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Research Article

Positive Integer Powers of Certain Tridiagonal Matrices and Corresponding Anti-Tridiagonal Matrices

Mohammad Beiranvand and Mojtaba Ghasemi Kamalvand

Department of Mathematics, Lorestan University, Khorramabad, Iran

Correspondence should be addressed to Mohammad Beiranvand; beiranvand.moh@fs.lu.ac.ir
and Mojtaba Ghasemi Kamalvand; ghasemi.m@lu.ac.ir

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In this paper, we firstly derive a general expression for the entries of the mth ($m \in \mathbb{N}$) power for two certain types of tridiagonal matrices of arbitrary order. Secondly, we present a method for computing the positive integer powers of the anti-tridiagonal matrix corresponding to these matrices. Also, we give Maple 18 procedures in order to verify our calculations.

1. Introduction

In recent years, computing the arbitrary positive integer powers of tridiagonal matrices has been a very popular problem for researchers. Tridiagonal matrices are used in different areas of science and engineering. Solution of difference systems [1], the numerical solution of PDE's [2], telecommunication system analysis [3, 4], texture modeling [5], image processing, and coding [6] are examples for applications of these matrices. In these areas, the computation of the powers of these matrices is necessary. There have been several papers on computing the integer powers of various kinds on tridiagonal matrices [7–16].

In this paper, we derive a general expression for the entries of the mth ($m \in \mathbb{N}$) power for two certain types of tridiagonal matrices of arbitrary order as follows:

$$A = \begin{bmatrix} a \pm b & b & & & & \\ b & a & b & & 0 & \\ & b & \ddots & \ddots & & \\ & & \ddots & & b & \\ & 0 & & b & a & b \\ & & & & b & a+b \end{bmatrix}, \tag{1}$$

where $b \neq 0$ and a and b are in the complex numbers. This model matrix has been known for a long time. For example, we have the influential works by D.E. Rutherford [3, 4]. The matrices that calculate their powers in [7, 8] are special cases of matrix (1), for ($a = 0$, $b = 1$), in cases that the $(1, 1)$th entry is $(a + b)$ and $(a - b)$, respectively.

2. Main Results

In this section, we firstly derive two formulas for calculating the mth power for matrix (1) in two cases, where $m \in \mathbb{N}$ and $\mathbb{N}$ denotes the set of natural numbers; then, we present a method for computing the positive integer powers for one type of the anti-tridiagonal matrix corresponding to the matrix (1). According to the following lemmas, we can find the eigenvalues and eigenvectors of the matrix (1).

Lemma 1 (see [17]). *The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix (1) are given as follows: when the $(1, 1)$th entry is $a + b$.*

$$\lambda_k = a + 2b \cos \frac{(k-1)\pi}{n}, \quad k = 1, 2, \cdots, n, \tag{2}$$

the corresponding eigenvectors $u^{(k)} = \left(u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}\right)^T, k = 1, 2, \cdots, n$, are given by

$$u_i^{(1)} = \sqrt{\frac{1}{n}}, \quad i = 1, 2, \cdots, n,$$

$$u_i^{(k)} = \sqrt{\frac{2}{n}} \cos \frac{(2i-1)(k-1)\pi}{2n}, \quad i = 1, 2, \cdots, n; k = 2, 3, \cdots, n. \tag{3}$$

Lemma 2 (see [17]). *The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix (1) are given as follows: when the $(1,1)$th entry is $a-b$.*

$$\lambda_k = a + 2b \cos \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \cdots, n, \tag{4}$$

the corresponding eigenvectors $v^{(k)} = (v_1^{(k)}, v_2^{(k)}, \cdots, v_n^{(k)})^T$, $k = 1, 2, \cdots, n$, are given by

$$v_i^{(k)} = \sqrt{\frac{2}{n}} \sin \frac{(2i-1)(2k-1)\pi}{4n}, \quad i = 1, 2, \cdots, n; k = 1, 2, \cdots, n. \tag{5}$$

Theorem 3 (see [5]). *If $A \in M_n$ has n distinct eigenvalues, then A is diagonalizable.*

Since $\theta^\circ \cos\theta$ is strictly decreasing on $[0,\pi]$, the eigenvalues of matrix (1), i.e., of A, are all distinct and the matrix is hence diagonalizable by [5], Theorem 1.3.9. The proof of [5], Theorem 1.3.7 shows that if we define the matrix $U = [u_0, u_1, \cdots, u_{n-1}]$, then $U^{-1}AU = \operatorname{diag}(\lambda_0, \lambda_1, \cdots, \lambda_{n-1}) = J$ will be a diagonal. Conversely, we will have $A^m = U \operatorname{diag}(\lambda_0^m, \lambda_1^m, \cdots, \lambda_{n-1}^m) U^{-1}$, for any integer m. We will use this to give explicit formulas for $[A^m]_{i,j}$. From Lemma 1, we can write the columns of the matrix U as

$$u^{(1)} = \sqrt{\frac{1}{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$u^{(k)} = \sqrt{\frac{2}{n}} \begin{bmatrix} \cos \frac{(k-1)\pi}{2n} \\ \cos \frac{3(k-1)\pi}{2n} \\ \vdots \\ \cos \frac{(2n-1)(k-1)\pi}{2n} \end{bmatrix}, \quad k = 2, 3, \cdots, n. \tag{6}$$

Hence,

$$U = \left[u^{(1)}, u^{(2)}, u^{(3)}, \cdots, u^{(n)}\right]. \tag{7}$$

Since matrix (1) is symmetric, the associated eigenvectors are orthogonal. Therefore, we can write

$$U^{-1} = U^T = \left[u^{(1)}, u^{(2)}, u^{(3)}, \cdots, u^{(n)}\right]^T. \tag{8}$$

By using following corollaries, we can calculate the m th power of the matrix (1).

Corollary 4. *Let A be a tridiagonal matrix defined in (1) in case $a+b$ and m is a positive integer; then,*

$$[A^m]_{i,j} = \frac{1}{n}\lambda_1^m + \frac{2}{n}\sum_{k=2}^{n} \lambda_k^m \cos \frac{(2i-1)(k-1)\pi}{2n} \cos \frac{(2j-1)(k-1)\pi}{2n}, \tag{9}$$

for $i, j = 1, 2, \cdots, n$, where $\lambda_k = a + 2b \cos(((k-1)\pi)/n)$, $k = 1, 2, \cdots, n$.

Corollary 5. *Let A be a tridiagonal matrix defined in (1) in case $a-b$ and m is a positive integer. Then,*

$$[A^m]_{i,j} = \frac{2}{n}\sum_{k=1}^{n} \lambda_k^m \sin \frac{(2k-1)(2i-1)\pi}{4n} \sin \frac{(2k-1)(2j-1)\pi}{4n}, \tag{10}$$

for $i, j = 1, 2, \cdots, n$, where $\lambda_k = a + 2b \cos(((2k-1)\pi)/2n)$, $k = 1, 2, \cdots, n$.

In the continuation, we present a method for computing the positive integer powers of the one type of the anti-tridiagonal matrices corresponding to the matrix (1), as follows:

$$B = \begin{bmatrix} & & & b & a+b \\ & 0 & b & a & b \\ & b & \iddots & \iddots & b & \\ b & a & b & \iddots & 0 & \\ a+b & b & & & & \\ & & & & & \end{bmatrix}. \tag{11}$$

Let

$$J = \begin{bmatrix} & & & 1 \\ & 0 & 1 & \\ & & \iddots & \\ & 1 & 0 & \\ 1 & & & \end{bmatrix}. \tag{12}$$

Lemma 6. *If matrices A, B, and J have the form (1), (11), and (12), respectively, then*

$$B = AJ. \tag{13}$$

Proof. According to the definition of the multiplication of matrices, we can conclude

$$\begin{aligned}[AJ]_{i,j} &= \sum_{k=1}^{n}[A]_{i,k}[J]_{k,j} \\ &= [A]_{i,n+1-j} = \begin{cases} a+b & \text{if} \quad i=1, j=n, \\ a+b & \text{if} \quad i=n, j=1, \\ a & \text{if} \quad n+1-(i+j)=0, \\ b & \text{if} \quad n+1-(i+j)=\pm 1, \\ 0 & \text{if} \quad \text{otherwise}. \end{cases} \\ &= [B]_{i,j}\,; i,j=1,2,\cdots,n,\end{aligned} \tag{14}$$

□

Lemma 7. *If the matrices A, B, and J have the form (1), (11), and (12), respectively, then the mth power* ($m \in \mathbb{N}$) *of matrix B is computed as follows:*

For $k = 1, 2, 3, \cdots$,

$$B^m = \begin{cases} A^m & \text{for} \quad r = 2k, \\ A^m J & \text{for} \quad r = 2k-1. \end{cases} \tag{15}$$

Proof. We prove this lemma by induction on k. The base case of $k = 1$ is true, because from Lemma 6and from that, $AJ = JA$ and $J = J^{-1}$ follow

$$\begin{aligned} B &= AJ, \\ B^2 &= (AJ)^2 = AJAJ = AJ^2A = AIA = A^2. \end{aligned} \tag{16}$$

□

From (16), we have $B^2 = A^2$ so $B^{2^k} = A^{2^k}$ and from $B^2 = A^2$ and $B = AJ$, we can conclude $B^3 = A^3J$. By repeating this method, we have $B^{2k-1} = A^{2k-1}J$.

Suppose that the result is true for $k > 1$ and consider case $k + 1$.

By the induction hypothesis, we have

$$\begin{aligned} B^{2k} &= A^{2k}, \\ B^{2k-1} &= A^{2k-1}J. \end{aligned} \tag{17}$$

We show that case $k + 1$ also is true. By using (17), we can write

$$B^{2k+2} = B^{2k}B^2 = A^{2k}A^2 = A^{2k+2}, \tag{18}$$

and also,

$$B^{2k+1} = B^{2k}B = A^{2k}AJ = A^{2k+1}J. \tag{19}$$

Thus, the formulas also hold for $k + 1$ and the induction arguments are completed.

We can compute the (i, j)th entry of the mth power for the matrix B in (11) by using formula (9) and Lemma 7.

We leave the calculation of the positive powers of the following antisymmetric matrix to the reader.

$$C = \begin{bmatrix} & & & & b & a-b \\ & 0 & & b & a & b \\ & b & \ddots & \ddots & b & \\ b & a & b & \ddots & 0 & \\ a+b & b & & & & \end{bmatrix}. \tag{20}$$

Appendix

Following Maple 18 procedures, calculate the mth power of n-square tridiagonal matrix given in (1) (in cases $(a + b)$ and $(a - b)$ of arbitrary order, respectively.

```
>restart:
with(ListTools):
power:=proc(n,m,a,b)
local kappa,lambda,i,j,A,power;
for kappa from 1to n
do
lambda[kappa] := a + 2 * b * cos ((kappa − 1) * Pi)/n;
end do;
power:=[]:
for i from 1 to n
do
for j from 1 to n
do
A[m, i, j] := (1/n) * (lambda[1])^m + (2/n) * sum((lambda
[k])^m) * cos ((2 * i − 1) * (k − 1) * Pi)/2 * n)cos ((2 * j − 1)
 * (k − 1) * Pi)/2 * n), k = 2..n);
power:=FlattenOnce([power,A[m,i,j]]);
od;
od;
print(simplify(Matrix(n,n,power)));
end proc:
>power(7,1,0,1)
```

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{A.1}$$

>power(7,2,0,1)

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \quad (A.2)$$

and

```
>restart:
with(ListTools):
power:=proc(n,m,a,b)
local kappa,lambda,i,j,A,power;
for kappa from 1to n
do
lambda[kappa] := a + 2 * b * cos  ((2 * kappa − 1) * Pi)/
2 * n;
end do;
power:=[]:
for i from 1 to n
do
for j from 1 to n
do
A[m, i, j] := (2/n) * sum((lambda[k])^m * sin  ((2 * k − 1)
* (2 * i − 1) * Pi)/4 * n)sin ((2 * k − 1) * (2 * j − 1) * Pi)/4
* n), k = 1..n);
power:=FlattenOnce([power,A[m,i,j]]);
od;
od;
print(simplify(Matrix(n,n,power)));
end proc:
>power(5,1,0,1)
```

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (A.3)$$

>power(5,3,0,1)

$$\begin{bmatrix} -3 & 3 & -1 & 1 & 0 \\ 3 & -1 & 3 & 0 & 1 \\ -1 & 3 & 0 & 3 & 1 \\ 1 & 0 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 & 3 \end{bmatrix} \quad (A.4)$$

Data Availability

Data is available on request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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