

Research Article

*-Ricci Tensor on α -Cosymplectic Manifolds

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In this paper, we study α -cosymplectic manifold M admitting *-Ricci tensor. First, it is shown that a *-Ricci semisymmetric manifold M is *-Ricci flat and a ϕ -conformally flat manifold M is an η -Einstein manifold. Furthermore, the *-Weyl curvature tensor \mathcal{W}^* on M has been considered. Particularly, we show that a manifold M with vanishing *-Weyl curvature tensor is a weak ϕ -Einstein and a manifold M fulfilling the condition $R(E_1, E_2) \cdot \mathcal{W}^* = 0$ is η -Einstein manifold. Finally, we give a characterization for α -cosymplectic manifold M admitting *-Ricci soliton given as to be nearly quasi-Einstein. Also, some consequences for three-dimensional cosymplectic manifolds admitting *-Ricci soliton and almost *-Ricci soliton are drawn.

1. Introduction

In the last few years, theory of almost contact geometry and related topics are an active branch of research due to elegant geometry and applications to physics. Nowadays, many attentions have been drawn towards the study of almost cosymplectic manifolds which are a special class of almost contact manifolds. This notion was initiated by Goldberg and Yano [1], in 1969, and then, a very systematic approach for the study of almost cosymplectic manifolds was carried forward by many geometers. A smooth manifold of $(2n + 1)$ -dimension with the condition $\eta \wedge d\eta^n \neq 0$ for a closed 1-form η is a cosymplectic manifold. A simple example of almost cosymplectic manifolds is given by the products of almost Kaehler manifolds and the real line \mathbb{R} or the circle \mathbb{S}^1 . At this moment, we refer the studies [2–5] and the references therein for a vast and exhaustive survey of the results on almost cosymplectic manifolds.

A new concept of the Ricci tensor named as *-Ricci tensor has been defined by Tachibana [6] and Hamada [7] in complex geometry. Similar to a complex case, the *-Ricci tensor of an almost contact metric manifold has been defined as follows:

$$S^*(E_1, E_2) = \frac{1}{2} Tr(E_3 \longrightarrow R(E_1, \phi E_2)\phi E_3), \quad (1)$$

for all $E_1, E_2 \in TM$, where R is the Riemannian curvature tensor. Naturally, Hamada also considered the notion of *-Einstein manifold. An Hermitian manifold is *-Einstein if we have $g(Q^*E_1, E_2) = \lambda g(E_1, E_2)$, where λ is a constant. Also, in the same study of Hamada, a classification of *-Einstein hypersurfaces was given. On the other hand, for an extension of Hamada's work, we refer to Ivey and Ryan [8]. The concept of the *-Ricci tensor has been studied in contact case. Venkatesha and his group ([9, 10]) recently studied some of the curvature properties on Sasakian manifold and contact metric generalized (κ, μ) -space form using the *-Ricci tensor.

In this study, the *-Ricci tensor within the framework of α -cosymplectic manifolds has been studied. In Section 2, we recall some basic formulas and results concerning α -cosymplectic manifold and *-Ricci tensor, which will be useful in further sections. An α -cosymplectic manifold satisfying *-Ricci semisymmetric and ϕ -conformally flat conditions are studied in Section 3 and shown that a ϕ -conformally flat α -cosymplectic manifold is η -Einstein

and a $*$ -Ricci semisymmetric α -cosymplectic manifold is $*$ -Ricci flat. In next section, the $*$ -Weyl curvature tensor has been studied in the background of α -cosymplectic manifold, and several consequences are noticed. In the last section, we studied a special type of metric called $*$ -Ricci soliton. Here, we have proved some important results of α -cosymplectic manifold admitting $*$ -Ricci soliton.

2. Preliminaries

Here, we are going to recall some general facts on α -cosymplectic manifolds which are relevant to our work.

An almost contact metric manifold of $(2n + 1)$ -dimension is a 5-tuple (M, ϕ, ξ, η, g) with the following resources [11].

$$\begin{aligned} \phi^2 E_1 &= -E_1 + \eta(E_1)\xi, \\ \eta(\xi) &= 1, \\ \phi\xi &= 0, \\ \eta(\phi E_1) &= 0, \\ g(\phi E_1, \phi E_2) &= g(E_1, E_2) - \eta(E_1)\eta(E_2), \\ E_1, E_2 &\in TM, \end{aligned} \quad (2)$$

for a $(1, 1)$ -tensor field ϕ , a characteristic vector field ξ , a 1-form η is dual of ξ , and g is a Riemannian metric. It is easily seen that

$$\begin{aligned} g(\phi E_1, E_2) &= -g(E_1, \phi E_2), \\ g(E_1, \xi) &= \eta(E_1), \\ E_1, E_2 &\in TM. \end{aligned} \quad (3)$$

It is well known that the fundamental 2-form ω is defined by $\omega(E_1, E_2) = g(\phi E_1, E_2)$ on M .

For an almost contact metric manifold M , we have the following classifications ([12, 13]):

- (1) If $d\eta = \omega$, then M is a contact metric manifold
- (2) If $d\eta = 0$ and $d\omega = 0$, then M is an almost cosymplectic manifold [1]
- (3) If $d\eta = 0$ and $d\omega = 2\alpha\eta\wedge\omega$, then M is an almost α -Kenmotsu manifold for a nonzero scalar α

In the contact geometry, the notion is normality that is a contact analogue of the integrability of an almost complex structure. An almost cosymplectic metric manifold being normal, if we have $[\phi, \phi] = 0$ which is the Nijenhuis tensor of the tensor field ϕ , is defined by

$$[\phi, \phi] = \phi^2[E_1, E_2] + [\phi E_1, \phi E_2] - \phi[\phi E_1, E_2] - \phi[E_1, \phi E_2], \quad (4)$$

for all $E_1, E_2 \in TM$. A normal almost cosymplectic manifold is a cosymplectic manifold.

Almost α -cosymplectic manifolds have been defined by Kim and Pak [14] by combining an almost α -Kenmotsu and almost cosymplectic structures by the following formula:

$$\begin{aligned} d\eta &= 0, \\ d\omega &= 2\alpha\eta\wedge\omega, \end{aligned} \quad (5)$$

for a constant α . On an α -cosymplectic manifold, we have

$$(\nabla_{E_1}\phi)E_2 = \alpha(g(\phi E_1, E_2)\xi - \eta(E_2)\phi E_1), \quad (6)$$

where ∇ denotes the Riemannian connection. From (6), it is easy to see that

$$\nabla_{E_1}\xi = -\alpha\phi^2 E_1 = \alpha[E_1 - \eta(E_1)\xi], \quad (7)$$

and

$$(\nabla_{E_1}\eta)E_2 = \alpha[g(E_1, E_2) - \eta(E_1)\eta(E_2)]. \quad (8)$$

On an α -cosymplectic manifold M of dimension $2n + 1$, the following relationships are valid:

$$R(\xi, E_1)E_2 = \alpha^2(\eta(E_2)E_1 - g(E_1, E_2)\xi), \quad (9)$$

$$R(E_1, E_2)\xi = \alpha^2(\eta(E_1)E_2 - \eta(E_2)E_1), \quad (10)$$

$$S(E_1, \xi) = -2n\alpha^2\eta(E_1), \quad (11)$$

where R and S are the curvature and Ricci tensors, respectively.

By the following lemma, we obtain some derivational features of α -cosymplectic manifold.

Lemma 1. *On an α -cosymplectic manifold of dimension $2n + 1$, we have*

$$(\nabla_{E_1}Q)\xi = -\alpha QE_1 - 2n\alpha^3 E_1, \quad (12)$$

$$(\nabla_{\xi}Q)E_1 = -2\alpha QE_1 - 4n\alpha^3 E_1, \quad (13)$$

$$\xi(r) = -2\alpha\{r + 2n(2n + 1)\alpha^2\}. \quad (14)$$

Proof. Note that (11) implies $Q\xi = -2n\alpha^2\xi$, for Q defined by $S(E_1, E_2) = g(QE_1, E_2)$. Differentiating this along E_1 and using (7), we get (12). Next, differentiation of (10) with respect to T gives

$$\begin{aligned} (\nabla_T R)(E_1, E_2)\xi &= -\alpha R(E_1, E_2)T \\ &+ \alpha^3\{g(E_1, T)E_2 - g(E_2, T)E_1\}. \end{aligned} \quad (15)$$

Let $\{e_i\}_{i=1}^{2n+1}$ be a local basis on M . Replacing $E_1 = T = e_i$ in the foregoing equation with summing over i gives

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i}R)(e_i, E_2)\xi, E_3) = \alpha S(E_2, E_3) + 2n\alpha^3 g(E_2, E_3). \quad (16)$$

Using second Bianchi's identity leads to

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} R)(E_3, \xi)E_2, e_i) = g((\nabla_{E_3} Q)\xi, E_2) - g((\nabla_{\xi} Q)E_3, E_2). \tag{17}$$

By considering (16) in (17) and then with the help of (12), we conclude

$$g((\nabla_{\xi} Q)E_1, E_2) = -2\alpha S(E_1, E_2) - 4n\alpha^3 g(E_1, E_2), \tag{18}$$

which proves (13). Finally, contraction of (13) gives (14). \square

From Riemannian geometry, the covariant derivative of a (1, s)-type of tensor field K is given by

$$(\operatorname{div} K)(E_1, E_1, \dots, E_1 s) = \sum_{i=1}^{2n+1} g((\nabla_{e_i} K)(E_1, E_1, \dots, E_1), e_i). \tag{19}$$

for all $E_1, E_1, \dots, E_1 \in TM$, where div is stated for the divergence [15].

By following descriptions, we present some classification facts which come from the Ricci tensor and have been stated.

- (1) An α -cosymplectic manifold M is called by weak ϕ -Einstein if we have

$$S^{\phi}(E_1, E_2) = \beta g^{\phi}(E_1, E_2), \quad \forall E_1, E_2 \in TM, \tag{20}$$

for some function β , where $g^{\phi}(E_1, E_2) = g(\phi E_1, \phi E_2)$, and S^{ϕ} is defined by

$$S^{\phi}(E_1, E_2) = \frac{1}{2} \{S^*(E_1, E_2) + Ric^*(E_2, E_1)\}, \tag{21}$$

$E_1, E_2 \in TM,$

In other words, S^{ϕ} denotes the symmetric part of S^* . If β is constant, then M is called ϕ -Einstein [16].

- (2) M is called near quasi-Einstein manifold if the Ricci tensor is of the form

$$S(E_1, E_2) = a g(E_1, E_2) + b E(E_1, E_2), \quad \forall E_1, E_2 \in TM, \tag{22}$$

where a and b are the nonzero scalars and E is a nonzero (0, 2) tensor [17].

- (3) M is called an η -Einstein manifold if we have

$$S(E_1, E_2) = \alpha g(E_1, E_2) + \gamma \eta(E_1)\eta(E_2), \quad \forall E_1, E_2 \in TM, \tag{23}$$

where α and γ are the constants [18].

By decomposition of Riemannian curvature tensor R , the Weyl conformal curvature tensor \mathcal{W} has been obtained in this way:

$$\begin{aligned} \mathcal{W}(E_1, E_2)E_3 &= R(E_1, E_2)E_3 - \frac{1}{2n-1} \{S(E_2, E_3)E_1 - S(E_1, E_3)E_2 + g(E_2, E_3)QE_1 - g(E_1, E_3)QE_2\} \\ &\quad + \frac{r}{2n(2n-1)} \{g(E_2, E_3)E_1 - g(E_1, E_3)E_2\}. \end{aligned} \tag{24}$$

for all $E_1, E_2, E_3 \in TM$ [15]. It is noted that, Weyl conformal curvature tensor vanishes whenever the metric is conformally identical to a flat metric, and it is one of the important curvature properties on a manifold.

3. *-Ricci Tensor on α -Cosymplectic Manifold

We are in a situation to confer the equation of the *-Ricci tensor in the framework of α -cosymplectic manifolds and then study its various properties. In [19], authors derived the expression of the *-Ricci tensor on α -cosymplectic manifold which is of the following form:

$$S^*(E_1, E_2) = S(E_1, E_2) + (2n-1)\alpha^2 g(E_1, E_2) + \alpha^2 \eta(E_1)\eta(E_2), \tag{25}$$

for all $E_1, E_2 \in TM$.

Note that S^* is not symmetric. By contraction of (25), the *-scalar curvature is specified by

$$r^* = r + 4n^2 \alpha^2. \tag{26}$$

If the *-Ricci tensor S^* is a constant multiple of the Riemannian metric g , then we say that the manifold is *-Einstein. Moreover, the *-scalar curvature is not constant on a *-Einstein manifold.

*3.1. *-Ricci Semisymmetric α -Cosymplectic Manifolds.* An α -cosymplectic manifold M satisfying the condition $R(E_1, E_2) \cdot S = 0$ for all $E_1, E_2 \in TM$ is called Ricci semi symmetric, where $R(E_1, E_2)$ acts as a derivation on S . This notion was introduced by Mirjoyan [20] for Riemann spaces and then studied by many authors. Analogous to this, an α -cosymplectic manifold is called *-Ricci semisymmetric if its *-Ricci tensor satisfies the condition $R(E_1, E_2) \cdot S^* = 0 \forall E_1, E_2 \in TM$.

Theorem 1. *If a $(2n+1)$ -dimensional α -cosymplectic manifold M is *-Ricci semisymmetric, then M is *-Ricci flat. Moreover, it is an η -Einstein manifold, and the Ricci tensor is to be exhibited as*

$$S(E_1, E_2) = (1 - 2n)\alpha^2 g(E_1, E_2) - \alpha^2 \eta(E_1)\eta(E_2), \quad (27)$$

$$E_1, E_2 \in TM.$$

Proof. Let us consider $*$ -Ricci semisymmetric α -cosymplectic manifold M . Then, condition $R(E_1, E_2).S^* = 0$ is equivalent to

$$S^*(R(E_1, E_2)E_3, E_4) + S^*(E_3, R(E_1, E_2)E_4) = 0. \quad (28)$$

Putting $E_1 = \xi$ in (28) and then recalling (9), we have

$$\alpha^2 \{S^*(E_2, E_4)\eta(E_3) - g(E_2, E_3)S^*(\xi, E_4)\} \\ + \alpha^2 \{S^*(E_3, E_2)\eta(E_4) - g(E_2, E_4)S^*(E_3, \xi)\} = 0. \quad (29)$$

It is well known that $Ric^*(E_1, \xi) = 0$. Making use of this in (29), we find

$$\alpha^2 \{S^*(E_2, E_4)\eta(E_3) + S^*(E_3, E_2)\eta(E_4)\} = 0. \quad (30)$$

Again, plugging E_4 by ξ in (30) shows that M is $*$ -Ricci flat, that is, $S^*(E_3, E_2) = 0, E_3, E_2 \in TM$. Moreover, in view of (25) and (30), we have the required result. \square

3.2. ϕ -Conformally Flat α -Cosymplectic Manifolds. An α -cosymplectic manifold M is said to be ϕ -conformally flat if we have

$$\phi^2 \mathcal{W}(\phi E_1, \phi E_2)\phi E_3 = 0, \quad (31)$$

for all $E_1, E_2, E_3 \in TM$. Sasakian manifolds which are ϕ -conformally flat have been studied in [21]. In the following, we study a ϕ -conformally flat α -cosymplectic manifold.

Theorem 2. *An ϕ -conformally flat α -cosymplectic manifold is $*$ - η -Einstein manifold. Moreover, M is weak ϕ -Einstein.*

Proof. Assume that an α -cosymplectic manifold is ϕ -conformally flat. So, it is easy to see that $\phi^2 C(\phi E_1, \phi E_2)\phi E_3 = 0$ carry if and only if $g(C(\phi E_1, \phi E_2)\phi E_3, \phi E_4) = 0 \forall E_1, E_2, E_3, E_4 \in TM$. Hence, ϕ -conformally flat means

$$g(R(\phi E_1, \phi E_2)\phi E_3, \phi E_4) = \frac{1}{2n-1} \{S(\phi E_2, \phi E_3)g(\phi E_1, \phi E_4) - S(\phi E_1, \phi E_3)g(\phi E_2, \phi E_4) + g(\phi E_2, \phi E_3)S(\phi E_1, \phi E_4) \\ - g(\phi E_1, \phi E_3)S(\phi E_2, \phi E_4)\} \\ - \frac{r}{2n(2n-1)} \{g(\phi E_2, \phi E_3)g(\phi E_1, \phi E_4) \\ - g(\phi E_1, \phi E_3)g(\phi E_2, \phi E_4)\}. \quad (32)$$

For a local orthonormal basis of TM with $\{e_1, \dots, e_{2n}, \xi\}$, if we put $E_1 = E_4 = e_i$ in (32) and sum up with respect to i , then we obtain

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi E_2)\phi E_3, \phi e_i) = \frac{1}{2n-1} \sum_{i=1}^{2n} \{g(\phi e_i, \phi e_i)S(\phi E_2, \phi E_3) - g(\phi E_2, \phi e_i)S(\phi e_i, \phi E_3) + g(\phi E_2, \phi E_3)S(\phi e_i, \phi e_i) \\ - g(\phi e_i, \phi E_3)S(\phi E_2, \phi e_i)\} \\ - \frac{r}{2n(2n-1)} \sum_{i=1}^{2n} \{g(\phi e_i, \phi e_i) \\ g(\phi E_2, \phi E_3) - g(\phi e_i, \phi E_3)g(\phi E_2, \phi e_i)\}, \quad (33)$$

and therefore,

$$S(\phi E_2, \phi E_3) - g(R(\xi, E_2)E_3, \xi) = \frac{2(n-1)}{2n-1} S(\phi E_2, \phi E_3) + \frac{1}{2n-1} \left[\frac{r}{2n} + \alpha^2 \right] g(\phi E_2, \phi E_3). \quad (34)$$

From (9), we obtain $g(R(\xi, E_2)E_3, \xi) = -\alpha^2 g(\phi E_2, \phi E_3)$, and hence, from (34), we get

$$S(\phi E_2, \phi E_3) = \left[\frac{r}{2n} + \alpha^2 \right] g(\phi E_2, \phi E_3). \quad (35)$$

Then, from (35), it follows from (32) that

$$g(R(\phi E_1, \phi E_2)\phi E_3, \phi E_4) = \frac{r + 4n\alpha^2}{2n(2n - 1)} \{g(\phi E_2, \phi E_3)g(\phi E_1, \phi E_4) - g(\phi E_1, \phi E_3)g(\phi E_2, \phi E_4)\}. \quad (36)$$

In an α -cosymplectic manifold, in view of (3) and (9) for all $E_1, E_2, E_3, E_4 \in TM$, we can verify that

$$R(\varphi^2 E_1, \varphi^2 E_2, \varphi^2 E_3, \varphi^2 E_4) = R(E_1, E_2, E_3, E_4) + \alpha^2 \{g(E_2, E_3)\eta(E_1)\eta(E_2) - g(E_1, E_3)\eta(E_2)\eta(E_4) + g(E_1, E_4)\eta(E_2)\eta(E_3) - g(E_2, E_4)\eta(E_1)\eta(E_3)\}. \quad (37)$$

Taking $\varphi E_1, \varphi E_2, \varphi E_3, \varphi E_4$ instead of E_1, E_2, E_3, E_4 in (37), respectively, and making use of (36), we obtain

$$R(E_1, E_2, E_3, E_4) = \frac{r + 4n\alpha^2}{2n(2n - 1)} \{g(E_1, E_4)g(E_2, E_3) - g(E_2, E_4)g(E_1, E_3)\} - \frac{r + 2n(2n + 1)\alpha^2}{2n(2n - 1)} \{(g(E_2, E_3)\eta(E_1) - g(E_1, E_3)\eta(E_2))\eta(E_4) + (g(E_1, E_4)\eta(E_2) - g(E_2, E_4)\eta(E_1))\eta(E_3)\}. \quad (38)$$

By the definition of S^* , direct computation yields

$$S^*(E_1, E_2) = \sum_{i=1}^{2n+1} R(E_1, e_i, \phi e_i, \phi E_2) = \beta g(E_1, E_2) - \beta \eta(E_1)\eta(E_2), \quad (39)$$

where $\beta = (r + 4n\alpha^2)/(2n(2n - 1))$ reveals that M is $*-\eta$ -Einstein. In view of (3), we have

$$S^*(E_1, E_2) = \frac{r + 4n\alpha^2}{2n(2n - 1)} g^\phi(E_1, E_2), \quad E_1, E_2 \in TM. \quad (40)$$

Thus, $S^* = S^\phi$, and hence, it is weak ϕ -Einstein. This completes the proof. \square

Next, for a constant scalar curvature of M , in view of (40), we state the following.

Corollary 1. *A ϕ -conformally flat α -cosymplectic manifold of constant scalar curvature is a ϕ -Einstein manifold.*

In an α -cosymplectic manifold, the $*-\text{Ricci}$ tensor is given by (25), and so in view of (40), we state the following.

Corollary 2. *A ϕ -conformally flat α -cosymplectic manifold is η -Einstein.*

Furthermore, an α -cosymplectic manifold M is called to have the η -parallel Ricci tensor if its Ricci tensor S satisfies the condition $(\nabla_{E_1} S)(\phi E_2, \phi E_3) = 0, E_1, E_2, E_3 \in TM$. This notion was introduced in 1976 by Kon [22] in the framework of Sasakian manifolds and then studied by many authors. Analogous to this notion, we state the following:

Definition 1. An α -cosymplectic manifold M is said to have a η -parallel $*-\text{Ricci}$ tensor if its $*-\text{Ricci}$ tensor satisfies the condition $(\nabla_{E_1} S^*)(\phi E_2, \phi E_3) = 0 \forall E_1, E_2, E_3 \in TM$.

Replacing E_1 by ϕE_1 and E_2 by ϕE_2 in (39), we obtain $S^*(\phi E_1, \phi E_2) = \beta g(\phi E_1, \phi E_2)$. Covariant derivative of the foregoing equation with respect to E_4 , we get $(\nabla_{E_4} S^*)(\phi E_1, \phi E_2) = dr(E_4)g(\phi E_1, \phi E_2)$. Therefore, from Definition 1, we have the following.

Corollary 3. *Let M be a $(2n + 1)$ -dimensional ϕ -conformally flat α -cosymplectic manifold. If M admits a*

η -parallel $*$ -Ricci tensor, then M has a constant scalar curvature.

defined recently by Kaimakamis and Panagiotidou [23] in the following way:

4. $*$ -Weyl Curvature Tensor on α -Cosymplectic Manifolds

The notion of $*$ -Weyl curvature tensor \mathcal{W}^* on real hypersurfaces of complex space forms (particularly, nonflat) is

$$\begin{aligned} \mathcal{W}^*(E_1, E_2)E_3 &= R(E_1, E_2)E_3 + \frac{r^*}{2n(2n-1)} \{g(E_2, E_3)E_1 - g(E_1, E_3)E_2\} \\ &\quad - \frac{1}{2n-1} \{g(Q^*E_2, E_3)E_1 - g(Q^*E_1, E_3)E_2 + g(E_2, E_3)Q^*E_1 - g(E_1, E_3)Q^*E_2\}, \end{aligned} \quad (41)$$

for all $E_1, E_2, E_3 \in TM$, where Q^* is the $*$ -Ricci operator and r^* is the $*$ -scalar curvature corresponding to Q^* .

Using (25), we can write

$$Q^*E_1 = QE_1 + \alpha^2\{(2n-1)E_1 + \eta(E_1)\xi\}. \quad (42)$$

With the help of (41), (32), and (42), we obtain the expression for the $*$ -Weyl curvature tensor on $(2n+1)$ -dimensional α -cosymplectic manifold M as

$$\begin{aligned} \mathcal{W}^*(E_1, E_2)E_3 &= \mathcal{W}(E_1, E_2)E_3 - \frac{2(n-1)\alpha^2}{2n-1} \{g(E_2, E_3)E_1 - g(E_1, E_3)E_2\} \\ &\quad - \frac{\alpha^2}{2n-1} \{\eta(E_2)\eta(E_3)E_1 - \eta(E_1)\eta(E_3)E_2 + g(E_2, E_3)\eta(E_1)\xi - g(E_1, E_3)\eta(E_2)\xi\}. \end{aligned} \quad (43)$$

4.1. α -Cosymplectic Manifold with Vanishing $*$ -Weyl Curvature Tensor

Theorem 3. An α -cosymplectic manifold with vanishing $*$ -Weyl curvature tensor is an η -Einstein manifold.

Proof. Let us consider an α -cosymplectic manifold M with vanishing $*$ -Weyl curvature tensor, that is, $\mathcal{W}^*(E_1, E_2)E_3 = 0$. Thus, (43) infers that

$$\begin{aligned} \mathcal{W}(E_1, E_2)E_3 &= \frac{2(n-1)\alpha^2}{2n-1} \{g(E_2, E_3)E_1 - g(E_1, E_3)E_2\} \\ &\quad + \frac{\alpha^2}{2n-1} \{\eta(E_2)\eta(E_3)E_1 - \eta(E_1)\eta(E_3)E_2 + g(E_2, E_3)\eta(E_1)\xi - g(E_1, E_3)\eta(E_2)\xi\}. \end{aligned} \quad (44)$$

Covariant differentiation of above relation along E_4 and then contracting the resultant equation over E_4 yields

$$\operatorname{div} \mathcal{W}(E_1, E_2)E_3 = 2(n-1)\alpha^3 \{g(E_1, E_3)\eta(E_2) - g(E_2, E_3)\eta(E_1)\}, \quad (45)$$

we easily obtain

where “div” denotes the divergence. On the other side, differentiating \mathcal{W} covariantly along E_4 and then contracting with the aid of following well-known formulas,

$$\operatorname{div} R(E_1, E_2)E_3 = g((\nabla_{E_1} Q)E_2, E_3) - g((\nabla_{E_2} Q)E_1, E_3),$$

$$\operatorname{div} Q E_1 = \frac{1}{2} E_1(r), \quad (46)$$

$$\operatorname{div} \mathcal{W}(E_1, E_2)E_3 = \frac{2(n-1)}{2n-1} \{g((\nabla_{E_1} Q)E_2, E_3) - g((\nabla_{E_2} Q)E_1, E_3)\} - \frac{(n-1)}{2n(2n-1)} \{g(E_2, E_3)E_1(r) - g(E_1, E_3)E_2(r)\}. \quad (47)$$

By virtue of (45) and (47), we have

$$\begin{aligned} \frac{2(n-1)}{2n-1} \{g((\nabla_{E_1} Q)E_2, E_3) - g((\nabla_{E_2} Q)E_1, E_3)\} &= 2(n-1)\alpha^3 \{g(E_1, E_3)\eta(E_2) - g(E_2, E_3)\eta(E_1)\} \\ &+ \frac{(n-1)}{2n(2n-1)} \{g(E_2, E_3)E_1(r) - g(E_1, E_3)E_2(r)\}. \end{aligned} \quad (48)$$

Replacing E_2 by ξ in (48), we obtain

$$\begin{aligned} \frac{2(n-1)}{2n-1} \{g((\nabla_{E_1} Q)\xi, E_3) - g((\nabla_{\xi} Q)E_1, E_3)\} &= 2(n-1)\alpha^3 \{g(E_1, E_3) - \eta(E_1)\eta(E_3)\} \\ &+ \frac{(n-1)}{2n(2n-1)} \{\eta(E_3)E_1(r) - g(E_1, E_3)\xi(r)\}. \end{aligned} \quad (49)$$

Recalling Lemma 1 to find

$$\frac{2(n-1)}{2n-1} \alpha \{g(QE_1, E_3) + \alpha^2 g(E_1, E_3)\} + 2(n-1)\alpha^3 \eta(E_1)\eta(E_3) = \frac{n-1}{2n(2n-1)} \{\eta(E_3)E_1(r) - g(E_1, E_3)\xi(r)\}. \quad (50)$$

Writing ξ instead of E_3 by ξ in the foregoing equation and by making use of (11), we derive

$$X(r) = \xi(r)\eta(E_1). \quad (51)$$

Making use of this equation in (50) yields

$$\alpha g(QE_1, E_3) = -\left[\alpha^3 + \frac{1}{4n}\xi(r)\right]g(E_1, E_3) + \left[-(2n-1)\alpha^3 + \frac{1}{4n}\xi(r)\right]\eta(E_1)\eta(E_3). \quad (52)$$

Using (14) in (52), we get

$$S(E_1, E_3) = \left\{ 2n\alpha^2 + \frac{r}{2n} \right\} g(E_1, E_3) + \left\{ -4n\alpha^2 - \frac{r}{2n} \right\} \eta(E_1)\eta(E_3). \quad (53)$$

This proves our result. \square

Substituting (53) in (25), we have

$$S^*(E_1, E_3) = \gamma \{ g(E_1, E_3) - \eta(E_1)\eta(E_3) \}, \quad (54)$$

where $\gamma = (1/2n) + (4n-1)\alpha^2$ shows that M is $*$ - η -Einstein. In view of (3), we obtain

$$S^*(E_1, E_3) = \left[\frac{r}{2n} + (4n-1)\alpha^2 \right] g^\phi(E_1, E_3), \quad E_1, E_3 \in TM. \quad (55)$$

Thus, $S^* = S^\phi$, and hence, it is weak ϕ -Einstein. Thus, we state the following.

Theorem 4. *An α -cosymplectic manifold with vanishing $*$ -Weyl curvature tensor is a weak ϕ -Einstein manifold.*

4.2. α -Cosymplectic Manifold Satisfying the Condition $R(X, Y) \cdot \mathcal{W}^* = 0$. An α -cosymplectic manifold M is called semisymmetric if its curvature tensor satisfies the condition $R \cdot R = 0$. In [24], Szabo studied the intrinsic classification of semisymmetric spaces thoroughly. In this context, Venkatesha and Kumara [21] studied Sasakian manifolds satisfying condition $R(E_1, E_2) \cdot \mathcal{W}^* = 0$. In this section, we make an attempt to study this condition in the framework of α -cosymplectic manifolds and prove the following.

Theorem 5. *An α -cosymplectic manifold satisfying the condition $R(E_1, E_2) \cdot \mathcal{W}^* = 0$ is an η -Einstein manifold.*

Proof. Let M be an $(2n+1)$ -dimensional α -cosymplectic manifold satisfying the condition $(R(E_1, E_2) \cdot \mathcal{W}^*)(U, V)E_4 = 0$. This infers that

$$R(E_1, E_2)\mathcal{W}^*(U, V)E_4 - \mathcal{W}^*(R(E_1, E_2)U, V)E_4 - \mathcal{W}^*(U, R(E_1, E_2)V)E_4 - \mathcal{W}^*(U, V)R(E_1, E_2)E_4 = 0. \quad (56)$$

Plugging ξ in place of E_1 in the previous equation and then picking inner product with ξ for the resultant equation, we obtain

$$\eta(R(\xi, E_2)\mathcal{W}^*(U, V)E_4) - \eta(\mathcal{W}^*(R(\xi, E_2)U, V)E_4) - \eta(\mathcal{W}^*(U, R(\xi, E_2)V)E_4) - \eta(\mathcal{W}^*(U, V)R(\xi, E_2)E_4) = 0. \quad (57)$$

In view of (9), it follows from (56) that

$$\alpha^2 \left\{ \begin{array}{l} \mathcal{W}^*(U, V, E_4, E_2) - g(E_2, U)\eta(\mathcal{W}^*(\xi, V)E_4) - \eta(\mathcal{W}^*(U, V)E_4)\eta(E_2) \\ -g(E_2, V)\eta(\mathcal{W}^*(U, \xi)E_4) + \eta(U)\eta(\mathcal{W}^*(E_2, V)E_4) + \eta(V)\eta(\mathcal{W}^*(U, E_2)E_4) \\ -g(E_2, E_4)\eta(\mathcal{W}^*(U, V)\xi) + \eta(E_4)\eta(\mathcal{W}^*(U, V)E_2) \end{array} \right\} = 0. \quad (58)$$

Replacing E_2 by U in the above equation, we have

$$\mathcal{W}^*(U, V, E_4, U) - g(U, U)\eta(\mathcal{W}^*(\xi, V)E_4) - g(U, V)\eta(\mathcal{W}^*(U, \xi)E_4) + \eta(E_4)\eta(\mathcal{W}^*(U, V)U) = 0, \quad (59)$$

provided $\alpha^2 \neq 0$. By virtue of (43), one can easily see that

$$\eta(\mathcal{W}^*(E_1, E_2)E_3) = -\frac{1}{2n-1}\{g(QE_2, E_3)\eta(E_1) - g(QE_1, E_3)\eta(E_2)\} + \left[\frac{r}{2n(2n-1)} - \frac{2(n-1)}{2n-1}\alpha^2\right] \{g(E_2, E_3)\eta(E_1) - g(E_1, E_3)\eta(E_2)\}, \eta(\mathcal{W}^*(E_1, E_2)\xi) = 0, \tag{60}$$

$$\eta(\mathcal{W}^*(E_1, \xi)E_3) = \frac{1}{2n-1}g(QE_1, E_3) + \frac{1}{2n-1}\left[2(n-1)\alpha^2 - \frac{r}{2n}\right]g(E_1, E_3) + \frac{1}{2n-1}\left[\frac{r}{2n} + 2\alpha^2\right]\eta(E_1)\eta(E_3), \tag{61}$$

$$\sum_{i=1}^{2n+1} \mathcal{W}^*(e_i, E_2, E_3, e_i) = -(2n-1)\alpha^2g(E_2, E_3) - \alpha^2\eta(E_2)\eta(E_3), \tag{62}$$

where $\{e_i\}_{i=1}^{2n+1}$ is an orthonormal basis of the tangent space at any point of the manifold. Taking $U = e_i$ in (59) and summing over i and making use of (56)–(61), we get

$$S(V, E_4) = \frac{1}{2n}\{r + \alpha^2\}g(V, E_4) - \frac{1}{2n}\{r + (4n^2 + 1)\alpha^2\}\eta(V)\eta(E_4). \tag{63}$$

This completes the proof. □

5. α -Cosymplectic Manifolds Admitting * -Ricci Solitons

Hamilton [25] introduced the notion of Ricci solitons as fixed points of the Ricci flows on a Riemannian manifold, and they are also self-similar solutions. These self-similar solutions also generalize Einstein metrics. Ricci solitons also correspond to self-similar solutions of Hamilton’s Ricci flow. A Ricci soliton with a potential vector field V is defined by

$$(E_Vg)(E_1, E_2) + 2S(E_1, E_2) + 2\lambda g(E_1, E_2) = 0, \tag{64}$$

for some constant λ . The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive, respectively. The study of Ricci solitons and almost Ricci solitons on three-dimensional cosymplectic manifolds have been carried out by Wang [26] and De and Dey [27], respectively.

By taking the necessary modification (64), Kaimakamis and Panagiotidou [28] introduced the notion of a special

type of metric called * -Ricci soliton on real hypersurfaces of nonflat complex space forms. A Riemannian metric g on M is called * -Ricci soliton, if the Lie derivative of a vector field V on M is given by

$$(E_Vg)(E_1, E_2) + 2S^*(E_1, E_2) + 2\lambda g(E_1 \cdot E_2) = 0, \forall E_1, E_2 \in TM. \tag{65}$$

Recently, the study of * -Ricci solitons within the context of almost contact and paracontact manifolds were carried out in the studies [18, 29–34] and drawn several interesting results. In this section, we intended to * -Ricci soliton on a α -cosymplectic manifold. Now, we prove the following result.

Theorem 6. *Let M be α -cosymplectic manifold admitting a * -Ricci soliton. If the potential vector field V is pointwise collinear with ξ , then M is a near quasi-Einstein manifold.*

Proof. Let V be a pointwise collinear vector field with ξ . Then, we have $V = b\xi$. From (7) and (65), we derive

$$(E_Vg)(E_1, E_2) = g(\nabla_{E_1}V, E_2) + g(\nabla_{E_2}V, E_1) = E_1(b)\eta(E_2) + E_2(b)\eta(E_1) + 2b\alpha\{g(E_1, E_2) - \eta(E_1)\eta(E_2)\}. \tag{66}$$

Let M be an α -cosymplectic manifold admitting a * -Ricci soliton. Then, from (61) and (62), we obtain

$$S^*(E_1, E_2) = -\{\lambda + b\alpha\}g(E_1, E_2) + b\alpha\eta(E_1)\eta(E_2) - \frac{1}{2}\{E_1(b)\eta(E_2) + E_2(b)\eta(E_1)\}. \tag{67}$$

Let Db be a gradient of smooth function b on M , that is, $E_1(b) = g(Db, E_1)$ and $E_2(b) = g(Db, E_2)$. Then, by denoting the dual form of Db by ν , we write

$$\begin{aligned} E_1(b) &= \nu(E_1), \\ E_2(b) &= \nu(E_2). \end{aligned} \quad (68)$$

By taking account of foregoing equations in (67), we get

$$S^*(E_1, E_2) = -\{\lambda + b\alpha\}g(E_1, E_2) + b\alpha\eta(E_1)\eta(E_2) - \frac{1}{2}\{\nu(E_1)\eta(E_2) + \nu(E_2)\eta(E_1)\}. \quad (69)$$

Then, from (25), equation (65) reduces to

$$S(E_1, E_2) = -\{\lambda + b\alpha + (2n-1)\alpha^2\}g(E_1, E_2) + \{b\alpha - \alpha^2\}\eta(E_1)\eta(E_2) - \frac{1}{2}\{\nu(E_1)\eta(E_2) + \nu(E_2)\eta(E_1)\}. \quad (70)$$

Let us take a nonvanishing symmetric $(0, 2)$ tensor E in (66), such that

$$\begin{aligned} S(E_1, E_2) &= \{b\alpha - \alpha^2\}\eta(E_1)\eta(E_2) - \frac{1}{2}\{\nu(E_1)\eta(E_2) \\ &\quad + \nu(E_2)\eta(E_1)\}. \end{aligned} \quad (71)$$

Then, equation (66) yields

$$S(E_1, E_2) = ag(E_1, E_2) + E(E_1, E_2), \quad (72)$$

where $a = -\{\lambda + b\alpha + (2n-1)\alpha^2\}$. So, M is a near quasi-Einstein. \square

As an immediate outcome of Theorem 6, we have the following corollary.

Corollary 4. *An α -cosymplectic manifold admitting a $*$ -Ricci soliton is an η -Einstein manifold if $V = \xi$.*

A near quasi-Einstein manifold is not a manifold of nearly quasicontant curvature. But, it is noted (Theorem 3.1 of [35]) that, a conformally flat near quasi-Einstein manifold is a manifold of nearly quasicontant curvature. Hence, as immediate consequence of this fact, we obtain the following corollary:

Corollary 5. *A conformally flat α -cosymplectic manifold admitting a $*$ -Ricci soliton is a manifold of near quasicontant curvature if V is a pointwise collinear with ξ .*

However, since a 3-dimensional Riemannian manifold is conformally flat, we have following.

Corollary 6. *A 3-dimensional α -cosymplectic manifold admitting $*$ -Ricci soliton is a manifold of nearly quasicontant curvature if V is a pointwise collinear with ξ .*

6. Conclusions

Einstein manifolds which are arisen from Einstein field equations are very important classes of Riemann manifolds. Some generalizations of Einstein manifolds have been defined in the literature, and there have been obtained some applications of these kinds of manifolds in theoretical physics. Contact manifolds are special Riemann manifolds with almost contact structures. In theoretical physics, there are valuable applications of contact manifolds. Contact manifolds divided into many subclasses via the certain properties of the structure. An important one is α -cosymplectic manifold. This structure is also a generalization of some different contact structures. Many different characteristic properties of manifolds with structures have been arisen from their special structures. One of important notion is the $*$ -Ricci tensor. This notion carries significant curvature features, and this feature gives valuable information about the geometry of the manifold. In this study, α -cosymplectic manifolds have been examined under the effect of the $*$ -Ricci tensor. Important results have been obtained on some generalized Einstein manifolds, which emerged with the effect of the $*$ -Ricci tensor. The notion of Ricci soliton comes from searching the solutions of Ricci flow equations. Ricci solitons have been effected from the structure of manifolds. We studied the concept of $*$ -Ricci soliton for α -cosymplectic manifolds. By the way, important physical results have been stated in this study.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

M.R.A., D.G.P, N.B.T, and I.U. conceptualized the study, developed methodology, performed formal analysis. D.G.P., N.B.T., and I.U. wrote the original draft. M.R.A. and N.B.T. reviewed and edited the article. M.R.A., D.G.P., and I.U. supervised the study. D.G.P. administered the project. A.M.R. and N.B.T acquired fund .

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