 + |P|)^{p-1}$ is continuous for $u$. More precisely, there exists a bounded, concave, and nondecreasing continuous modulus $\omega : [0, \infty) \to [0, 1]$ with $\lim_{s \to 0} \omega(s) = 0 = \omega(0)$ such that

$$\left| A^k_j(x, u, P) - A^k_j(x, u_0, P) \right| \leq \lambda \omega(|u - u_0|) (1 + |P|)^{p-1}. \quad (9)$$

(H3). Let $A^k_j$ be differentiable for the variable $P$ with continuous derivatives, that is, there exists a bounded, concave, and nondecreasing continuous modulus $\theta : [0, \infty) \to [0, 1]$ with $\theta(s) \leq s$, $\lim_{s \to 0} \theta(s) = 0 = \theta(0)$ such that

$$\left| D_P A^k_j(x, u, P) - D_P A^k_j(x, u, P_0) \right| \leq \Lambda \theta(|P - P_0|) (1 + |P| + |P_0|)^{p-2}. \quad (10)$$

(H4). For all $x \in B(x_0), A^k_j(x, u, P)/(1 + |P|)^{p-1}$ satisfies the following VMO condition:

$$\left| A^k_j(x, u, P) - \left( A^k_j(x, u, P) \right)_{x_0, r} \right| \leq \nu(x, r)(1 + |P|)^{p-1}, \quad (11)$$

where $\nu : \mathbb{R}^{2m+1} \times [0, r_0] \to [0, 2A]$ is a bounded function and satisfies

$$\lim_{r \to 0} \nu(r_0) = \limsup_{r \to 0} \sup_{x_0 \in \Omega} \left( \nu(x, r) \right)_{B(x_0)r_0} = 0. \quad (12)$$

Here, we have used in (11) and (12) the notation

$$(f)_{x_0, r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} f dx. \quad (13)$$

(HC) (controllable growth condition). The nonhomogeneous term $B^k$ satisfies the following controllable growth condition

$$|B^k(x, u, P)| \leq c \left( 1 + |u|^{p-1} + |P|^{p(1 - (1/p^*) - 1)} \right), \quad (14)$$

where $c$ is a positive constant,

$$p^* = \frac{p\phi}{\varphi - p} \quad \text{for} \quad 1 < p < \varphi, \quad (15)$$

and $\varphi$ denotes the homogeneous dimension of the Heisenberg group.

Obviously, we can see that system (5) includes the system

$$-\sum_{i=1}^{2n} X_i \left( A^k_j(x) \left( 1 + |\nabla_H u|^2 \right)^{(p-2)/2} X_i \right)^k - Tu = B^k(x, u, \nabla_H u), \quad x \in \Omega, k = 1, 2, \cdots, N. \quad (16)$$

We state the main result.

**Theorem 1.** Assume that $A^k_j(x, u, \nabla_H u)$ and $B^k(x, u, \nabla_H u)$ satisfy the assumptions (H1)-(H4) and (HC). If $1 < p < 2$ and $u \in H^{1,2}_0(\Omega, \mathbb{R}^N)$ is a weak solution to system (5), i.e., for all $\varphi \in H^{1,2}_0(\Omega),$

$$\int_{\Omega} A^k_j(x, u, \nabla_H u) \nabla_H \varphi dx + \int_{\Omega} u \cdot T \varphi dx = \int_{\Omega} B^k(x, u, \nabla_H u) \varphi dx, \quad (17)$$

then, there exists a relatively closed singular set $\Omega_0 \subset \Omega$ such that for any $\alpha \in (0, 1)$, we have
u ∈ \( C^{0,\alpha}_{\text{loc}}(Ω \setminus Ω_0, \mathbb{R}^N) \).

(18)

Moreover, for any \( \beta = \varphi - p(1 - \alpha) \), we have

\[ \nabla_{H} u \in L^{p,\beta}_{\text{loc}}(Ω \setminus Ω_0, \mathbb{R}^{2n+2N}) \],

(19)

where \( L^{p,\beta}_{\text{loc}} \) is a local Morrey space. The singular set \( Ω_0 \) satisfies

\[ |Ω_0| = 0, Ω_0 \subset \bigcup_i \bigcup_j Ω_{ij} \]

(20)

where\n
\[ \sum_i = \left\{ x_0 ∈ Ω : \limsup_{r→0} \frac{1}{r} \left( \left| \nabla_{H} u \right|^{\beta} \right)_{x_0} \right\} = \infty \}, \]

\[ \sum_j = \left\{ x_0 ∈ Ω : \liminf_{r→0} \frac{1}{r} \left( \left| \nabla_{H} u \right|^{\beta} \right)_{x_0} \right\} \] \( dx > 0 \} \}.

(21)

Corollary 2. Assume that \( A^k(x, u, \nabla_{H} u) \) and \( B^k(x, u, \nabla_{H} u) \) satisfy the assumptions (H1)-(H4) and the following assumption:

\[ (HN). \] The nonhomogeneous term \( B^k(x, u, \nabla_{H} u) \) satisfies the following \( p \) natural growth condition

\[ |B^k(x, u, P)| ≤ a|P|^p + b \]

(22)

for \( |u| ≤ M \), where \( a \) and \( b \) are constants depending only on \( M \).

Then, we have

\[ u ∈ C^{0,\alpha}_{\text{loc}}(Ω \setminus Ω_0, \mathbb{R}^N) \] \( \text{and} \) \( \nabla_{H} u \in L^{p,\beta}_{\text{loc}}(Ω \setminus Ω_0, \mathbb{R}^{2n+2N}) \)

(23)

for weak solution \( u \) in \( HW^{1,2}(Ω, \mathbb{R}^N) \) to system (5) under the assumption \( 2αM < \lambda \), where \( 1 < p < 2 \) and \( Ω_0 \) is same as in Theorem 1.

Its proof is direct by combining the proof of Theorem 1 in this paper with the proof of Theorem 1.2 in [15].

Let us recall that the A-harmonic approximation method was first introduced by Duzaar and Steffen in [16] and then extended to other cases by some authors, see [17–19]. In this paper, we use the A-harmonic approximation method described in [15] to conclude Theorem 1. Different from [15], the system considered by us has a drift term, which brings new challenges to our research. Actually, the processing of drift term are different from that the processings of other terms in the system. Moreover, Lemmas 11–14 in Section 3 used in proving Theorem 1 are different from the corresponding lemmas in [15] and will be rebuilt.

This paper is organized as follows: in Section 2, we introduce the related knowledge of the Heisenberg group, some function spaces on the Heisenberg group, horizontal affine functions, and some necessary lemmas. In Section 3, we show a Caccioppoli-type inequality for weak solution to (5), the approximately A-harmonic lemma, the decay estimate, and iteration relations. In Section 4, the proof of Theorem 1 is given.

2. Preliminaries

2.1. The Heisenberg Group \( \mathbb{H}^n \) and Some Function Spaces on \( \mathbb{H}^n \).

The Euclidean space \( \mathbb{R}^{2n+1} \), \( n ≥ 1 \) with the group multiplication

\[ x * y = \left( x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n, t + s + \frac{1}{2} \sum_{i=1}^{2n} (x_i' y_{n+i} - x_{n+i} y_i) \right), \]

(24)

where \( x = (x_1, x_2, \cdots, x_{2n+1}), y = (y_1, y_2, \cdots, y_{2n+1}, s) \in \mathbb{R}^{2n+1} \) leads to the Heisenberg group \( \mathbb{H}^n \). The left invariant vector fields generated by commutation the Lie algebra on \( \mathbb{H}^n \) are

\[ X_i = \partial_{x_i} - \frac{x_{n+i}}{2} \partial_t, X_{n+i} = \partial_{x_{n+i}}, 1 ≤ i ≤ n, \]

(25)

and the only nontrivial commutator of such fields is

\[ T = \partial_{x_1} = \{ X_1, X_{n+1} \} = \{ X_1, X_{n+2} \} = \{ X_{n+1}, X_{n+2} \}, 1 ≤ i ≤ n. \]

(26)

We call that \( X_1, X_2, \cdots, X_{2n} \) are the vertical vector fields on \( \mathbb{H}^n \) and \( T \) the vertical vector field. Denote the horizontal gradient of a smooth function \( u \) on \( \mathbb{H}^n \) by

\[ \nabla_{H} u = (X_1 u, X_2 u, \cdots, X_{2n} u). \]

(27)

The homogeneous dimension of \( \mathbb{H}^n \) is \( ρ = 2n + 2 \). The Haar measure in \( \mathbb{H}^n \) is equivalent to the Lebesgue measure in \( \mathbb{R}^{2n+1} \). We denote the Lebesgue measure of a measurable set \( E \subset \mathbb{H}^n \) by \( |E| \).

The Carnot-Carathéodory metric (C-C metric) between two points in \( \mathbb{H}^n \) is the shortest length of the horizontal curve joining them, denoted by \( d \). The ball induced by the C-C metric is

\[ B_ρ(x) = \left\{ y ∈ \mathbb{H}^n : d(y, x) < ρ \right\}. \]

(28)

For \( x = (x_1, x_2, \cdots, x_{2n+1}, t) ∈ \mathbb{H}^n \), its Korányi metric is denoted by

\[ ||x||_{\mathbb{H}^n} = \left( \sum_{i=1}^{2n} x_i^2 + t^2 \right)^{1/4}. \]

(29)

The C-C metric \( d \) is equivalent to the Korányi metric

\[ d(x, y) = ||x^{-1} * y||_{\mathbb{H}^n}. \]

(30)

For \( 1 ≤ p < ∞ \), \( Ω ⊂ \mathbb{H}^n \), the horizontal Sobolev space \( HW^{k,p}(Ω) \) is defined as
which is a Banach space under the norm

$$\|u\|_{H^k_p(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{m=1}^{k} \|\nabla^m u\|_{L^q(\Omega)}.$$  

(32)

The local horizontal Sobolev space $H^k_p_{\text{loc}}(\Omega)$ is defined as

$$H^k_p_{\text{loc}}(\Omega) = \{u : L^p(\Omega), \forall \Omega' \subset \subset \Omega\},$$

(33)

and the space $H^k_p(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^k_p(\Omega)$.

Similar to the definition in [20], Morrey space and Campanato space on Heisenberg group are defined as follows.

**Definition 3 (Morrey space).** Let $1 \leq p < \infty$ and $\beta \geq 0$. For the function $g \in L^p(\Omega)$, if

$$\|g\|_{L^p(\Omega)} = \left( \sup_{x \in \Omega, 0 < r < \text{diam} \Omega} r^{-\beta} \int_{\Omega(x,r)} |g(y)|^p \, dy \right)^{1/p} \in C^\infty(\Omega),$$

(34)

then, we say that $g$ belongs to the Morrey space denoted by $L^p_\beta(\Omega)$, where $\Omega(x,r) = \Omega \cap B_r(x)$.

**Definition 4 (Campanato space).** Let $1 \leq p < \infty$ and $\beta \geq 0$. For the function $g \in L^p(\Omega)$, if

$$\|g\|_{L^p(\Omega)} = \left( \sup_{x \in \Omega, 0 < r < \text{diam} \Omega} r^{-\beta} \int_{\Omega(x,r)} |g(y) - (g)_{\Omega(x,r)}|^p \, dy \right)^{1/p} \in C^\infty(\Omega),$$

(35)

then, we say that $g$ belongs to the Campanato space denoted by $L^p_\beta(\Omega)$, and its norm is defined as

$$\|g\|_{L^p_\beta(\Omega)} = \|g\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$  

(36)

**Lemma 5 (see [21, 22]).** If for any $1 < p < \infty$, $0 < \alpha < 1$, we have $g \in L^{p+\alpha p}(\Omega)$, then $g \in C^{\alpha p}(\Omega)$.

**Lemma 6 (Sobolev inequality, [23]).** For $B_r \subset \mathbb{H}^n$, $1 \leq q < \varphi = 2n + 2$ and for any $u \in H^k_p(\Omega)$, it holds

$$\left( \frac{1}{|B_r|} \int_{B_r} |u|^{(p\varphi)/(\varphi-q)} \, dx \right)^{1/q} \leq C \left( \frac{1}{|B_r|} \int_{B_r} |\nabla u|^q \, dx \right)^{1/q},$$

(37)

where $c = c(p,q) > 0$.

Then, the following four lemmas are true.

For the proof of Lemma 6, see [24] and [25].

2.2. Horizontal Affine Function and Some Lemmas. Let $u \in L^2(B_r(x_0), \mathbb{R}^N)$, $x$ and $x_0 \in \mathbb{R}^{2n+1}$. Denote the horizontal components of $x$, $x_0$ by

$$x' = (x_1, \cdots, x_n),$$

$$x_0' = (x_0^0, \cdots, x_0^n).$$

(38)

Let $l : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^N$ be a horizontal affine function. Following [13], if the horizontal affine function

$$l(x_0', x') = l_{x_0}(x') + \nabla_H l_{x_0}(x' - x_0'),$$

(39)

then, we have

$$l_{x_0}(x_0') = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \, dx,$$

$$\nabla_H l_{x_0}(x_0') = \frac{c_0}{c_0^p} \int_{B_r(x_0)} u \, dx,$$

(40)

where $u \otimes (x' - x_0')$ stands for the matrix $[u^k (x_0^i - x_0^j)]_{N \times 2n}$, $k = 1, \cdots, N, i = 1, \cdots, n$, and $c_0$ is a positive constant defined as

$$c_0 = \frac{1}{\pi} \frac{\int_0^n \sin \theta \cos \theta \, d\theta}{\int_0^n \sin \theta \, d\theta} = \begin{cases} \frac{0}{\pi}, & n = 2m - 1, \\ \frac{2}{(2m - 1)!}, & n = 2m, \end{cases}$$

(42)

According to the meaning of $l_{x_0}$, one has the following Poincaré inequality ([13]):

$$\left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u - l_{x_0}(x')|^p \, dx \right)^{1/p} \leq C_p \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\nabla_H u - \nabla_H l_{x_0}(x')|^q \, dx \right)^{1/q},$$

(43)

where $1 < q < \varphi$, $1 \leq p \leq (\varphi q)/(q - \varphi)$.

Throughout the paper, we define

$$V(\zeta) = (1 + |\zeta|^2)^{(p-2)/4}, 1 < p < 2, \zeta \in \mathbb{R}^N.$$  

(44)

**Lemma 7 (see [26]).** For any $\zeta_1, \zeta_2 \in \mathbb{R}^N$ and $s > 0$, it holds
Lemma 8 (Sobolev-Poincaré-type inequality, [15]). Let \( 1 < p < 2 \) and \( u \in H^{1,p}(B_r(x_0), \mathbb{R}^N) \) with \( B_r(x_0) \subset O \). Then, it follows

\[
\left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(u-u_{x_0})|^2 \frac{p}{p+2} dx \right)^{\frac{p}{p+2}} \leq C_p \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(V_H u)|^2 dx \right)^{\frac{1}{2}}
\]

where \( p^* = \frac{p}{p-1} \) and \( C_p \) depends only on \( \varphi, N, p \). In particular, we have

\[
\left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} V(u-u_{x_0})^2 dx \right)^{\frac{1}{2}} \leq C_p \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(V_H u)|^2 dx \right)^{\frac{1}{2}}
\]

Let \( A \in \text{Bil}(O \times \mathbb{R}^N \times \mathbb{R}^{2n+N}, \mathbb{R}^{2n+N}) \) be a bilinear form with constant tensorial coefficients. We recall that a map \( h \in C^\infty(B_r(x_0), \mathbb{R}^N) \) is \( A \)-harmonic if and only if it holds

\[
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} A(V_H h, V_H \phi) dx = 0
\]

for any testing function \( \phi \in C_0^\infty(B_r(x_0), \mathbb{R}^N) \).

Lemma 9 (see [15]). Let \( h \in H^{1,1}(O, \mathbb{R}^N) \) be a weak solution of the constant coefficient system

\[
- \sum_{i=1}^{2n} X_i A_i^k (V_H h) = 0, \quad k = 1, \ldots, N.
\]

Then, \( h \) is smooth and there exists \( c \geq 1 \) such that for any \( B_r(x_0) \subset \Omega \),

\[
\sup_{B_{cr}(x_0)} \left( |V_H h|^2 + |V_{H}^2 h|^2 \right) \leq c r^{-2} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V_H h|^2 dx.
\]

Lemma 10 (see [15]). Given \( 0 < v \leq L, 1 \leq p < 2 \), for any \( \epsilon > 0 \), there exist constants \( \rho \in [0, 1] \) and \( \delta = \delta(p, N, \rho, v, L, \epsilon) \in (0, 1) \) and a bilinear form \( A \) on \( \mathbb{R}^{2n+N} \) satisfying that for \( P, \bar{P} \in \mathbb{R}^{2n+N} \),

\[
A(P, P) \geq v |P|^2, \quad A(P, \bar{P}) \leq L |P||\bar{P}|.
\]

If \( w \in H^{1,p}(B_r(x_0), \mathbb{R}^N) \) is an approximate \( A \)-harmonic map, i.e., for any \( \phi \in C_0^\infty(B_r(x_0), \mathbb{R}^N) \), it holds

\[
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} A(V_H w, V_H \phi) dx \leq \delta \rho \sup_{B_r(x_0)} |V_H \phi|,
\]

\[
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(V_H w)| dx \leq \rho^2,
\]

then, there exists an \( A \)-harmonic map \( h \in C^\infty(B_r(x_0), \mathbb{R}^N) \) satisfying

\[
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(w - \rho h)|^2 dx \leq \rho^2 \epsilon, \quad \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(V_H h)|^2 dx \leq 1.
\]

3. Some Lemmas

For convenience, we introduce some notations:

\[
p' = \frac{p}{p-1},
\]

\[
p^* = \frac{pp}{p-p'},
\]

\[
(p^*)' = \frac{p^*}{p^*-1},
\]

\[
f(x_0, r) = r^2 \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V_H u|^p + |u|^{p^*} + 1 \right) dx \left( \frac{p}{p(p^*)'} \right).
\]

\[
\Phi(r) = \Phi(x_0, r, l) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |V(V_H u - V_H l)|^2 dx,
\]

\[
\Psi(r) = \Psi(x_0, r, l) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( \frac{u - l}{r} \right)^2 dx + \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \frac{|u - l|^2}{r} dx.
\]
\[
\Psi_*(r) = \Psi_*(x_0, r, l) = \Psi(x_0, r, l) + \omega \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u - l(x'_0)|^p \right) dx + \nu(r) + f(x_0, r),
\]  

where

\[
\nu(r) = \sup_{x_0 \in \Omega_{0(r)}} \sup_{\Omega} \left( v_{x_0}(x, r') \right) B_*(x_0) \cap \Omega. \tag{55}
\]

**Lemma 11 (Caccioppoli-type inequality).** Let \( u \in H^{1,2}(\Omega, \mathbb{R}^N) \) be a weak solution to (5) under the assumptions (H1)-(H4) and (HC). Then, for any \( x_0 = (x'_0, t_0) \in \Omega, B_{r}(x_0) \subset \subset \Omega \) and the known affine function \( l : \mathbb{R}^m \rightarrow \mathbb{R}^N \) with \( |l(x'_0)| + |\nabla_{\Omega} h| \leq M_0 \), we have

\[
\Phi \left( \frac{x_0}{r}, \frac{l}{r} \right) \leq C \Psi_*(x_0, r, l), \tag{56}
\]

where \( C_\epsilon \) is a positive constant depending on \( \varphi, p, \lambda, \Lambda, M_0 \).

**Proof.** We choose a standard cut-off function \( \eta \in C_0^\infty(B_{r/2}(x_0), [0, 1]) \) with \( \eta \equiv 1 \) on \( B_{r/2}(x_0) \) and

\[
|\nabla_{\Omega} \eta| \leq \frac{4}{r}, \quad |T \eta| \leq \frac{c}{r^2}. \tag{57}
\]

Taking a testing function \( \varphi = \eta^2 (u - l) \) in (17), we have

\[
\begin{aligned}
\int_{B_{r/2}(x_0)} \eta^2 A^k_\epsilon(x, u, \nabla_{\Omega} h)(\nabla_{\Omega} h - \nabla_{l}) dx &= -2 \int_{B_{r/2}(x_0)} \eta A^k_\epsilon(x, u, \nabla_{\Omega} h)(u - l) \nabla_{\Omega} \eta dx + \int_{B_{r/2}(x_0)} \eta^2 B_\epsilon^k(x, u, \nabla_{\Omega} h)(u - l) dx \\
&- \int_{B_{r/2}(x_0)} u \cdot T(\eta^2(u - l)) dx.
\end{aligned} \tag{58}
\]

Dividing the equality above by the measure of the ball, it yields

\[
\begin{aligned}
\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 A^k_\epsilon(x, u, \nabla_{\Omega} h)(\nabla_{\Omega} h - \nabla_{l}) dx &= -2 \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta A^k_\epsilon(x, u, \nabla_{\Omega} h)(u - l) \nabla_{\Omega} \eta dx + \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 B_\epsilon^k(x, u, \nabla_{\Omega} h)(u - l) dx \\
&- \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} u \cdot T(\eta^2(u - l)) dx.
\end{aligned} \tag{59}
\]

Note that \( (A^k_\epsilon, l(l(x'_0), \nabla_{\Omega} h))_{x_0} \), is a constant, so it infers by using the integration by parts that

\[
0 = \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left( A^k_\epsilon \left( \cdot, l(l(x'_0), \nabla_{\Omega} h) \right) \right)_{x_0} \nabla_{\epsilon} \phi dx. \tag{60}
\]

Owing to

\[
\begin{aligned}
&- \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 A^k_\epsilon(x, u, \nabla_{\Omega} h)(\nabla_{\Omega} h - \nabla_{l}) dx = 2 \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta A^k_\epsilon(x, u, \nabla_{\Omega} h)(u - l) \nabla_{\Omega} h \eta dx \\
&- \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} A^k_\epsilon(x, u, \nabla_{\Omega} h) \nabla_{\epsilon} \phi dx,
\end{aligned} \tag{61}
\]

we substitute the left and right hand sides of (60) and (61) into the left and right hand sides of (59), respectively, to obtain

\[
\begin{aligned}
&\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 \left( A^k_\epsilon(x, u, \nabla_{\Omega} h) - A^k_\epsilon(x, u, \nabla_{\Omega} h) \right)(\nabla_{\Omega} h - \nabla_{l}) dx \\
&= 2 \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta A^k_\epsilon(x, u, \nabla_{\Omega} h)(u - l) \nabla_{\Omega} h \eta dx - \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} A^k_\epsilon(x, u, \nabla_{\Omega} h) \nabla_{\epsilon} \phi dx \\
&+ \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left( A^k_\epsilon \left( l(l(x'_0), \nabla_{\Omega} h) \right) \right)_{x_0} \nabla_{\epsilon} \phi dx + \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 B_\epsilon^k(x, u, \nabla_{\Omega} h)(u - l) dx \\
&- \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} u \cdot T(\eta^2(u - l)) dx.
\end{aligned} \tag{62}
\]

Then, adding and subtracting the same term \( (1/|B_{r/2}(x_0)|) \int_{B_{r/2}(x_0)} A^k_\epsilon(x, l(l(x'_0), \nabla_{\Omega} h)) \nabla_{\epsilon} \phi dx \) on the right hand side of the above equality, it gets

\[
\begin{aligned}
I_5 &= \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 \left( A^k_\epsilon(x, u, \nabla_{\Omega} h) - A^k_\epsilon(x, u, \nabla_{\Omega} h) \right)(\nabla_{\Omega} h - \nabla_{l}) dx \\
&= 2 \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta A^k_\epsilon(x, u, \nabla_{\Omega} h)(u - l) \nabla_{\Omega} h \eta dx - \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} A^k_\epsilon(x, u, \nabla_{\Omega} h) \nabla_{\epsilon} \phi dx \\
&+ \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left( A^k_\epsilon \left( l(l(x'_0), \nabla_{\Omega} h) \right) \right)_{x_0} \nabla_{\epsilon} \phi dx + \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \eta^2 B_\epsilon^k(x, u, \nabla_{\Omega} h)(u - l) dx \\
&- \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} u \cdot T(\eta^2(u - l)) dx = 2I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{63}
\]

The treatments to the terms \( I_0, I_1, \cdots, I_5 \) in (63) are similar to that of Lemma 4.1 in [15], and we simply write the processes of proofs. By (7), (44) and the known inequality \( (1 + |a| + |b - a|)^2 \leq 3 (1 + |a|^2 + |b - a|^2) \), it gains
By (8), \(|\nabla_H \eta| \leq 4/r\), Young's inequality and Lemma 7, we have

\[
I_1 \leq 2 \epsilon \frac{1}{|B(x_0)|} \int_{B(x_0)} \eta^2 |(\nabla_H u - \nabla_H l)|^2 \, dx
\]

\[
+ c(p, \lambda, M_0) \epsilon^{1/(1-p)} \frac{1}{|B(x_0)|} \int_{B(x_0)} \left| \frac{u-l}{r} \right|^2 \, dx.
\]

Using (11), Young's inequality and Lemma 7, it follows

\[
I_2 \leq 2 \epsilon \frac{1}{|B(x_0)|} \int_{B(x_0)} \eta^2 |(\nabla_H u - \nabla_H l)|^2 \, dx
\]

\[
+ 2 \epsilon \frac{1}{|B(x_0)|} \int_{B(x_0)} \left| \frac{u-l}{r} \right|^2 \, dx
\]

\[
+ c(p, \lambda, M_0) \epsilon^{1/(1-p)} \nu(r).
\]

We have by using (14), Hölder's inequality, Lemma 6, Young's inequality and Lemma 7 that

\[
I_4 \leq 4 \epsilon^2 \frac{1}{|B(x_0)|} \int_{B(x_0)} \eta^2 |(\nabla_H u - \nabla_H l)|^2 \, dx
\]

\[
+ 4 \epsilon^2 \frac{1}{|B(x_0)|} \int_{B(x_0)} \left| \frac{u-l}{r} \right|^2 \, dx
\]

\[
+ c(p, \epsilon) \rho^{p/(p')} \left( \frac{1}{|B(x_0)|} \right) \int_{B(x_0)} \left| \nabla_H u^p + |u|^p + 1 \right| \, dx^{2/(p')}.
\]

The remaining task is to deal with \(I_5\). Noting \(l\) is independent of \(t\) and so

\[
I_{5} = \int_{B(x_0)} u \cdot T(\eta^2(u-l)) \, dx
\]

\[
- \int_{B(x_0)} (u-l) \cdot T(\eta^2(u-l)) \, dx
\]

\[
\leq \frac{c}{|B(x_0)|} \int_{B(x_0)} \left| \frac{u-l}{r} \right|^2 \, dx.
\]

Now, substituting (64)–(70) into (63), and taking \(\epsilon\) small enough, it implies

\[
\frac{1}{|B(x_0)|} \int_{B(x_0)} |(\nabla_H u - \nabla_H l)|^2 \, dx
\]

\[
\leq C_1 \left[ \frac{1}{|B(x_0)|} \int_{B(x_0)} \left| \frac{u-l}{r} \right|^2 \, dx \right]^{1/2} \left[ \frac{1}{|B(x_0)|} \int_{B(x_0)} \left| \frac{u-l}{r} \right|^2 \, dx \right]^{1/2}
\]

\[
+ C_2 \epsilon^{p/(p')} \left( \int_{B(x_0)} \left| \nabla_H u^p + |u|^p + 1 \right| \, dx \right)^{1/(p')}
\]

\[
+ C_3 \epsilon^2 \left( \int_{B(x_0)} \left| \nabla_H u^p + |u|^p + 1 \right| \, dx \right)^{2/(p')}.
\]

Then, (56) is proved.

**Lemma 12** (approximately A-harmonic lemma). Assume the assumptions of Theorem 1 are satisfied. For \(B_{r_0}(x_0) \subset \Omega \) with \(r \leq r_0\) and a horizontal affine function \(l : \mathbb{R}^n \rightarrow \mathbb{R}^N \) with \(|l(x_0')| + |\nabla_H l| \leq M_0\), we define

\[
A = \left( D_p A_k^l \left( \cdot, l(x_0'), \nabla_H l \right) \right)_{x_0'},
\]

\[
w = u - l
\]

then, for all \(\phi \in C_{0}^{\infty}(B_{r_0}(x_0), \mathbb{R}^N)\), it follows

\[
\left| \frac{1}{|B_{r_0}(x_0)|} \int_{B_{r_0}(x_0)} A(\nabla_H w, \nabla_H \phi) \, dx \right| \leq c_1 \left[ \Psi_{\epsilon}(2r) + \Psi_{\epsilon}(2r) + \delta(\Psi_{\epsilon}(2r)) + \delta(\Psi_{\epsilon}(2r)) \right] \sup_{B_{r_0}(x_0)} |\nabla_H \phi|,
\]

where \(c_1 = C(p, M_0, \lambda, C_4)\). Here, we say that \(w\) is an approximately A-harmonic map.
The treatment of \( J_1 \) is similar to that of Lemma 4.2 in [15]. In fact, we use (10), the monotonicity of \( \delta \), Lemma 7, Young’s inequality, Jensen’s inequality and Hölder’s inequality to gain

\[
J_1 \leq C(p, M_0, A) \left[ \Phi(r) + \delta(\Phi^{1/2}(r)) + \delta(\Phi^{1/3}(r)) \right] \sup_{B_r(x_0)} |\nabla_H \phi|.
\]

(75)

Now, let us estimate \( J_2 \). Since

\[
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( A^\delta(x, \cdot, \nabla_H u) \right)_{x_0,r} \nabla_H \phi dx = 0,
\]

(76)

we see

\[
J_2 = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( A^\delta(x, \cdot, \nabla_H u) \right)_{x_0,r} \nabla_H \phi dx = 0.
\]

(77)

Noting from (17) that

\[
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} A^\delta(x, u, \nabla_H u) \nabla_H \phi dx + \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \cdot T \phi dx
\]

\[
- \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} B^\delta(x, u, \nabla_H u) \phi dx = 0,
\]

(78)

so we have

\[
J_2 = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( A^\delta(x, \cdot, \nabla_H u) \right)_{x_0,r} \nabla_H \phi dx
\]

\[
- \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} A^\delta(x, \cdot, \nabla_H u) \nabla_H \phi dx
\]

\[
+ \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} B^\delta(x, u, \nabla_H u) \phi dx = 0.
\]

(79)

The treatments of \( J_{21} \) and \( J_{22} \) are similar to that of Lemma 4.2 in [15]. To be specific, by (11), Lemma 7, Young’s inequality and \(|l(x')| + |\nabla_H l| \leq M_0\), one has

\[
J_{21} \leq C(p, M_0, A) |\psi(r) + \Phi(r) - \Phi(r)| \sup_{B_r(x_0)} |\nabla_H \phi|.
\]

(80)

We use (9), Young’s inequality, Jensen’s inequality and Lemma 7 to get

\[
J_{22} \leq C(p, M_0, A) \left[ \omega \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u - l(x')|^p dx \right) + \Phi(r) \right] \sup_{B_r(x_0)} |\nabla_H \phi|.
\]

(81)

It is worth noting that the treatments of \( J_{23} \) and \( J_{24} \) are different from that in [15]. Using the assumption (14), Hölder’s inequality and Lemma 6, we obtain

\[
J_{23} \leq c \epsilon \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( |\nabla_H u|^p + |u|^p + 1 \right)^{1/p} \right) \frac{\phi dx}{\epsilon^{1/p}}
\]

\[
\leq \epsilon \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( |\nabla_H u|^p + |u|^p + 1 \right) dx \right)^{1/p}
\]

\[
\cdot \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( \phi \right)^{1/p} dx \right)^{1/p}
\]

\[
\leq \epsilon \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left( |\nabla_H u|^p + |u|^p + 1 \right) dx \right)^{1/p}
\]

\[
\cdot \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\nabla_H \phi|^p dx \right)^{1/p} \leq c \nu^{1/2}(r) \sup_{B_r(x_0)} |\nabla_H \phi|.
\]

(82)

Noting

\[
T = X_{x^{n+1}} - X_{x^{n+1}}
\]

(83)
In order to deal with (1/|B_r(x_0)|) \int_{B_r(x_0)} |\nabla_H(u-l)| dx, we denote
\begin{align*}
\Omega_1 &= \{ |\nabla_H(u-l)| \leq 1 \} \cap B_r(x_0), \\
\Omega_2 &= \{ |\nabla_H(u-l)| > 1 \} \cap B_r(x_0),
\end{align*}
so
\begin{align*}
\frac{1}{|B_r(x_0)|} \int_{\Omega_1} |\nabla_H(u-l)| dx &= \frac{1}{|B_r(x_0)|} \int_{\Omega_1} |\nabla_H(u-l)| dx \\
+ \frac{1}{|B_r(x_0)|} \int_{\Omega_2} |\nabla_H(u-l)| dx &\leq \left( \frac{1}{\Omega_1} \int_{\Omega_1} |\nabla_H(u-l)|^2 dx \right)^{1/2} \\
+ \left( \frac{1}{\Omega_2} \int_{\Omega_2} |\nabla_H(u-l)|^2 dx \right)^{1/2} \\
&\leq \left( \frac{1}{|B_r(x_0)|} \int_{\Omega_1} |\nabla_H(u-l)|^2 dx \right)^{1/2} \\
+ \left( \frac{1}{|B_r(x_0)|} \int_{\Omega_2} |\nabla_H(u-l)|^2 dx \right)^{1/2} \\
&\leq c(\Phi^{1/2}(r) + \Phi^{1/p}(r)).
\end{align*}

Then,
\begin{equation}
J_{24} \leq c(\Phi^{1/2}(r) + \Phi^{1/p}(r)) \sup_{B_r(x_0)} |\nabla_H \Phi|.
\end{equation}

Finally, we substitute (75) and (88) into (74) and then use Lemma 11 to get
\begin{align*}
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} A(\nabla_H u, \nabla_H \Phi) dx &\leq C(p, M_0, \Lambda) \left[ \Phi(r) + \Phi^{1/2}(r) + \Phi^{1/p}(r) \right] \\
+ \Phi^{1/2}(r) + \Phi^{1/p}(r) + \Phi^{1/2}(r) + \Phi^{1/p}(r) \sup_{B_r(x_0)} |\nabla_H \Phi|,
\end{align*}
where (ii) \( \rho = 0 \).

**Lemma 13** (decay estimate). Assume the assumptions of Theorem 1 are satisfied and \( B_r(x_0) \subset 2 \) with \( r \leq r_0 \). For constants \( \theta \in (0, 1/4], \delta = \delta(\varphi, N, p, v, L, \theta) \in (0, 1] \) from the A-harmonic approximation Lemma 10, we impose the following smallness conditions:
\begin{itemize}
  \item[(i)] \( \Psi_0^{1/2}(r) < \delta/2 \);
  \item[(ii)] \( \rho = \sqrt{\Psi_0(r) + \delta/(2^8)} \Phi^{1/2}(r) + \Phi^{1/2}(r) + \Phi^{1/p}(r) + \Phi^{1/p}(r) \leq 1 \).
\end{itemize}

Then, it holds
\begin{equation}
\Psi(x_0, \theta r, l_{x_0, \theta r}, \tilde{\omega}) = c_0 \theta^2 \Psi_0(x_0, r, l_{x_0, r}),
\end{equation}
where \( l_{x_0, \theta r} \) and \( l_{x_0, r} \) denote the minimizing horizontal affine functions with \( |l_{x_0, \theta r}(x_0')| + |\nabla_H l_{x_0, \theta r}| \leq M_0 \) and \( |l_{x_0, r}(x_0')| + |\nabla_H l_{x_0, r}| \leq M_0 \), respectively, and the constant \( c_0 \) depends only on \( n, N, p, \lambda, \Lambda, \delta \).

**Proof.** We divide several steps to prove (90).

**Step 1.** Let us take
\begin{equation}
\tilde{\omega} = \frac{u - l_{x_0, r}}{c_2},
\end{equation}
where \( l_{x_0, r} = u_{x_0, r} + \nabla_H l_{x_0, r}(x' - x'_0), c_2 = 0 \). We first claim that \( \tilde{\omega} \) satisfies the assumptions (51) and (52) Lemma 10.

In fact, for \( l = l_{x_0, r} \) and any \( \phi \in \mathcal{C}_0^\infty(B_r(x_0), \mathbb{R}^N) \), we have by using Lemma 12, Lemma 7 (2), and assumptions (ii) and (i) that
\begin{align*}
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} A(\nabla_H \tilde{\omega}, \nabla_H \phi) dx &\leq c_1 \\
+ \rho \delta \sup_{\tilde{\omega}} |\nabla_H \phi|, \end{align*}
where (ii) \( \rho = 0 \).
Now, it deduces from Lemma 7 (2) and Lemma 11 that
\[
\frac{1}{|B_r(z_0)|} \int_{B_r(z_0)} |V(\nabla H \tilde{w})|^2 dx \\
\leq \frac{1}{c} \left( \frac{1}{|B_r(z_0)|} \int_{B_r(z_0)} |V(H_t u - \nabla H t_{x_r})|^2 dx \right)
\leq \Phi(r/2, I_{2r}) \leq \frac{C_r}{c} \leq \rho^2.
\] (93)

Then, the assumptions (51) and (52) of Lemma 10 are satisfied by (92) and (93).

Using Lemma 10, it follows that there exists an A-harmonic function \( h \in C^\infty(B_{r/2}(x_0), \mathbb{R}^N) \) satisfies
\[
\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |V(\nabla H)^2| dx \leq 1.
\] (94)

and
\[
\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |V(\nabla H)^2| dx \leq 1.
\] (95)

**Step 2.**
We estimate \( \left(1/|B_{r/2}(x_0)|\right) \int_{B_{r/2}(x_0)} V(u - I_{x_0, \theta} / \rho) dx \).

Let us denote
\[
I^b(x) = h_{x_0, \theta} + (\nabla H)_{x_0, \theta} (x' - x_0),
\] (96)

and compute by (3) and (2) of Lemma 7, Lemma 8 and (94) that
\[
\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \frac{\tilde{w} - \rho h}{\theta} \right) \right|^2 dx \\
= \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \frac{\tilde{w} - \rho h}{\theta} - \rho \left( h_{x_0, \theta} - (\nabla H)_{x_0, \theta} (x' - x_0) \right) \right) \right|^2 dx \\
\leq C^2 \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( h_{x_0, \theta} - (\nabla H)_{x_0, \theta} (x' - x_0) \right) \right|^2 dx \\
+ C^2 \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \tilde{w} - \rho h \right) \right|^2 dx
\leq C_p \theta^{-2(2\theta - p)} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \frac{\tilde{w} - \rho h}{\theta} \right) \right|^2 dx \right) \\
+ C_p \theta^{-2(2\theta - p)} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \tilde{w} - \rho h \right) \right|^2 dx \right)
\leq c_p \theta^{-2(2\theta - p)} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \tilde{w} - \rho h \right) \right|^2 dx \right) + c_p \theta^{-2(2\theta - p)} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \tilde{w} - \rho h \right) \right|^2 dx \right).
\] (97)

In order to estimate (97), we need to deal with \( \left(1/|B_{r/2}(x_0)|\right) \int_{B_{r/2}(x_0)} |V(\nabla H)^2| dx \). Noting Lemma 9, it derives \( h \in C^\infty(\Omega, \mathbb{R}^N) \) and
\[
\sup_{B_{r/2}(x_0)} \left( |V H|^2 + |\nabla H|^2 \right) \leq cr^{-2} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \right).
\] (98)

so we have
\[
\sup_{B_{r/2}(x_0)} |\nabla H|^2 \leq cr^{-2} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \right),
\] (99)

\[
|\nabla H| \leq \left( cr^{-2} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \right)^{1/2} \right) = M \text{ in } B_{r/2}(x_0).
\] (100)

Hence, for \( \theta \in (0, 1/4) \), we use Lemma 7 (1), (99), (100), Hölder’s inequality and (95) to obtain
\[
\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |V(\nabla H)^2| dx \leq \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \\
\leq \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \\
\leq cr^{-2} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \right)^{1/2} \\
\leq cr^{-2} M \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^{1/2} dx \right)^{1/2} \\
\leq 2c^2 \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^p dx \right)^{1/2}
\] (101)

By substituting (101) into (97), we get
\[
\frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} \left| V \left( \frac{\tilde{w} - \rho h}{\theta} \right) \right|^2 dx \\
\leq c_p \theta^{-2(2\theta - p)} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \right) + c_p \theta^{-2(2\theta - p)} \left( \frac{1}{|B_{r/2}(x_0)|} \int_{B_{r/2}(x_0)} |\nabla H|^2 dx \right)
\] (102)

Since \( \varepsilon > 0 \) in (102) is arbitrary, we take especially \( \varepsilon = \theta^{p+4} \). Noting
\[
\Psi_*(r) < 1 \text{ and } \frac{2}{p} > 1,
\] (103)

we use (ii) and the monotonicity of \( \Psi \) to obtain
\[ \rho^2 = \Psi_*(r) + \left( \frac{\delta}{2} \right)^2 \left[ \Psi_*^{1/2}(r) + \Psi_*^{1/\rho}(r) \right]^2 + \Theta \Psi_*^{1/2}(r) \right)^2 \leq \Psi_*(r) + 16\delta^{-2} \left( \Psi_*^{1/2}(r) + \Psi_*^{1/\rho}(r) \right)^2 \leq \Psi_*(r) + 64\delta^{-2} \Psi_*(r). \] 

Therefore, it shows from (102) that

\[ \frac{1}{|B_{\theta R}(x_0)|} \int_{B_{\theta R}(x_0)} \left| V \left( \frac{\tilde{w} - \rho h}{\theta r} \right) \right|^2 \, dx \leq c(p, \rho, C) \theta^2 (1 + 64\delta^{-2}) \Psi_*(r). \] 

Now, we substitute \( \tilde{w} = (u - l_{x_0, R})/c_2 \) into (105) and use Lemma 7 (2) to gain

\[ \frac{1}{|B_{\theta R}(x_0)|} \int_{B_{\theta R}(x_0)} \left| V \left( \frac{u - l_{x_0, R} - c_2 \rho h}{\theta r} \right) \right|^2 \, dx \leq \frac{c_2^2 c(p, \rho, C) \theta^2 (1 + 64\delta^{-2}) \Psi_*(r)}{2} \leq c_2^2 c(p, \rho, C) \theta^2 (1 + 64\delta^{-2}) \Psi_*(r). \]

Step 3. We estimate \( \left( 1/|B_{\theta R}(x_0)| \right) \int_{B_{\theta R}(x_0)} (u - l_{x_0, R}) \theta r^2 \, dx \). To do so, let us first deal with \( \left( 1/|B_{\theta R}(x_0)| \right) \int_{B_{\theta R}(x_0)} \left| \tilde{w} - \rho h \right| \theta r^2 \, dx \). Since \( V\left( \tilde{w} - \rho h \right) \) is bounded almost everywhere by (94), we denote its upper bound by \( M_1 \). It implies by Lemma 7 (1) that

1. when \( |\tilde{w} - \rho h|/r \leq 1 \), it follows

\[ \left| \frac{\tilde{w} - \rho h}{r} \right| \leq \sqrt{2} \left| V \left( \frac{\tilde{w} - \rho h}{r} \right) \right| \leq \sqrt{2} M_1 \]

2. when \( |\tilde{w} - \rho h|/r > 1 \), we have

\[ \left| \frac{\tilde{w} - \rho h}{r} \right| \leq \left( \sqrt{2} \left| V \left( \frac{\tilde{w} - \rho h}{r} \right) \right| \right)^{2p} \leq \left( \sqrt{2} M_1 \right)^{2p}. \]

Hence,

\[ \frac{1}{|B_{\theta R}(x_0)|} \int_{B_{\theta R}(x_0)} \left| \frac{u - l_{x_0, R} - c_2 \rho h}{\theta r} \right|^2 \, dx \leq \frac{c_2^2 c(p, \rho, C) \theta^2 (1 + 64\delta^{-2}) \Psi_*(r)}{2} \leq c_2^2 c(p, \rho, C) \theta^2 (1 + 64\delta^{-2}) \Psi_*(r). \]
Step 4. Combining (107) with (114), we derive (90). Lemma 13 is proved.

Before stating a new lemma, we introduce Campanato-type functions. For the fixed Hölder exponent $\alpha \in (0, 1)$, define a Campanato-type function by

$$Y_\alpha(x_0, r) = r^{-\alpha} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u - u_{x_0,r}|^p \, dx, \quad 1 < p < 2.$$  

(115)

We can prove the following lemma from Lemma 13.

Lemma 14 (iteration relations). Assume the assumptions of Theorem 1 are satisfied. For any $\alpha \in (0, 1)$, there exist constants $\epsilon_*, \kappa_*$, $r_*$ and $\theta \in (0, 1/\alpha]$, such that if for $0 < r < r_* \Omega$, one has

$$\Psi(x_0, r, l_{x_0,r}) < \epsilon_* \text{ and } Y_\alpha(x_0, r) < \kappa_* \quad \text{(D_0)}$$

then, for any $k \in \mathbb{N}$, it holds

$$\Psi(x_0, \theta^k r, l_{x_0,\theta^k r}) < \epsilon_* \text{ and } Y_\alpha(x_0, \theta^k r) < \kappa_* \quad \text{(D_k)}$$

Proof. Its proof is similar to Lemma 4.4 in [15]. Actually, \(\Psi_*(\theta^k r) \leq \epsilon_*\) has been proved in [15], so we only need to take $\epsilon_* \leq \min \{(\theta^{k\alpha})^p/8, \theta^{k/4}/(4\delta^2 + 256)\}$ and change the estimate of $\rho(\theta^k r)$ in [15] to

$$\rho(\theta^k r) = \sqrt{\Psi_*(\theta^k r) + \frac{\delta}{2} \left[ \Psi_*(\theta^k r) + \Psi_*(\theta^{2k} r) \right]^2}$$

where $\delta > 0$. For any $0 < r_0 < \text{dist}(x_0, \partial \Omega)$, we use Lemma 8, (43) and Lemma 7 (1) to gain

$$\Psi(x_0, r_0, l_{x_0,r_0}) = \frac{1}{|B_{r_0}(x_0)|} \int_{B_{r_0}(x_0)} \left| V \left( \frac{u - u_{x_0,r_0}}{r_0} \right) \right|^2 dx$$

$$+ \frac{1}{|B_{r_0}(x_0)|} \int_{B_{r_0}(x_0)} \left| \frac{u - u_{x_0,r_0}}{r_0} \right|^2 dx$$

$$\leq C \left[ \frac{1}{|B_{r_0}(x_0)|} \int_{B_{r_0}(x_0)} \left| V H u - V H l_{x_0,r_0} \right|^2 dx \right]$$

$$+ \frac{1}{|B_{r_0}(x_0)|} \int_{B_{r_0}(x_0)} \left| \nabla H u - \nabla H l_{x_0,r_0} \right|^2 dx.$$  

(117)

For any $\alpha \in (0, 1)$ and $r_0 \leq 1$, using

$$l_{x_0,r_0} = u_{x_0,r_0} + \nabla H l_{x_0,r_0} (x' - x_0')$$

with $B_r(x_0) \subset \Omega$, one has

$$\Psi(x_0, r, l_{x_0,r}) < \epsilon_* \text{ and } Y_\alpha(x_0, r) < \kappa_* \quad \text{(D_0)}$$

(118)

4. Proof of Theorem 1

Proof of Theorem 1 is finished with two steps.

Step 1. We prove $u \in C^{\alpha,\beta}_{\text{loc}}(\Omega \setminus \Omega_0, \mathbb{R}^N)$. In fact, by Lebesgue’s differentiation theorem ([27]), we get $\int_1 \int_2 = 0$, so our aim is to show that $u$ is Hölder continuous for every $x_0 \in \Omega \setminus (\sum_1 \cup \sum_2)$. For any $0 < r_0 < \text{dist}(x_0, \partial \Omega)$, we use Lemma 8, (43) and Lemma 7 (1) to gain

(119)
Using the continuity of integrals, it follows that there exists a neighborhood $U \subseteq \Omega$ of $x_0$ so that for any $x \in U$,

$$\Psi(x, r', l_{x_0}, \theta r') < \epsilon_\alpha,$$

$$Y_\alpha(x, r') < \kappa_\alpha.$$ (121)

Then, Lemma 14 shows

$$\Psi(x, \theta r', l_{x_0}, \theta r') < \epsilon_\alpha, Y_\alpha(x, \theta r') < \kappa_\alpha, \forall x \in U, k \in \mathbb{N},$$ (122)

so

$$\sup_{x \in U \cap \{0\}} \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |u - u_{l_{x_0}}|^p \, dy = \sup_{x \in U \cap \{0\}} Y_\alpha(x, r) < \kappa_\alpha < \infty,$$ (123)

i.e., $u \in L^{p_\alpha}(U, \mathbb{R}^{2nN})$. Therefore, we have $u \in C^0_{\text{loc}}(U, \mathbb{R}^N)$ from Lemma 5.

Step 2. We prove $\nabla_H u \in L^p(\Omega \setminus \Omega_k, \mathbb{R}^{2nN})$. For $\alpha \in (0, 1)$, it implies by Lemma 7 (1) that

(1) if $|\nabla_H u - \nabla_H l_{x_0}| > 1$, then $|\nabla_H u - \nabla_H l_{x_0}|^p \leq 2 |V(V_H u - V_H l_{x_0})|^2$,

(2) if $|\nabla_H u - \nabla_H l_{x_0}| \leq 1$, then $|\nabla_H u - \nabla_H l_{x_0}|^p \leq 1$.

so

$$\sup_{x \in U \cap \{0\}} r^{p(1-\alpha)} \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |V_H u - V_H l_{x_0}|^p \, dy \leq \sup_{x \in U \cap \{0\}} r^{p(1-\alpha)}$$

\begin{align*}
&\cdot \sup_{x \in U \cap \{0\}} \frac{2}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |V(V_H u - V_H l_{x_0})|^2 \, dy + 1 \\
&\leq \sup_{x \in U \cap \{0\}} r^{p(1-\alpha)} \frac{2}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |V(V_H u - V_H l_{x_0})|^2 \, dy + 1 \\
&\leq \sup_{x \in U \cap \{0\}} r^{p(1-\alpha)} [2C_{\Psi}(Y(x, r), l_{x_0}) + \omega(Y(x, r)) + \nu(r)] + f(x, r) + 1 < \infty.
\end{align*} (124)

Thus,

$$\sup_{x \in U \cap \{0\}} r^{-p(1-\alpha)} \int_{B_\epsilon(x)} |V_H u|^p \, dy \leq C_p \sup_{x \in U \cap \{0\}} r^{p(1-\alpha)}$$

\begin{align*}
&\cdot \left[ \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |V_H u - V_H l_{x_0}|^p \, dy + |V_H l_{x_0}|^p \right] \\
&\leq C_p \sup_{x \in U \cap \{0\}} r^{p(1-\alpha)} \left[ \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |V_H u - V_H l_{x_0}|^p \, dy + M_{\alpha}^0 \right] < \infty.
\end{align*} (125)

i.e., $\nabla_H u \in L^p(\Omega \setminus \Omega_k, \mathbb{R}^{2nN})$, where $\beta = p - p(1 - \alpha)$. Therefore, Theorem 1 is proved.

**Data Availability**

No data is used.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This work was supported by the National Natural Science Foundation of China (No. 11771354 and No. 12061010) and the National Natural Science Foundation of Jiangxi Province grant 20202BAB201004.

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