Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 8751037, 11 pages
https://doi.org/10.1155/2022/8751037

Research Article

Strictly or Semitrivial Principal Eigensurface for (p, q)-Biharmonic Systems

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Received 26 October 2021; Accepted 1 March 2022; Published 16 March 2022

Academic Editor: Francesco Toppan

This paper extends the eigensurface of p-bilaplacian operator to examine existence and simplicity of the first eigensurface for the third-order spectrum of (p, q)-biharmonic systems subject to boundary conditions.

1. Introduction and Preliminary Results

We wish to investigate existence and simplicity of the first eigensurface for the following system:

$$(\Sigma): \begin{cases} \text{Find } ((u, v), \beta, \lambda) \in Y_{pq}(\Omega) \setminus \{(0,0)\} \times \mathbb{R}^N \times \mathbb{R} \text{ such that} \\ \Delta_p^2 u + H_1(\beta, u) - \lambda m_1(x)|u|^{p-2}u = m(x)|v|^{\alpha_2+1}|u|^{\alpha_1-1}u \text{ in } \Omega, \\ \Delta_q^2 v + H_2(\beta, v) - \lambda m_2(x)|v|^{q-2}v = m(x)|u|^{\alpha_1+1}|v|^{\alpha_2-1}v \text{ in } \Omega, \\ u = \Delta u = v = \Delta v = 0 \text{ on} \partial\Omega, \end{cases} \tag{1}$$

where

(i) $\Omega \subset \mathbb{R}^N$ (with $N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$

(ii) α_1, α_2, p, and q are constants such that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $p > 1$, $q > 1$, and $((\alpha_1 + 1)/p) + ((\alpha_2 + 1)/q) = 1$

(iii) $H_1(\beta, u) = 2\beta \cdot \nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2 |\Delta u|^{p-2}\Delta u$

(iv) $H_2(\beta, v) = 2\beta \cdot \nabla(|\Delta v|^{q-2}\Delta v) + |\beta|^2 |\Delta v|^{q-2}\Delta v$

(v) $Y_{pq}(\Omega) = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)]$

(vi) the weights m, m_1, $m_2 \in L^\infty(\Omega)$ are assumed to be nonnegatives in Ω

(vii) $\Delta_r^2 w = \Delta(|\Delta w|^{r-2}\Delta w)$ with $(r, w) \in \{(p, u), (q, v)\}$ denoting the r-biharmonic operator

Note that this operator describes a huge class of physical phenomena and its linear part $\Delta^2 := \Delta \circ \Delta$ is often seen as a prototypical example of biharmonic operator well known in the theory of elasticity. The spectrum of p-biharmonic operator has drawn much attention in recent works, and problems like (Σ) type appear in several branches of pure and applied mathematics, such as surface diffusion on solids, interface dynamics, thin plate theory, electrorheological fluids, thermorheological fluids, image restoration, and other phenomena related to electrical resistivity and polycrystal plasticity (see, for example, [1–10]).

In [11], Ben Haddouch et al. have investigated the scalar version of problem (Σ) with $m \equiv 0$, which reads

$$(P_{a,p}): \begin{cases} \text{Find } (u, \beta, \Gamma) \in X_p \setminus \{0\} \times \mathbb{R}^N \times \mathbb{R} \text{ such that} \\ \Delta\left(|\Delta u|^{p-2}\Delta u\right) + 2\beta \cdot \nabla\left(|\Delta u|^{p-2}\Delta u\right) + |\beta|^2 |\Delta u|^{p-2}\Delta u = \Gamma a(x)|u|^{p-2}u \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $a \in L^\infty(\Omega)$ and $X_p = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. They proved that $(P_{a,p})$ possesses at least one sequence of positive eigensurfaces $(\Gamma_n^p(.,a))_n$ defined by

$$\forall \beta \in \mathbb{R}^N, \Gamma_n^p(\beta, a) = \inf_{K \subset \mathcal{B}_n} \sup_{u \in K} \int_\Omega e^{\beta.x} |\Delta u|^p dx, \tag{3}$$

where

$$\mathcal{B}_n = \{K \subset \mathcal{N}_\beta : \text{K is compact, Symmetric and } \gamma(K) \geq n\},$$
$$\mathcal{N}_\beta = \left\{u \in X_p : \int_\Omega a e^{\beta.x} |u|^p dx = 1\right\}, \tag{4}$$

and $\Gamma_n^p(\beta, a) \longrightarrow \infty \text{as } n \longrightarrow \infty$.

The authors in [12] gave the first eigensurface $\Gamma_1^p(.,a)$ and showed that if $a \geq 0$ a.e. in Ω, then $\Gamma_1^p(.,a)$ is simple (i.e., the associated eigenfunctions are a constant multiple of one another) and principal, i.e., the associated eigenfunction, denoted by $\varphi_{p,a}$ is positive or negative on Ω with

$$\forall \beta \in \mathbb{R}^N, \Gamma_1^p(\beta, a) = \inf_{u \in \mathcal{N}_\beta} \int_\Omega e^{\beta.x} |\Delta u|^p dx. \tag{5}$$

It is of interest to know that (u, β, λ) is a solution of problem $(P_{m_1,p})$ if and only if $[(u, 0)\,; \beta, \lambda]$ is a solution of (Σ). This kind of solution is called "semitrivial solution" of (Σ). Furthermore, if $[(u, 0)\,; \beta, \lambda(\beta)]$ is a solution of (Σ) with u of one sign on Ω, then $\lambda(.)$ is called "semitrivial principal eigensurface" of (Σ). Consequently, there are two forms of semitrivial solutions for problem (Σ): one of the type $[(u, 0)\,; \beta, \lambda(\beta)]$ with $u\equiv 0$ and $(u, \beta, \lambda(\beta))$ solution of the problem $(P_{m_1,p})$ and the second of the type $[(0, v)\,; \beta, \lambda(\beta)]$ with $v\equiv 0$ and $(v, \beta, \lambda(\beta))$ solution of the problem $(P_{m_2,q})$. In particular, $\Gamma_1^p(.,m_1)$ and $\Gamma_1^q(.,m_2)$ are semitrivial principal eigensurfaces of (Σ).

Motivated by the recent work in [13] where (Σ) was considered in the case $\beta = 0$, our main goal in this paper is to show, in the presence of β, the simplicity and the existence of the first strictly principal eigensurface or semitrivial principal eigensurface of the system (Σ).

Throughout this work, the Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|.\|_r$, $\forall r \in (1,\infty]$ and the norm in a normed space X by $\|.\|_X$. We denote by $Y_{pq}(\Omega) = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)]$ which is a reflexive Banach space endowed with the norm [7, 14]

$$\|(u, v)\| = \|\Delta u\|_p + \|\Delta v\|_q. \tag{6}$$

The weak convergence in $Y_{pq}(\Omega)$ is denoted by $\rightharpoonup$. The positive and negative parts of a function w are denoted by $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Equalities (and inequalities) between two functions must be understood a.e., and for all $f \in L^r(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

is uniquely solvable in $X_r = W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ (see [15]). The following lemma gives us some properties of the inverse operator of $-\Delta : X_r \mapsto L^r(\Omega)$ denoted by Λ.

Lemma 1 [16, 17].

(1) Continuity: there exists a constant $c_r > 0$ such that

$$\|\Lambda f\|_{W^{2,r}} \leq c_r \|f\|_r \tag{8}$$

holds for all $r \in (1,\infty)$ and $f \in L^r(\Omega)$.

(2) Continuity: given $k \in \mathbb{N}^*$, there exists a constant $c_{r,k} > 0$ such that

$$\|\Lambda f\|_{W^{k+2,r}} \leq c_{r,k} \|f\|_{W^{k,r}} \tag{9}$$

holds for all $r \in (1,\infty)$ and $f \in W^{k,r}(\Omega)$.

(3) Symmetry: the identity

$$\int_\Omega \Lambda u \cdot v dx = \int_\Omega u.\Lambda v dx \quad (10)$$

holds for $u \in L^r(\Omega)$ *and* $v \in L^{r'}(\Omega)$ *with* $r' = r/(r-1)$ *and* $r \in (1,\infty)$.

(4) *Regularity: given* $f \in L^\infty(\Omega)$, *we have* $\Lambda f \in C^{1,\nu}(\bar{\Omega})$ *for all* $\nu \in (0,1)$. *Moreover, there exists* $c_\nu > 0$ *such that*

$$\|\Lambda f\|_{C^{1,\nu}(\Omega)} \le c_\nu \|f\|_\infty. \quad (11)$$

(5) *Regularity and Hopf-type maximum principle: let* $f \in C(\bar{\Omega})$ *and* $f \ge 0$ *then*

$w = \Lambda f \in C^{1,\nu}(\bar{\Omega})$, *for all* $\nu \in (0,1)$ *and* w *satisfies:* $w > 0$ *in* Ω, $\partial w/\partial n < 0$ *on* $\partial\Omega$.

(6) *Order-preserving property: given* $f, g \in L^r(\Omega)$ *if* $f \le g$ *in* Ω, *then* $\Lambda f < \Lambda g$ *in* Ω.

The rest of the paper is organized as follows. We construct an eigensurface curve associated to system (Σ) in Section 2, and Section 3 is devoted to the study of strictly principal eigensurface or semitrivial principal eigensurface of (Σ) followed by simplicity results.

2. An Eigensurface Curve Associated to System (Σ)

To address the question of existence of solutions of (Σ), our analysis is partly based upon the "λ-dependent approach" used in several recent works, for instance, [18–22] and references therein. To this end, let us fix $\lambda \in \mathbb{R}$ and embed the system (Σ) into a λ-dependent system (Σ_λ) by introducing a new real parameter μ as follows:

$$(\Sigma_\lambda): \begin{cases} \text{Find } ((u,v),\beta,\mu) \in Y_{pq}(\Omega) \setminus \{(0,0)\} \times \mathbb{R}^N \times \mathbb{R} \text{ such that} \\ \Delta_p^2 u + H_1(\beta,u) - m(x)|v|^{\alpha_2+1}|u|^{\alpha_1-1}u - \lambda m_1(x)|u|^{p-2}u = \mu|u|^{p-2}u \text{ in } \Omega, \\ \Delta_q^2 v + H_2(\beta,v) - m(x)|u|^{\alpha_1+1}|v|^{\alpha_2-1}v - \lambda m_2(x)|v|^{q-2}v = \mu|v|^{q-2}v \text{ in } \Omega, \\ u = \Delta u = v = \Delta v = 0 \text{ on } \partial\Omega. \end{cases} \quad (12)$$

We now give equivalent versions of both systems (Σ) and (Σ_λ) in the following lemma which is straightforward from [11].

Lemma 2. *For* $\beta \in \mathbb{R}^N$, *the system* (Σ_λ)) *(resp.* (Σ)) *is equivalent to the system*

$$(S_\lambda): \begin{cases} \text{Find } ((u,v),\mu) \in Y_{pq}(\Omega) \setminus \{(0,0)\} \times \mathbb{R} \text{ such that} \\ \Delta_p^{2,\beta} u - e^{\beta\cdot x} m(x)|v|^{\alpha_2+1}|u|^{\alpha_1-1}u - \lambda e^{\beta\cdot x} m_1(x)|u|^{p-2}u = \mu e^{\beta\cdot x}|u|^{p-2}u \text{ in } \Omega, \\ \Delta_q^{2,\beta} v - e^{\beta\cdot x} m(x)|u|^{\alpha_1+1}|v|^{\alpha_2-1}v - \lambda e^{\beta\cdot x} m_2(x)|v|^{q-2}v = \mu e^{\beta\cdot x} u|v|^{q-2}v \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega, \end{cases} \quad (13)$$

$$\left(\text{resp.} (S): \begin{cases} \text{Find } ((u,v),\lambda) \in Y_{pq}(\Omega) \setminus \{(0,0)\} \times \mathbb{R} \text{ such that} \\ \Delta_p^{2,\beta} u - e^{\beta\cdot x} m(x)|v|^{\alpha_2+1}|u|^{\alpha_1-1}u = \lambda e^{\beta\cdot x} m_1(x)|u|^{p-2}u \text{ in } \Omega \\ \Delta_q^{2,\beta} v - e^{\beta\cdot x} m(x)|u|^{\alpha_1+1}|v|^{\alpha_2-1}v = \lambda e^{\beta\cdot x} m_2(x)|v|^{q-2}v = \text{ in } \Omega \\ u = \Delta u = 0 \text{ on} \partial\Omega \end{cases} \right), \quad (14)$$

where $\Delta_p^{2,\beta} u = \Delta(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u)$ *and* $\Delta_q^{2,\beta} v = \Delta(e^{\beta\cdot x}|\Delta v|^{q-2}\Delta v)$.

For the sake of clarity, we give a series of definitions that is used throughout this work.

Definition 1.

(1) The set of couples $(\beta,\lambda) \in \mathbb{R}^N \times \mathbb{R}(\text{resp.}(\beta,\mu) \in \mathbb{R}^N \times \mathbb{R})$ such that there exists a solution $((u,v),\beta,\lambda) \in Y_{pq}(\Omega) \setminus \{(0,0)\} \times \mathbb{R}^N \times \mathbb{R}$ (resp. $((u,v),\beta,\mu) \in$

$Y_{pq}(\Omega)\setminus\{(0,0)\}\times\mathbb{R}^N\times\mathbb{R})$ of (Σ) (resp. (Σ_λ)) is called the third-order spectrum of the (p,q)-biharmonic operator plus potential. The couple (β,λ) (resp. (β,μ)) is then called a third-order eigenvalue and (u,v) is said to be an associated eigenfunction of (Σ) (resp. (Σ_λ)). Moreover, a set of third-order eigenvalues of the form $(\beta,f(\beta))$, for $\beta\in\mathbb{R}^N$ and some function $f:\mathbb{R}^N\longrightarrow\mathbb{R}$, is called an eigensurface.

(2) For $\beta\in\mathbb{R}^N$, $[(u,v);\mu]\in Y_{p,q}(\Omega)\times\mathbb{R}$ is a (weak) solution to problem (S_λ) if

$$\begin{aligned}&\int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\Delta\varphi_1dx\\&\quad-\int_\Omega me^{\beta\cdot x}|v|^{\beta+1}|u|^{\alpha-1}u\varphi_1dx-\lambda\int_\Omega m_1e^{\beta\cdot x}|u|^{p-2}u\varphi_1dx\\&\quad=\mu\int_\Omega e^{\beta\cdot x}|u|^{p-2}u\varphi_1dx,\end{aligned}\tag{15}$$

$$\begin{aligned}&\int_\Omega e^{\beta\cdot x}|\Delta v|^{q-2}\Delta v\Delta\varphi_2dx\\&\quad-\int_\Omega me^{\beta\cdot x}|u|^{\alpha+1}|v|^{\beta-1}v\varphi_2dx-\lambda\int_\Omega m_2e^{\beta\cdot x}|v|^{q-2}v\varphi_2dx\\&\quad=\mu\int_\Omega e^{\beta\cdot x}|v|^{q-2}v\varphi_1dx,\end{aligned}\tag{16}$$

for all $(\varphi_1,\varphi_2)\in Y_{pq}(\Omega)$.

(3) For $\beta\in\mathbb{R}^N$, $[(u,v);\lambda]\in Y_{p,q}(\Omega)\times\mathbb{R}$ is a (weak) solution to problem (Σ) if

$$\begin{aligned}&\int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\Delta\varphi_1dx-\int_\Omega me^{\beta\cdot x}|v|^{\beta+1}|u|^{\alpha-1}u\varphi_1dx=\lambda\int_\Omega m_1e^{\beta\cdot x}|u|^{p-2}u\varphi_1dx,\\&\int_\Omega e^{\beta\cdot x}|\Delta v|^{q-2}\Delta v\Delta\varphi_2dx-\int_\Omega me^{\beta\cdot x}|u|^{\alpha+1}|v|^{\beta-1}v\varphi_2dx=\lambda\int_\Omega m_2e^{\beta\cdot x}|v|^{q-2}v\varphi_2dx,\end{aligned}\tag{17}$$

for all $(\varphi_1,\varphi_2)\in Y_{pq}(\Omega)$.

(4) If $[(u,v);\lambda]\in Y_{p,q}(\Omega)\times\mathbb{R}$ (resp. $[(u,v);\mu]\in Y_{p,q}(\Omega)\times\mathbb{R}$) is a (weak) solution to problem (S) (resp. (S_λ)), (u,v) shall be called an eigenfunction of the problem (S) (resp. (S_λ)) associated to the eigenvalue λ (resp. $\mu(\lambda)$). Let us agree to say that an eigenvalue of (S) or (S_λ) is strictly principal (resp., semitrivial principal) if it is associated to an eigenfunction (u,v) such that $u>0$ or $u<0$ and $v>0$ or $v<0$ (resp. $[u>0$ and $v\equiv0$ or $u<0$ and $v\equiv0]$ or $[u\equiv0$ and $v>0$ or $u\equiv0$ and $v<0]$).

(5) If $((u,v),\mu(\beta,\lambda))\in Y_{pq}(\Omega)\times\mathbb{R}$(resp.$((u,v),\lambda(m,\beta,m_1,m_2))\in Y_{pq}(\Omega)\times\mathbb{R}$) is a weak solution to problem (S_λ) (resp. (S)), (u,v) shall be called an eigenfunction of the problem (S_λ) (resp. (S)) associated to the eigenvalue $\mu(\beta,\lambda)$ (resp. $\lambda(m,\beta,m_1,m_2)$). So (u,v) shall be called an eigenfunction of the problem (Σ) (resp. (Σ_λ)) associated to the eigensurface $\lambda(m,.,m_1,m_2)$ (resp., $\mu(.,\lambda)$). We can say that an eigensurface of (Σ) or (Σ_λ) is strictly principal (resp., semitrivial principal) if it is associated to an eigenfunction (u,v) such that $u>0$ or $u<0$ and $v>0$ or $v<0$ (resp. $[u>0$ and $v\equiv0$ or $u<0$ and $v\equiv0]$ or $[u\equiv0$ and $v>0$ or $u\equiv0$ and $v<0]$).

In a context of fixing λ and finding μ in terms of λ, we focus on the smallest eigenvalue $\mu(.)$ of system (Σ_λ). For this, we define the energy functional

$$\begin{aligned}&J_{\lambda,\beta}:Y_{p,q}(\Omega)\longrightarrow\mathbb{R},\\&(u,v)\mapsto J_{\lambda,\beta}(u,v)=E_\beta(u,v)-V_\beta(u,v)-\lambda M_\beta(u,v),\end{aligned}\tag{18}$$

where

$$\begin{aligned}E_\beta(u,v)&=\frac{\alpha_1+1}{p}\int_\Omega e^{\beta\cdot x}|\Delta u|^pdx+\frac{\alpha_2+1}{q}\int_\Omega e^{\beta\cdot x}|\Delta v|^qdx,\\V_\beta(u,v)&=\int_\Omega me^{\beta\cdot x}|u|^{\alpha_1+1}|v|^{\alpha_2+1}dx,\\M_\beta(u,v)&=\frac{\alpha_1+1}{p}M_{1,\beta}(u)+\frac{\alpha_2+1}{q}M_{2,\beta}(v),\end{aligned}\tag{19}$$

with

$$\begin{aligned}M_{1,\beta}(u)&=\int_\Omega m_1e^{\beta\cdot x}|u|^pdx,\\M_{2,\beta}(v)&=\int_\Omega m_2e^{\beta\cdot x}|v|^qdx,\forall(u,v)\in Y_{pq}(\Omega).\end{aligned}\tag{20}$$

Equalities (13) and (15) are equivalent to

$$\nabla J_{\lambda,\beta}(u,v)=\mu\nabla I_\beta(u,v),\tag{21}$$

where

$$\begin{aligned}I_\beta(u,v)&=\frac{\alpha_1+1}{p}\int_\Omega e^{\beta\cdot x}|u|^pdx\\&\quad+\frac{\alpha_2+1}{q}\int_\Omega e^{\beta\cdot x}|v|^qdx,\forall(u,v)\in Y_{pq}(\Omega).\end{aligned}\tag{22}$$

Lemma 3. *Let* $(\omega_1, \omega_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. *If* $\omega_1, \omega_2 > 0$ *on* Ω, *then there exist three positive constants* $c_{1,\beta}$, $c_{2,\beta}$, *and* $c_{3,\beta}$ *such that*

$$\|\Delta u\|_p^p + \|\Delta v\|_q^q \le c_{1,\beta} J_{\lambda,\beta}(u,v) + c_{2,\beta}\int_\Omega \omega_1 e^{\beta\cdot x}|u|^p dx + c_{3,\beta}\int_\Omega \omega_2 e^{\beta\cdot x}|v|^q dx, \tag{23}$$

for every $(u,v) \in Y_{pq}(\Omega)$ *and* $\beta \in \mathbb{R}^N$.

Proof. We know, following ideas of [13], that there exist three positive constants c_1, c_2, and c_3 such that

$$\|\Delta u\|_p^p + \|\Delta v\|_q^q \le c_1 J_{\lambda,0}(u,v) + c_2\int_\Omega \omega_1|u|^p dx + c_3\int_\Omega \omega_2|v|^q dx. \tag{24}$$

Considering the following function:

$$f_\beta : \bar{\Omega} \longrightarrow \mathbb{R}, \quad x \mapsto f_\beta(x) = e^{\beta\cdot x}. \tag{25}$$

We can deduce the existence of two positive constants $k_{1,\beta} = \min_{x\in\bar{\Omega}} f_\beta(x)$ and $k_{2,\beta} = \max_{x\in\bar{\Omega}} f_\beta(x)$ satisfying

$$\begin{gathered} k_{1,\beta}\int_\Omega w_1|u|^p dx \le \int_\Omega w_1 e^{\beta\cdot x}|u|^p dx \le k_{2,\beta}\int_\Omega w_1|u|^p dx, \\ k_{1,\beta}\int_\Omega w_2|v|^q dx \le \int_\Omega w_2 e^{\beta\cdot x}|v|^q dx \le k_{2,\beta}\int_\Omega w_2|v|^q dx, \\ k_{1,\beta} J_{\lambda,0}(u,v) \le J_{\lambda,\beta}(u,v) \le k_{2,\beta} J_{\lambda,0}(u,v). \end{gathered} \tag{26}$$

This gives

$$k_{1,\beta}\left[\|\Delta u\|_p^p + \|\Delta v\|_q^q\right] \le c_1 J_{\lambda,\beta}(u,v) + c_2\int_\Omega \omega_1 e^{\beta\cdot x}|u|^p dx + c_3\int_\Omega \omega_2 e^{\beta\cdot x}|v|^q dx, \tag{27}$$

leading to $c_{1,\beta} = c_1/k_{1,\beta}$, $c_{2,\beta} = c_2/k_{1,\beta}$, and $c_{3,\beta} = c_3/k_{1,\beta}$ and the result follows. □

Let us now consider the manifold

$$\mathcal{M}_\beta = \left\{(u,v) \in Y_{pq}(\Omega): I_\beta(u,v) = 1\right\}. \tag{28}$$

In the following, we construct for each $\lambda : \beta \in \mathbb{R}^N \mapsto \lambda(\beta) \in \mathbb{R}$ the smallest eigensurface $\beta \in \mathbb{R}^N \mapsto \mu_1(\beta, \lambda(\beta))$ from which we get the eigensurface curve $\lambda(\beta) \in \mathbb{R} \mapsto \mu_1(\beta, \lambda(\beta))$. This is the main result of this section.

Proposition 1. $\mu_1(.,\lambda(.))$ *is the smallest eigensurface of* (Σ_λ) *with*

$$\mu_1(\beta, \lambda(\beta)) \coloneqq \inf\left\{J_{\lambda,\beta}(u,v): (u,v) \in \mathcal{M}_\beta\right\}, \tag{29}$$

for $\beta \in \mathbb{R}^N$.

Proof. In the first step, Lemma 3 ensures the existence of $\mu_1(\beta, \lambda(\beta))$ for all $\beta \in \mathbb{R}^N$. Furthermore, any sequence (u_n, v_n) that minimizes $J_{\lambda,\beta}$ on $\mathcal{M}_\beta$ is bounded in $Y_{pq}(\Omega)$. Thus, there exists $(u_0, v_0) \in Y_{pq}(\Omega)$ such that, up to a subsequence, the sequence (u_n, v_n) converges weakly to (u_0, v_0) in $Y_{pq}(\Omega)$ and strongly in $L^p(\Omega) \times L^q(\Omega)$. Therefore,

$$J_{\lambda,\beta}(u_0, v_0) \le \lim_{n\longrightarrow\infty} J_{\lambda,\beta}(u_n, v_n) = \mu_1(\beta, \lambda(\beta)), (u_0, v_0) \in \mathcal{M}, \tag{30}$$

and consequently, $J_{\lambda,\beta}(u_0, v_0) = \mu_1(\beta, \lambda(\beta))$. By the Lagrange multipliers rule, $\mu_1(\beta, \lambda(\beta))$ is an eigenvalue for (S_λ) and (u_0, v_0) is an associated eigenfunction. As a result, the value $\mu_1(.,\lambda(.))$ is an eigensurface for (Σ_λ) and (u_0, v_0) is an associated eigenfunction.

In the second step, for any eigenvalue $\mu(\beta, \lambda(\beta))$ associated to $(u_\lambda, v_\lambda) \in Y_{pq}(\Omega) \setminus \{(0,0)\}$, one has $J_{\lambda,\beta}(u_\lambda, v_\lambda) = \mu(\beta, \lambda(\beta)) I_\beta(u_\lambda, v_\lambda)$ with $I_\beta(u_\lambda, v_\lambda) > 0$. Then,

$$\mu_1(\beta, \lambda(\beta)) \le J_{\lambda,\beta}\left(\frac{u_\lambda}{I_\beta(u_\lambda, v_\lambda)^{1/p}}, \frac{v_\lambda}{I_\beta(u_\lambda, v_\lambda)^{1/q}}\right) = \frac{J_{\lambda,\beta}(u_\lambda, v_\lambda)}{I_\beta(u_\lambda, v_\lambda)} = \mu(\beta, \lambda(\beta)), \tag{31}$$

and we can draw the conclusion that $\mu_1(.,\lambda(.))$ is the smallest eigensurface of (Σ_λ) putting an end to the proof. □

For $m = m_1 = m_2 \equiv 0$, we denote by

$$\mu_0(\beta) = \mu_1(\beta, \lambda(\beta)) = \inf\left\{\frac{\alpha_1+1}{p}\int_\Omega e^{\beta\cdot x}|\Delta u|^p dx + \frac{\alpha_2+1}{q}\int_\Omega e^{\beta\cdot x}|\Delta v|^q dx : (u,v) \in \mathcal{M}_\beta\right\}. \tag{32}$$

Since the space $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ with $r \in \{p,q\}$ does not contain any constant nontrivial function, one has $\mu_0(\beta) > 0$. The following properties are well known (see details of their proof in [13]) and are useful in proving the main result of the next section.

Proposition 2.

(1) $\mu_1(\beta,.)$: $\mathbb{R} \longrightarrow \mathbb{R}$ *is concave differentiable with*

$$\mu_1'(\beta, \lambda) = \frac{\partial \mu_1(\beta, \lambda)}{\partial \lambda} = -M_\beta(u_0, v_0), \tag{33}$$

where (u_0, v_0) is some eigenfunction of (S_λ) associated to $\mu_1(\beta, \lambda)$ for all $\lambda \in \mathbb{R}$.

$$\lim_{\lambda \longrightarrow \infty} \mu_1(\beta, \lambda) = -\infty. \tag{34}$$

(2) $\mu_1(\beta, .)$ is strictly decreasing.

3. Strictly or Semitrivial Principal Eigensurface for System (Σ)

In the following lemma, we build a sufficient condition for $\mu_1(.,\lambda(.))$ to vanish as it is known that its zeros solve (Σ).

Lemma 4. *If $\|m\|_\infty < \mu_0(.)$, then $\mu_1(.,0) > 0$ and $\mu_1(.,\lambda(.)) \equiv 0$ has a unique positive solution $\lambda(.)$ (eigensurface of (Σ)).*

Proof. We have $V_\beta(u, v) \le \|m\|_\infty I_\beta(u, v)$, $\forall (u, v) \in Y_{pq}(\Omega)$ so that

$$E_\beta(u, v) - \|m\|_\infty I_\beta(u, v) \le E_\beta(u, v) - V_\beta(u, v), \forall (u, v) \in Y_{pq}(\Omega). \tag{35}$$

Then,

$$\begin{gathered} \mu_0(\beta) \le E_\beta(u, v) - V_\beta(u, v) + \|m\|_\infty, \forall (u, v) \in \mathcal{M}_\beta, \\ \mu_0(\beta) - \|m\|_\infty \le \inf \left\{ E_\beta(u, v) - V_\beta(u, v), (u, v) \in \mathcal{M}_\beta \right\} \le \mu_1(\beta, 0). \end{gathered} \tag{36}$$

Thus, $\mu_1(.,0) > 0$, and from Proposition 1, $\mu_1(\beta, .)$ is strictly decreasing. Consequently $\mu_1(.,\lambda(.)) \equiv 0$ has a unique positive solution $\lambda(.)$ which is an eigensurface of (Σ). □

We will denote by

$$\begin{aligned} L(\Omega) &:= ([L^p(\Omega) \times L^q(\Omega)] \setminus \{(0, 0)\}) \times \mathbb{R}, \\ L_0(\Omega) &:= ([L^p(\Omega) \times L^q(\Omega)] \setminus \{(0, 0)\}) \times \{0\}. \end{aligned} \tag{37}$$

We apply some results proved by Drábek and Ôtani [16] and Talbi and Tsouli [17] and some ideas used by Leadi and Toyou [13] to make the following remarks.

Remark 1.

(1) $\forall u \in X_r$, $\forall v \in L^r(\Omega)$ with $r \in (1,\infty)$: $v = -\Delta u \Leftrightarrow u = \Lambda v$

(2) Let N_r be the Nemytskii operator with $r \in (1,\infty)$, defined by

$$N_r(u)(x) = \begin{cases} |u(x)|^{r-2} u(x), & \text{if } u(x) \ne 0, \\ 0, & \text{if } u(x) = 0. \end{cases} \tag{38}$$

We have

$$\forall v \in L^r(\Omega), \forall w \in L^{r'}(\Omega): N_r(v) = w \Leftrightarrow v = N_{r'}(w), \tag{39}$$

with $r' = r/(r-1)$.

(3) If (u, v) is an eigenfunction of (Σ_λ) associated with $\mu(.)$, then for $\beta \in \mathbb{R}^N$, (u, v) is an eigenfunction of (S_λ) associated with $\mu(\beta)$ and $\varphi = -\Delta u$, $w = -\Delta v$ satisfy

$$\begin{cases} e^{\beta \cdot x} N_p(\varphi) = \Lambda\Big([\mu(\lambda) + \lambda m_1] e^{\beta \cdot x} N_p(\Lambda\varphi) + m e^{\beta \cdot x} |\Lambda w|^{\alpha_2+1} |\Lambda\varphi|^{\alpha_1-1} \Lambda\varphi\Big), \\ e^{\beta \cdot x} N_q(w) = \Lambda\Big([\mu(\lambda) + \lambda m_2] e^{\beta \cdot x} N_q(\Lambda w) + m e^{\beta \cdot x} |\Lambda\varphi|^{\alpha_1+1} |\Lambda w|^{\alpha_2-1} \Lambda w\Big). \end{cases} \tag{40}$$

Hence,

(a) $[(u_0, v_0) ; \beta, \mu(\lambda)]$ is a solution of (Σ_λ) if and only if $[(\varphi_0, w_0) ; \mu(\lambda)]$ is a solution of problem

$$\left(S'_\lambda\right): \begin{cases} \text{Find}[(\varphi, w) ; \mu(\lambda)] \in L(\Omega) \text{ such that} \\ e^{\beta \cdot x} N_p(\varphi) = \Lambda\Big([\mu(\lambda) + \lambda m_1] e^{\beta \cdot x} N_p(\Lambda\varphi) + m e^{\beta \cdot x} |\Lambda w|^{\alpha_2+1} |\Lambda\varphi|^{\alpha_1-1} \Lambda\varphi\Big), \\ e^{\beta \cdot x} N_q(w) = \Lambda\Big([\mu(\lambda) + \lambda m_2] e^{\beta \cdot x} N_q(\Lambda w) + m e^{\beta \cdot x} |\Lambda\varphi|^{\alpha_1+1} |\Lambda w|^{\alpha_2-1} \Lambda w\Big), \end{cases} \tag{41}$$

with $\varphi_0 = -\Delta u_0$ and $w_0 = -\Delta v_0$.

(b) $[(u_0, v_0) ; \beta, \mu(\lambda)]$ is a solution of (Σ) if and only if $[(\varphi_0, w_0) ; \lambda]$ is a solution of problem

$$\left(S'\right): \begin{cases} \text{Find}[(\varphi, w) ; \lambda] \in L(\Omega) \text{ such that} \\ e^{\beta\cdot x} N_p(\varphi) = \Lambda\left(\lambda m_1 e^{\beta\cdot x} N_p(\Lambda\varphi) + m e^{\beta\cdot x} |\Lambda w|^{\alpha_2+1} |\Lambda\varphi|^{\alpha_1-1}\Lambda\varphi\right), \\ e^{\beta\cdot x} N_q(w) = \Lambda\left(\lambda m_2 e^{\beta\cdot x} N_q(\Lambda w) + m e^{\beta\cdot x} |\Lambda\varphi|^{\alpha_1+1} |\Lambda w|^{\alpha_2-1}\Lambda w\right), \end{cases} \tag{42}$$

with $\varphi_0 = -\Delta u_0$ and $w_0 = -\Delta v_0$.

(c) $[(\varphi_0, w_0) ; \mu(\lambda)] \in L_0(\Omega)$ is a solution of (S'_λ) if and only if $[(\varphi_0, w_0) ; \lambda] \in L(\Omega)$ is a solution of problem (S')

$$\mu_1(\beta, \lambda(\beta)) := \inf\left\{F_{\beta,\lambda}(\varphi, w) : (\varphi, w) \in L^p(\Omega) \times L^q(\Omega), R_\beta(\varphi, w) = 1\right\}, \tag{43}$$

where

$$\begin{aligned} F_{\beta,\lambda}(\varphi, w) = {} & \frac{\alpha_1+1}{p}\left[\int_\Omega e^{\beta\cdot x}|\varphi|^p dx - \lambda\int_\Omega m_1 e^{\beta\cdot x}|\Lambda\varphi|^p dx\right] \\ & + \frac{\alpha_2+1}{q}\left[\int_\Omega e^{\beta\cdot x}|w|^q dx - \lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda w|^q dx\right] \\ & - \int_\Omega m e^{\beta\cdot x}|\Lambda\varphi|^{\alpha_1+1}|\Lambda w|^{\alpha_2+1} dx, \end{aligned}$$

$$R_\beta(\varphi, w) = \frac{\alpha_1+1}{p}\int_\Omega e^{\beta\cdot x}|\Lambda\varphi|^p dx + \frac{\alpha_2+1}{q}\int_\Omega e^{\beta\cdot x}|\Lambda w|^q dx. \tag{44}$$

In a sequel, we name previous sufficient condition of solvability of (Σ)

$$(H_m): \|m\|_\infty < \mu_0(.). \tag{45}$$

Lemma 5. *If $[(u, v) ; \beta, \mu(\lambda)]$ is a solution of (Σ_λ), then $-\Delta u$, $-\Delta v \in C(\bar{\Omega})$, and u, $v \in C^{1,\nu}(\bar{\Omega})$, for all $\nu \in (0, 1)$.*

Proof. An easy adaptation of ideas of the proof of Lemma 4 in [13] and we just omit it. □

Lemma 6. *$[(\varphi_1, w_1) ; \mu_1(\beta, \lambda)] \in L(\Omega)$ is a solution of problem (S'_λ), if and only if*

$$G_{\beta,\lambda}(\varphi_1, w_1) = 0 = \min_{(\varphi,w)\in L^*(\Omega)} G_{\beta,\lambda}(\varphi, w), \tag{46}$$

where

$$\begin{aligned} G_{\beta,\lambda}(\varphi, w) &= F_{\beta,\lambda}(\varphi, w) - \mu_1(\beta, \lambda) R_\beta(\varphi, w), \\ L^*(\Omega) &= [L^p(\Omega) \times L^q(\Omega)] \setminus \{(0, 0)\}. \end{aligned} \tag{47}$$

Proof. Assuming $F_{\beta,\lambda}(\varphi_1, w_1) = \mu_1(\beta, \lambda) R_\beta(\varphi_1, w_1)$, then

$$G_{\beta,\lambda}(\varphi_1, w_1) = F_{\beta,\lambda}(\varphi_1, w_1) - \mu_1(\beta, \lambda) R_\beta(\varphi_1, w_1) = 0, \tag{48}$$

and one can set $\bar{\varphi} = \varphi/([R_\beta(\varphi, w)]^{1/p})$ and $\bar{w} = w/([R_\beta(\varphi, w)]^{1/q})$ for every $(\varphi, w) \in L^*(\Omega)$. We readily check that $R_\beta(\bar{\varphi}, \bar{w}) = 1$ and deduce

$$\begin{aligned} \mu_1(\beta, \lambda) \le F_{\beta,\lambda}(\bar{\varphi}, \bar{w}) &= \frac{F_{\beta,\lambda}(\varphi, w)}{R_\beta(\varphi, w)}, \\ G_{\beta,\lambda}(\varphi, w) = F_{\beta,\lambda}(\varphi, w) &- \mu_1(\beta, \lambda) R_\beta(\varphi, w) \ge 0, \end{aligned} \tag{49}$$

for all $(\varphi, w) \in L^*(\Omega)$. This proves (45).

Conversely, let us suppose that (45) holds. We deduce that $\nabla G_{\beta,\lambda}(\varphi_1, w_1) = (0, 0)$ and then

$$\left\langle \frac{\partial G_{\beta,\lambda}}{\partial\varphi}(\varphi_1, w_1), \Psi \right\rangle = \left\langle \frac{\partial G_{\beta,\lambda}}{\partial w}(\varphi_1, w_1), \theta \right\rangle = 0, \tag{50}$$

for all $(\Psi, \theta) \in [L^p(\Omega) \times L^q(\Omega)]$ which proves that $[(\varphi_1, w_1) ; \mu_1(\beta, \lambda)] \in L(\Omega)$ is a solution of (S'_λ). □

Lemma 7. *If (H_m) is satisfied and $[(\varphi_1, w_1) ; \mu_1(\beta, \lambda)] \in L_0(\Omega)$ is a solution of problem (S'_λ), then $[(|\varphi_1|, |w_1|) ; \mu_1(\beta, \lambda)] \in L_0(\Omega)$ is a solution of problem (S'_λ).*

Proof. From the assumptions $G_{\beta,\lambda}(\varphi_1, w_1) = 0$, $\mu_1(\beta, \lambda) = 0$, $\lambda > 0$, and $(|\varphi_1|, |w_1|) \in [L^p(\Omega) \times L^q(\Omega)] \setminus \{(0, 0)\}$; we have

$G_{\beta,\lambda}(|\varphi_1|, |w_1|) \geq 0$. In addition, $|\Lambda(|\varphi_1|)|^r \geq |\Lambda\varphi_1|^r$ and $|\Lambda(|w_1|)|^r \geq |\Lambda w_1|^r$, for all $r \in (1;\infty)$ and imply

$$\begin{aligned}
-\lambda\int_\Omega m_1 e^{\beta\cdot x}|\Lambda(|\varphi_1|)|^p dx &\leq -\lambda\int_\Omega m_1 e^{\beta\cdot x}|\Lambda\varphi_1|^p dx,\\
-\lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda(|w_1|)|^q dx &\leq -\lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda w_1|^q dx,\\
-\int_\Omega m e^{\beta\cdot x}|\Lambda(|\varphi_1|)|^{\alpha+1}|\Lambda(|w_1|)|^{\beta+1}dx &\leq -\int_\Omega m e^{\beta\cdot x}|\Lambda\varphi_1|^{\alpha+1}|\Lambda w_1|^{\beta+1}dx.
\end{aligned} \tag{51}$$

Therefore, $F_{\beta,\lambda}(|\varphi_1|, |w_1|) \leq F_{\beta,\lambda}(\varphi_1, w_1)$ and $G_{\beta,\lambda}(|\varphi_1|, |w_1|) \leq G_{\beta,\lambda}(\varphi_1, w_1) = 0$. It then reads $G_{\beta,\lambda}(|\varphi_1|, |w_1|) = 0$ and $[(|\varphi_1|, |w_1|) ; \mu_1(\beta, \lambda)]$ is a solution of (S'_λ). □

Proposition 3. *Assume that (H_m) holds and $\mu_1(.,\lambda(.)) \equiv 0$. Then, $\lambda(.)$ is a semitrivial principal eigensurface or strictly principal eigensurface of system (Σ).*

Proof. The value $\lambda(.)$ is an eigensurface of problem (Σ) associated with $(u, v) \in Y_{pq}(\Omega) \setminus \{(0, 0)\}$. Let $\beta \in \mathbb{R}^N$ with $\mu_1(\beta, \lambda(\beta)) = 0$.

First, if $u\equiv 0$ and $v\equiv 0$, then $[(\varphi, w) ; \mu_1(\beta, \lambda(\beta))]$ and $[(|\varphi|, |w|) ; \mu_1(\beta, \lambda(\beta))]$ belong to $L_0(\Omega)$ and are solutions of problem (S'_λ) with $\varphi = -\Delta u\equiv 0$ and $w = -\Delta v\equiv 0$. Since $|\varphi| \geq 0$ and $|w| \geq 0$, and then, $\Lambda(|\varphi|) > 0$ and $\Lambda(|w|) > 0$. Therefore, $N_p(\Lambda|\varphi|) > 0$, $N_q(\Lambda|w|) > 0$, $|\Lambda(|w|)|^{\alpha_2+1}|\Lambda(|\varphi|)|^{\alpha_1} > 0$, $|\Lambda(|\varphi|)|^{\alpha_1+1}|\Lambda(|w|)|^{\alpha_2} > 0$, and

$$\begin{cases}
|\varphi| = N_{p'}\left(e^{\beta\cdot x}\Lambda\left[\lambda(\beta)m_1 e^{\beta\cdot x}N_p(\Lambda|\varphi|) + m e^{\beta\cdot x}|\Lambda(|w|)|^{\alpha_2+1}|\Lambda(|\varphi|)|^{\alpha_1-1}\Lambda(|\varphi|)\right]\right) > 0,\\
|w| = N_{q'}\left(e^{\beta\cdot x}\Lambda\left[\lambda(\beta)m_2 e^{\beta\cdot x}N_q(\Lambda|w|) + m e^{\beta\cdot x}|\Lambda(|\varphi|)|^{\alpha_1+1}|\Lambda(|w|)|^{\alpha_2-1}\Lambda(|w|)\right]\right) > 0.
\end{cases} \tag{52}$$

We then conclude that $[(\varphi, w) ; \mu_1(\beta, \lambda(\beta))]$ is a solution of problem (S'_λ) with both φ and w positive in Ω or negative in Ω. On the other hand, Lemma 5 expresses that $\varphi, w \in C(\bar{\Omega})$ yielding to $u = \Lambda\varphi$ positive in Ω or negative in Ω and $v = \Lambda w$ positive in Ω or negative in Ω (see Lemma 1). It follows immediately that $\lambda(.)$ is a strictly principal eigensurface of (Σ).

Second, if $[u \equiv 0$ and $v\equiv 0]$ or $[u\equiv 0$ and $v \equiv 0]$, then we also prove that $[u \equiv 0$ and $v > 0$ in Ω or $v < 0$ in $\Omega]$ or $[u > 0$ in Ω or $u < 0$ in Ω and $v \equiv 0]$. We conclude that $\lambda(.)$ is a semitrivial principal eigensurface of (Σ). □

Lemma 8 [13]. *Let A, B, C, and r be real numbers satisfying $A \geq 0$, $B \geq 0$, $C \geq \max\{B - A, 0\}$, and $r \in [1,+\infty)$. Then,*

$$|A + C|^r + |B - C|^r \geq A^r + B^r. \tag{53}$$

Lemma 9. *Suppose that (H_m) holds. If (φ_1, w_1) and (φ_2, w_2) are positive eigenfunctions of problem (S'_λ) associated with $\mu_1(\beta, \lambda) = 0$, then (φ_{12}, w_{12}), (φ_{12}, w_{21}), (φ_{21}, w_{12}), and (φ_{21}, w_{21}) with*

$$\begin{cases}
\varphi_{12}(x) \coloneqq \max\{\varphi_1(x), \varphi_2(x)\} = \varphi_1(x) + (\varphi_2 - \varphi_1)^+(x),\\
w_{12}(x) \coloneqq \max\{w_1(x), w_2(x)\} = w_1(x) + (w_2 - w_1)^+(x),\\
\varphi_{21}(x) \coloneqq \min\{\varphi_1(x), \varphi_2(x)\} = \varphi_2(x) - (\varphi_2 - \varphi_1)^+(x),\\
w_{21}(x) \coloneqq \min\{w_1(x), w_2(x)\} = w_2(x) - (w_2 - w_1)^+(x),
\end{cases} \tag{54}$$

for all $x \in \Omega$, are eigenfunctions of (S'_λ) associated with $\mu_1(\beta, \lambda) = 0$.

Proof. From Lemma 8, we have

$$\begin{cases}
|\Lambda\varphi_{12}|^p + |\Lambda\varphi_{21}|^p \geq |\Lambda\varphi_1|^p + |\Lambda\varphi_2|^p,\\
|\Lambda w_{12}|^q + |\Lambda w_{21}|^q \geq |\Lambda w_1|^q + |\Lambda w_2|^q,\\
|\Lambda\varphi_{12}|^{\alpha_1+1} + |\Lambda\varphi_{21}|^{\alpha_1+1} \geq |\Lambda\varphi_1|^{\alpha_1+1} + |\Lambda\varphi_2|^{\alpha_1+1},\\
|\Lambda w_{12}|^{\alpha_2+1} + |\Lambda w_{21}|^{\alpha_2+1} \geq |\Lambda w_1|^{\alpha_2+1} + |\Lambda w_2|^{\alpha_2+1}.
\end{cases} \tag{55}$$

Then,

$$\begin{aligned}
&-\lambda\int_\Omega m_1 e^{\beta\cdot x}\left|\Lambda\varphi_{12}\right|^p dx - \lambda\int_\Omega m_1 e^{\beta\cdot x}\left|\Lambda\varphi_{21}\right|^p dx \leq\\
&-\lambda\int_\Omega m_1 e^{\beta\cdot x}\left|\Lambda\varphi_1\right|^p dx - \lambda\int_\Omega m_1 e^{\beta\cdot x}\left|\Lambda\varphi_2\right|^p dx,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&-\lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda w_{12}|^q dx - \lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda w_{21}|^q dx \leq\\
&-\lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda w_1|^q dx - \lambda\int_\Omega m_2 e^{\beta\cdot x}|\Lambda w_2|^q dx.
\end{aligned} \tag{57}$$

Similarly, we have

$$\begin{aligned}
Z_1(\varphi, w) \leq Z_2(\varphi, w) \leq &-\int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_1\right|^{\alpha_1+1}\left|\Lambda w_1\right|^{\alpha_2+1}dx\\
&-\int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_2^{\alpha_1+1}\right|\Lambda w_2\Big|^{\alpha_2+1}dx,
\end{aligned} \tag{58}$$

with

$$Z_1(\varphi, w) = -\int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_{12}\right|^{\alpha_1+1}\left|\Lambda w_{12}\right|^{\alpha_2+1}dx - \int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_{12}{}^{\alpha_1+1}\right|\Lambda w_{21}\Big|^{\alpha_2+1}dx, \quad (59)$$

$$-\int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_{21}\right|^{\alpha_1+1}\left|\Lambda w_{12}\right|^{\alpha_2+1}dx - \int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_{21}{}^{\alpha_1+1}\right|\Lambda w_{21}\Big|^{\alpha_2+1}dx, \quad (60)$$

$$Z_2(\varphi, w) = -\int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_1\right|^{\alpha_1+1}\left|\Lambda w_1\right|^{\alpha_2+1}dx - \int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_1{}^{\alpha_1+1}\right|\Lambda w_2\Big|^{\alpha_2+1}dx, \quad (61)$$

$$-\int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_2\right|^{\alpha_1+1}\left|\Lambda w_1\right|^{\alpha_2+1}dx - \int_\Omega m e^{\beta\cdot x}\left|\Lambda\varphi_2\right|^{\alpha_1+1}\left|\Lambda w_2\right|^{\alpha_2+1}dx. \quad (62)$$

Moreover,

$$\int_\Omega e^{\beta\cdot x}\left|\varphi_{12}\right|^p dx + \int_\Omega e^{\beta\cdot x}\left|\varphi_{21}\right|^p dx = \int_\Omega e^{\beta\cdot x}\left|\varphi_1\right|^p dx + \int_\Omega e^{\beta\cdot x}\left|\varphi_2\right|^p dx, \quad (63)$$

$$\int_\Omega e^{\beta\cdot x}|w_{12}|^q dx + \int_\Omega e^{\beta\cdot x}|w_{21}|^q dx = \int_\Omega e^{\beta\cdot x}|w_1|^q dx + \int_\Omega e^{\beta\cdot x}|w_2|^q dx. \quad (64)$$

Combining (54), (56), (57), (59), and (63), we establish

$$\begin{gathered} F_{\beta,\lambda}(\varphi_{12}, w_{12}) + F_{\beta,\lambda}(\varphi_{12}, w_{21}) + F_{\beta,\lambda}(\varphi_{21}, w_{12}) + F_{\beta,\lambda}(\varphi_{21}, w_{21}) \\ \le F_{\beta,\lambda}(\varphi_1, w_1) + F_{\beta,\lambda}(\varphi_2, w_2), \\ G_{\beta,\lambda}(\varphi_{12}, w_{12}) + G_{\beta,\lambda}(\varphi_{12}, w_{21}) + G_{\beta,\lambda}(\varphi_{21}, w_{12}) + G_{\beta,\lambda}(\varphi_{21}, w_{21}) \\ \le G_{\beta,\lambda}(\varphi_1, w_1) + G_{\beta,\lambda}(\varphi_2, w_2) = 0, \end{gathered} \quad (65)$$

which imply

$$G_{\beta,\lambda}(\varphi_{12}, w_{12}) = G_{\beta,\lambda}(\varphi_{12}, w_{21}) = G_{\beta,\lambda}(\varphi_{21}, w_{12}) = G_{\beta,\lambda}(\varphi_{21}, w_{21}) = 0, \quad (66)$$

and ensure that (φ_{12}, w_{12}), (φ_{12}, w_{21}), (φ_{21}, w_{12}), and (φ_{21}, w_{21}) are eigenfunctions of (S'_λ) associated with $\mu_1(\beta,\lambda) = 0$. □

Proposition 4. *Assume that (H_m) holds and $\mu_1(.,\lambda(.)) \equiv 0$. Then $\lambda(.)$ is a semitrivial principal eigensurface or strictly principal eigensurface of problem (Σ) and simple.*

Proof. From Proposition 3, the value $\lambda(.)$ is a semitrivial principal eigensurface or strictly principal eigensurface of problem (Σ). The rest of the proof falls naturally into two cases as follows. □

Case 1: $\lambda(.)$ is a strictly principal eigensurface of (Σ).

Let (u_{11}, u_{12}) and (u_{21}, u_{22}) be two positive eigenfunctions of (Σ) associated with $\lambda(.)$. Then, $[(v, w)\,;0]$, $[(\varphi, \psi)\,;0]$, $[(|v|, |w|)\,;0]$, $[(|\varphi|, |\psi|)\,;0] \in L_0(\Omega)$, are solutions of (S'_λ) with $v = -\Delta u_{11} > 0$, $w = -\Delta u_{12} > 0$, $\varphi = -\Delta u_{21} > 0$, and $\psi = -\Delta u_{22} > 0$. Let us fix $x_0 \in \Omega$ and set

$$\begin{aligned} k = \frac{\varphi(x_0)}{v(x_0)}, \omega_1(x) = \max\{\varphi(x), kv(x)\} \text{ and } \omega_2(x) \\ = \max\left\{\psi(x), k^{p/q}w(x)\right\}, \end{aligned} \quad (67)$$

for all $x \in \Omega$.

Using Lemma 9, $[(\omega_1, \omega_2)\,;0]$ is a solution of problem (S'_λ) as $[(kv, k^{p/q}w)\,;0]$ and $[(\varphi, \psi)\,;0]$ are solutions of (Σ_λ). We infer that $N_p(v)$, $N_q(w)$, $N_p(\varphi)$, $N_q(\psi)$, $N_p(\omega_1)$, $N_q(\omega_2) \in C^{1,v}(\bar{\Omega})$, and $N_p(\varphi)/N_p(v)$, $N_q(\psi)/N_q(w) \in C^1(\Omega)$, and for any unit vector $e = (0, \cdots, e_i, \cdots, 0)$ with $i \in \{1, \cdots, N\}$ and $t \in \mathbb{R}$, we obtain

$$\begin{cases} N_p(\varphi)(x_0 + te) - N_p(\varphi)(x_0) \le N_p(\omega_1)(x_0 + te) - N_p(\omega_1)(x_0), \\ N_p(kv)(x_0 + te) - N_p(kv)(x_0) \le N_p(\omega_1)(x_0 + te) - N_p(\omega_1)(x_0). \end{cases} \quad (68)$$

Dividing these inequalities by $t > 0$ and $t < 0$ and letting t tend to $0^\pm$, we get

$$\begin{gathered} \begin{cases} \dfrac{\partial}{\partial x_i}\left[N_p(\varphi)\right](x_0) \le \dfrac{\partial}{\partial x_i}\left[N_p(\omega_1)\right](x_0), \\ \dfrac{\partial}{\partial x_i}\left[N_p(kv)\right](x_0) \le \dfrac{\partial}{\partial x_i}\left[N_p(\omega_1)\right](x_0), \end{cases} \\ \begin{cases} \dfrac{\partial}{\partial x_i}\left[N_p(\varphi)\right](x_0) \ge \dfrac{\partial}{\partial x_i}\left[N_p(\omega_1)\right](x_0), \\ \dfrac{\partial}{\partial x_i}\left[N_p(kv)\right](x_0) \ge \dfrac{\partial}{\partial x_i}\left[N_p(\omega_1)\right](x_0), \end{cases} \end{gathered} \quad (69)$$

for all $i \in \{1, \cdots, N\}$. That is

$$\begin{cases} \dfrac{\partial}{\partial x_i}\left[N_p(\varphi)\right](x_0) = \dfrac{\partial}{\partial x_i}\left[N_p(\omega_1)\right](x_0), \\ \dfrac{\partial}{\partial x_i}\left[N_p(kv)\right](x_0) = \dfrac{\partial}{\partial x_i}\left[N_p(\omega_1)\right](x_0), \end{cases} \quad (70)$$

for all $i \in \{1, \cdots, N\}$. In other words,

$$\nabla N_p(\varphi)(x_0) = \nabla N_p(\omega_1)(x_0) = \nabla N_p(kv)(x_0) = k^{p-1}\nabla N_p(v)(x_0). \quad (71)$$

Furthermore,

$$\nabla\left(\frac{N_p(\varphi)}{N_p(v)}\right)(x_0) = \frac{\nabla\left(N_p(\varphi)\right)(x_0)N_p(v)(x_0) - N_p(\varphi)(x_0)\nabla\left(N_p(v)\right)(x_0)}{\left[N_p(v)(x_0)\right]^2} = \frac{\left[N_p(v)(x_0) - k^{1-p}N_p(\varphi)(x_0)\right]\nabla\left(N_p(\varphi)\right)(x_0)}{\left[N_p(v)(x_0)\right]^2} = 0, \tag{72}$$

for all $x_0 \in \Omega$. Then, $N_p(\varphi/v) = N_p(\varphi)/N_p(v) = \text{cst} = k^{p-1}$ in Ω, i.e., $\varphi = kv$ in Ω. In the same manner, we can see that $\psi = hw$ if for $x_0 \in \Omega$). Setting

$$h = \frac{\psi(x_0)}{w(x_0)}, \bar{\omega}_1(x) = \max\{\psi(x), hw(x)\} \text{ and } \bar{\omega}_2(x) = \max\left\{\varphi(x), k^{q/p}v(x)\right\}, \tag{73}$$

for all $x \in \Omega$, we can write $(\varphi, \psi) = (kv, hw)$ with $k = h^{q/p}$. We deduce that $(u_{21}, u_{22}) = (ku_{11}, hu_{12})$ with $k = h^{q/p}$. Further, let (u_{11}, u_{12}) and (u_{21}, u_{22}) be two eigenfunctions of (Σ) associated with $\lambda(.)$. If there exist $i, j \in \{1, 2\}$ such that $u_{ij} < 0$, then we can set $\bar{u}_{ij} = -u_{ij}$ and the result follows.

Case 2: $\lambda(.)$ is a semitrivial principal eigensurface of (Σ).

Let $[(u_{11}, 0)$ and $(u_{21}, 0)]$ or $[(0, u_{12})$ and $(0, u_{22})]$ be two eigenfunctions of (Σ) associated with $\lambda(.)$. It is obvious to see that there exist $[k \neq 0$ real number$]$ or $[h \neq 0$ real number$]$ such that $[u_{11} = ku_{21}]$ or $[u_{12} = hu_{22}]$. The proof is complete.

We are now ready to state the main result of this section concerning (Σ).

Theorem 1. *Assume that* (H_m) *holds. The lowest positive eigensurface of problem* (Σ) *is* $\lambda_1(m,.,m_1, m_2)$ *defined by*

$$\lambda_1(m, \beta, m_1, m_2) = \min_{(u,v)\in\mathcal{S}_\beta} E_{\beta,m}(u, v), \tag{74}$$

for all $\beta \in \mathbb{R}^N$ *and*

$$\mathcal{S}_\beta = \left\{(u, v) \in Y_{pq}(\Omega): \quad M_\beta(u, v) = 1\right\}. \tag{75}$$

Moreover,

(1) $\lambda_1(m,.,m_1, m_2) \leq \min\{\Gamma_1^p(.,m_1),\Gamma_1^q(.,m_2)\}$

(2) $\lambda_1(m,.,,m_1, m_2)$ *is semitrivial principal eigensurface or strictly principal eigensurface*

(3) $\lambda_1(m,.,m_1, m_2)$ *is simple*

Proof. Combining Proposition 2 and Lemma 4, there exists a unique $\lambda_1(m,.,m_1, m_2)$ solution of equation $\mu_1(.,\lambda) \equiv 0$, that is, $\lambda_1(m,.,m_1, m_2)$ is an eigensurface of (Σ) and

$$\mu_1'(\beta, \lambda_1(m, \beta, m_1, m_2)) = -M_\beta(u_0, v_0) < 0 = \mu_1(\beta, \lambda_1(m, \beta, m_1, m_2)) = E_{\beta,m}(u_0, v_0) - \lambda_1(m, \beta, m_1, m_2)M_\beta(u_0, v_0), \tag{76}$$

for all $\beta \in \mathbb{R}^N$, with $(u_0, v_0) \in \mathcal{M}_\beta$. Then, $E_{\beta,m}(u_0, v_0) = \lambda_1(m, \beta, m_1, m_2)M_\beta(u_0, v_0) > 0$ and we can set

$$\bar{u}_0 = \frac{u_0}{\left[M_\beta(u_0, v_0)\right]^{1/p}}, \quad \bar{v}_0 = \frac{v_0}{\left[M_\beta(u_0, v_0)\right]^{1/q}}. \tag{77}$$

We easily prove that $(\bar{u}_0, \bar{v}_0) \in \mathcal{S}_\beta$ and infer $E_{\beta,m}(\bar{u}_0, \bar{v}_0) = \lambda_1(m, \beta, m_1, m_2)$. On the other hand, for each $(u, v) \in \mathcal{S}_\beta$, we have

$$E_{\beta,m}\left(\frac{u}{\left[I_\beta(u, v)\right]^{1/p}}, \frac{v}{\left[I_\beta(u, v)\right]^{1/q}}\right) \geq \lambda_1(m, \beta, m_1, m_2)M_\beta \cdot \left(\frac{u}{\left[I_\beta(u, v)\right]^{1/p}}, \frac{v}{\left[I_\beta(u, v)\right]^{1/q}}\right), \tag{78}$$

i.e., $E_{\beta,m}(u, v) \geq \lambda_1(m, \beta, m_1, m_2)$. Therefore, (73) holds, and applying Proposition 4, we get that $\lambda_1(m,.,m_1, m_2)$ is a strictly principal eigenvalue or semitrivial principal eigenvalue and simple.

Finally, what is left is to show that $\lambda_1(m,.,m_1, m_2) \leq \min\{\Gamma_1^p(.,m_1),\Gamma_1^q(.,m_2)\}$. To do this, consider $\varphi_p = (p/(\alpha_1 + 1))^{1/p}\varphi_{p,m_1}$ and $\varphi_q = (q/(\alpha_2 + 1))^{1/q}\varphi_{q,m_2}$. Then, for all $\beta \in \mathbb{R}^N$, we have

$$\frac{\alpha_1 + 1}{p}M_{1,\beta}\left(\varphi_p\right) + \frac{\alpha_2 + 1}{q}M_{2,\beta}(0) = 1, \quad \frac{\alpha_1 + 1}{p}M_{1,\beta}(0) + \frac{\alpha_2 + 1}{q}M_{2,\beta}\left(\varphi_q\right) = 1. \tag{79}$$

Consequently,

$$\begin{cases} \lambda_1(m, \beta, m_1, m_2) \leq E_{\beta,m}\left(\varphi_p, 0\right) = \dfrac{\alpha_1 + 1}{p}E_\beta\left(\varphi_p, 0\right) = \Gamma_1^p(\beta, m_1), \\ \lambda_1(m, \beta, m_1, m_2) \leq E_{\beta,m}\left(0, \varphi_q\right) = \dfrac{\beta + 1}{q}E_\beta\left(0, \varphi_q\right) = \Gamma_1^q(\beta, m_2), \end{cases} \tag{80}$$

and the result follows. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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