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Research Article

Darboux Vector in Four-Dimensional Space-Time

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As the space-time model of the theory of relativity, four-dimensional Minkowski space is the basis of the theoretical framework for the development of the theory of relativity. In this paper, we introduce Darboux vector fields in four-dimensional Minkowski space. Using these vector fields, we define some new planes and curves. We find that the new planes are the instantaneous rotation planes of rigid body moving in four-dimensional space-time. In addition, according to some characteristics of Darboux vectors in geometry, we define some new space curves in four-dimensional space-time and describe them with curvature functions. Finally, we give some examples.

1. Introduction

On the basis of the principle of relativity and Lorentz transformation, in 1907, Minkowski proposed to add a time dimension on the basis of three space dimensions, thus forming a four-dimensional space-time, and this space-time is also called Minkowski 4-space. The metric tensor g in $\mathbb{E}_1^4$ is given by

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2, \tag{1}$$

where (x_1, x_2, x_3, x_4) is a standard rectangular coordinate system in $\mathbb{E}_1^4$. Minkowski space is not only closely related to physics but also provides theoretical and methodological support for the study of astrophysics and cosmology [1–4]. The study of submanifolds in Minkowski space is of interest in relativity theory; therefore, more and more geometers and physicists are committed to the study of submanifolds in Minkowski space. For example, in [5], the authors studied some local properties of slant geometry on spacelike submanifolds of codimension two in Lorentz-Minkowski space and investigate spacelike curves in Lorentz-Minkowski 3-space from different viewpoints as another special case. In [6], the authors studied null helices of 1-dimensional lightlike submanifolds and gave some characterizations of null helices in $\mathbb{R}_1^3$. We refer the reader to [7–17] and the references therein for more related works.

The Darboux vector is the local speed vector of the Frenet frame of space curves, which was discovered and named after Gaston Darboux [18]. If an object moves along a regular curve, we can use the Frenet frame of space curves to describe the motion of the object in terms of two vectors: the translation vector and the rotation vector, where the rotation vector is the Darboux vector. Because the Darboux vector is directly related to the angular momentum, it is also called the angular momentum vector.

In the past few decades, many researchers have mainly studied Darboux vectors in 3-dimensional space [19–26] and have obtained some interesting conclusions. For example, in 2012, Ziplar introduced and studied Darboux helices in Euclidean 3-space and proved that Darboux helices coincide with slant helices [19]. In [20], Öztürk and Nešovic′ defined the pseudo null and null Cartan Darboux helices in Minkowski 3-space and obtained the relationship between pseudo null, null Cartan Darboux helices, and slant helices. In [21], the quasi Darboux vector field of null curve in Minkowski 3-space was defined, and some interesting conclusions about osculating developable of null curve which is defined by quasi Darboux vector field of null curve were obtained. Wang and Pei defined the Darboux vector of the null curve in [23] and described the direction of the rotation axis of the Cartan frame in Minkowski 3-space. Later, in 2017, Düldül [27] extended the Darboux frame field to four-dimensional Euclidean space and gave the relationship between the curvature of Frenet frame and Darboux frame. In [28], Düldül defined some new vector fields in four-dimensional Euclidean space and showed that the determined new planes play the role of the Darboux vector. İlarslan and Yildirim [29] defined the Darboux helices in four-dimensional Euclidean space as a curve whose Darboux vector makes a constant angle with some fixed direction and obtained relation between the curves Darboux helix, general helix, and V_4-slant helix in a special case.

Motivated by those ideas, in this paper, we construct four new vector fields along the space curve whose curvatures do not disappear in four-dimensional space-time. Based on these vector fields, we define some new planes and helices in four-dimensional space-time. The corresponding curvature functions are given when the position vectors of the curves lie on different planes. Moreover, we define Darboux helices in Minkowski 4-space and give some descriptions of their curvature functions.

2. Preliminaries

Four-dimensional space-time $\mathbb{E}_1^4$ is the real four-dimensional vector space $\mathbb{R}^4$ equipped with the standard flat metric given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4. \tag{2}$$

For any three vectors $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$, and $z = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}_1^4$, their exterior product is given by

$$x \times y \times z = \begin{vmatrix} e_1 & e_2 & e_3 & -e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \tag{3}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthogonal basis in $\mathbb{E}_1^4$, that is,

$$e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1). \tag{4}$$

A vector v in $\mathbb{E}_1^4$ is called spacelike, timelike, or null (lightlike), if $\langle v, v \rangle > 0$ or $\langle v, v \rangle < 0$, $\langle v, v \rangle = 0$, respectively. In particular, the vector $v = 0$ is said to be spacelike. A curve $\gamma(s)$: $I \longrightarrow \mathbb{E}_1^4$ is called spacelike, timelike, or null (lightlike) if all of its velocity vectors $\gamma'(s)$ satisfy $\langle \gamma', \gamma' \rangle > 0$, $\langle \gamma', \gamma' \rangle < 0$, or $\langle \gamma', \gamma' \rangle = 0$, respectively. The norm of a vector v in $\mathbb{E}_1^4$ is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$ [15].

Definition 1 (see [30]). Let $\gamma(s)$ be a null curve parameterized by null arc length s (i.e., $\|\gamma''(s)\| = 1$) in $\mathbb{E}_1^4$. Then, $\gamma(s)$ can be framed by a Cartan Frenet frame $\{T, N, B_1, B_2\}$ such that

$$\begin{bmatrix} T' \\ N' \\ B_{1'} \\ B_{2'} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_2 & 0 & -1 & 0 \\ 0 & -k_2 & 0 & k_3 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned} \langle N, B_2 \rangle &= \langle T, B_2 \rangle = \langle N, B_1 \rangle = \langle T, N \rangle = \langle T, T \rangle \\ &= \langle B_1, B_1 \rangle = \langle B_1, B_2 \rangle = 0, \\ \langle N, N \rangle &= \langle B_2, B_2 \rangle = \langle T, B_1 \rangle = 1, \\ T \times N \times B_1 &= B_2, N \times B_1 \times B_2 = B_1, B_1 \times B_2 \times T \\ &= N, B_2 \times T \times N = T. \end{aligned} \tag{6}$$

In sequence, T, N, B_1, B_2 are called the tangent, principal normal, first binormal, and second binormal vector field of $\gamma(s)$ and k_2 and k_3 are first curvature and second curvature of the curve $\gamma(s)$, respectively.

Definition 2 (see [16]). Let$\gamma(s)$ be a pseudo null curve parameterized by arc length s (i.e., $\|\gamma''(s)\| = 0$) in $\mathbb{E}_1^4$. Then, the Frenet equation is defined by

$$\begin{bmatrix} T' \\ N' \\ B_{1'} \\ B_{2'} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 0 & k_2 & 0 & -k_1 \\ -1 & 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} \langle B_2, B_2 \rangle &= \langle B_1, B_2 \rangle = \langle N, B_1 \rangle = \langle T, B_1 \rangle \\ &= \langle T, N \rangle = \langle T, B_2 \rangle = \langle N, N \rangle = 0, \\ \langle N, B_2 \rangle &= \langle B_1, B_1 \rangle = \langle T, T \rangle = 1, \\ T \times N \times B_1 &= N, N \times B_1 \times B_2 = T, B_1 \times B_2 \times T \\ &= B_2, B_2 \times T \times N = B_1, \end{aligned} \tag{8}$$

and k_1 and k_2 are first curvature and second curvature of the curve $\gamma(s)$, respectively.

3. Darboux Helix and Planes of Null Curve

When the Frenet frame $\{T, N, B_1, B_2\}$ of a nongeodesic null curve makes an instantaneous helix motion in $\mathbb{E}_1^4$, there exists an axis of the frame's rotation. The direction of such axis is given by the vector

$$D_1 = k_2 T + B_1, D_2 = B_2, D_3 = T, D_4 = k_3 N + k_2 B_2, \tag{9}$$

and we call them the Darboux vectors for the null curves in $\mathbb{E}_1^4$. The Darboux vectors satisfy the Darboux equations

$$\begin{aligned} T' &= D_1 \times D_2 \times T, \\ N' &= D_2 \times D_1 \times N, \\ B_{1'} &= D_3 \times D_4 \times B_1, \\ B_{2'} &= D_4 \times D_3 \times B_2. \end{aligned} \tag{10}$$

From (10), we know that Frenet vectors T and N rotate around the D_1D_2 plane, and Frenet vectors B_1 and B_2 rotate around the D_3D_4 plane. We find that the D_1D_2 plane and D_3D_4 plane play the role of Darboux vector in three-dimensional space. We also note that D_2 and D_3 are Frenet vectors of the null curve, $\{D_1, D_2, D_3, D_4\}$ is linearly independent, and D_1 is orthogonal to D_2 and D_4. We are going to use the subspace spanned by $\{D_1, D_2\}$ and $\{D_1, D_4\}$ to represent D_1D_2 plane and D_1D_4 plane, respectively.

Inspired by [10, 28], we discuss the situation when the curve $\gamma(s)$ lies in D_1D_2 and D_1D_4 planes.

Theorem 3. *Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a null curve parameterized by null arc length s. k_2, k_3 are the curvature functions of the null curve $\gamma(s)$. If $\gamma(s)$ lies in D_1D_2 plane, then the curvature functions k_2, k_3 satisfy*

$$\left[\frac{(1/k_3)'}{k_3 + (k_{2'}/k_3)'}\right]' = 0, \tag{11}$$

and in addition, the curve $\gamma(s)$ can be expressed as

$$\gamma(s) = cD_1 + \frac{ck_{2'} - 1}{k_3} D_2, \tag{12}$$

where c is nonzero constant.

Proof. We may assume that

$$\gamma(s) = \lambda(s)D_1 + \mu(s)D_2, \tag{13}$$

and we take the derivative of (13) according to s, and we obtain

$$T = \gamma'(s) = \left((\lambda k_2)' - \mu k_3\right)T + \lambda' B_1 + \left(\lambda k_3 + \mu'\right)B_2. \tag{14}$$

Hence,

$$\begin{cases} (\lambda k_2)' - \mu k_3 = 1, \\ \lambda' = 0, \\ \lambda k_3 + \mu' = 0. \end{cases} \tag{15}$$

From the second equation of (15), we get

$$\lambda = c, \tag{16}$$

and substituting (16) into the first equation of (15), we have

$$\mu = \frac{ck_{2'} - 1}{k_3}. \tag{17}$$

Then, the curve $\gamma(s)$ can be denoted as

$$\gamma(s) = cD_1 + \frac{ck_{2'} - 1}{k_3} D_2. \tag{18}$$

From the third equation of (15), we get

$$c = \frac{(1/k_3)'}{k_3 + (k_{2'}/k_3)'}, \tag{19}$$

that is,

$$\left[\frac{(1/k_3)'}{k_3 + (k_{2'}/k_3)'}\right]' = 0. \tag{20}$$

This ends the proof. □

Corollary 4. *In particular, when $k_3 = a = constant \neq 0$, we have $k_2 = (a^2/2)s^2 + b$, and the curve $\gamma(s)$ can be expressed as*

$$\gamma(s) = cD_1 + \frac{ca^2 s - 1}{a} D_2, \tag{21}$$

where a and b are constants.

Theorem 5. *Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a null curve parameterized by null arc length s. k_2, k_3 are the curvature functions of the pseudonull curve $\gamma(s)$. If the $\gamma(s)$ lies in D_1D_4 plane, then the curvature functions k_2, k_3 satisfy*

$$k_2 = \frac{s + c_3}{c_1 s + c_2},\ k_3^2 = \frac{c_1 (s + c_3)^2}{(c_1 s + c_2)^2 \left(1 - (s + c_3)^2\right)}, \tag{22}$$

and in addition, the curve $\gamma(s)$ can be expressed as

$$\gamma(s) = (c_1 s + c_2)D_1 + \frac{c_1}{k_3} D_4, \tag{23}$$

where c_1, c_2, and c_3 are constants.

Proof. Assume that

$$\gamma(s) = \lambda D_1 + \mu D_4. \tag{24}$$

Differentiating equation (24) with respect to s, we have

$$\begin{aligned} T = \gamma' &= (\lambda k_2)' T + (\mu k_3)' N + \left(\lambda' - \mu k_3\right)B_1 \\ &\quad + \left((\mu k_2)' + \lambda k_3\right)B_2. \end{aligned} \tag{25}$$

Then, we obtain the system of differential equations

$$\begin{cases} (\lambda k_2)' = 1, \\ (\mu k_3)' = 0, \\ \lambda' - \mu k_3 = 0, \\ (\mu k_2)' + \lambda k_3 = 0. \end{cases} \tag{26}$$

From the second equation of (26), we get

$$\mu k_3 = c_1. \tag{27}$$

Substituting (27) into the third equation of (26), we have

$$\lambda = c_1 s + c_2. \tag{28}$$

Then, the curve $\gamma(s)$ can be denoted as

$$\gamma(s) = (c_1 s + c_2) D_1 + \frac{c_1}{k_3} D_4. \tag{29}$$

Substituting (28) into the first equation of (26), we have

$$k_2 = \frac{s + c_3}{c_1 s + c_2}. \tag{30}$$

Substituting (27) and (28) into the fourth equation of (26), we can calculate that

$$k_3^2 = \frac{c_1 (s + c_3)^2}{(c_1 s + c_2)^2 \left(1 - (s + c_3)^2\right)}. \tag{31}$$

This ends the proof. □

Definition 6. Let $\gamma(s)$: $I \longrightarrow \mathbb{E}_1^4$ be a null curve with parameterized by null arc length s. If there exists a fixed direction $V \neq 0$ such that

$$\langle D_1, V \rangle = a, a \in \mathbb{R}, \tag{32}$$

then the null curve $\gamma(s)$ is called the null Darboux helix, and the fixed direction V is called an axis of the null Darboux helix.

Theorem 7. *Let $\gamma(s)$: $I \longrightarrow \mathbb{E}_1^4$ be a null curve with parameterized by null arc length s. If $\gamma(s)$ is a null Darboux helix in $\mathbb{E}_1^4$ whose fixed direction V satisfies*

$$\langle D_1, V \rangle = a, a \in \mathbb{R}, \tag{33}$$

then V is given by

$$V = \left(a - b k_2 e^{\int \xi ds}\right) T + b \xi e^{\int \xi ds} N + b e^{\int \xi ds} B_1 - \frac{b k_{2'}}{k_3} e^{\int \xi ds} B_2, \tag{34}$$

and the curvature functions k_2, k_3 satisfy

$$\left(2k_2 - \xi^2 - \xi'\right) e^{\int \xi ds} = \frac{a}{b}, \tag{35}$$

where

$$\xi = \frac{k_3 - (k_{2'}/k_3)'}{k_{2'}/k_3}, \tag{36}$$

and $b \in \mathbb{R}_0, k_3 \neq 0, k_2 \neq const.$

Proof. Let $\gamma(s)$: $I \longrightarrow \mathbb{E}_1^4$ be a null Darboux helix with parameterized by null arc length s. Then, for a fixed direction V satisfying

$$\langle D_1, V \rangle = a, a \in \mathbb{R}, \tag{37}$$

we can assume

$$V = u_1 T + u_2 N + u_3 B_1 + u_4 B_2. \tag{38}$$

By using (5), we can obtain

$$\langle D_1, V \rangle = k_2 u_3 + u_1 = a, \tag{39}$$

$$\langle D_{1'}, V \rangle = k_{2'} u_3 + k_3 u_4 = 0. \tag{40}$$

Taking the derivative of equation (39) according to s, we obtain

$$u_{1'} = -(k_2 u_3)'. \tag{41}$$

Differentiating equation (38) and using the Frenet equation (5), we have

$$\begin{cases} u_{1'} - k_3 u_4 + k_2 u_2 = 0, \\ u_1 + u_{2'} - k_2 u_3 = 0, \\ u_2 - u_{3'} = 0, \\ u_{4'} + k_3 u_3 = 0. \end{cases} \tag{42}$$

By (40), we can obtain

$$u_4 = -\frac{k_{2'}}{k_3} u_3, \tag{43}$$

Substituting (43) into the fourth equation of (42), we can obtain

$$u_3 = b e^{\int \xi ds}, \xi = \frac{k_3 - (k_{2'}/k_3)'}{k_{2'}/k_3}. \tag{44}$$

From (39), (44), and the third equation of (42), we have

$$\begin{aligned} u_2 &= u_{3'} = b \xi e^{\int \xi ds}, \\ u_1 &= a - k_2 u_3 = -b \xi e^{\int \xi ds} + a. \end{aligned} \tag{45}$$

Thus,

$$V=\left(a-bk_2e^{\int\xi ds}\right)T+b\xi e^{\int\xi ds}N+be^{\int\xi ds}B_1-\frac{bk_{2'}}{k_3}e^{\int\xi ds}B_2. \quad (46)$$

From the second equation of (42), the relationship between k_2 and k_3 can be expressed as

$$\left(2k_2-\xi^2-\xi'\right)e^{\int\xi ds}=\frac{a}{b}, \quad (47)$$

where b is given by the relation (44), and if $b=0$, the axis $V=0$, which is a contradiction. Hence, $b\neq 0$, which completes the proof. □

Corollary 8. *In particular, when $a=0, b=1$, we have*

$$V=\left(-k_2T+\xi N+B_1-\frac{k_{2'}}{k_3}B_2\right)e^{\int\xi ds}, \quad (48)$$

and the curvature functions k_2, k_3 satisfy

$$\xi'+\xi^2-2k_2=0, \quad (49)$$

where

$$\xi=\frac{k_3-(k_{2'}/k_3)'}{k_{2'}/k_3}. \quad (50)$$

Some examples of null Darboux helix in $\mathbb{E}_1^4$ are given below.

Example 1. Let $\gamma(s): I\longrightarrow\mathbb{E}_1^4$ be a null curve with the arc length s and the curvature

$$k_2=\frac{s^2+1}{2}, k_3=\frac{s}{\sqrt{e^{-s^2}+1}}, \quad (51)$$

and then, $\gamma(s)$ is a Darboux helix whose fixed direction is given by

$$V=\left(-\frac{s^2+1}{2}T+sN+B_1-\sqrt{s^2+1}B_2\right)e^{s^2/2}. \quad (52)$$

Example 2. Let $\gamma(s): I\longrightarrow\mathbb{E}_1^4$ be a null curve with the arc length s and the curvature

$$k_2=\frac{\sec^2 s+\tan^2 s}{2}, k_3=2\sec s\tan s, \quad (53)$$

and then, $\gamma(s)$ is a Darboux helix whose fixed direction is given by

$$V=\left(-\frac{1+\sin^2 s}{2\cos^2 s}T+\tan sN+B_1-\sec sB_2\right)\sec s. \quad (54)$$

Example 3. Let $\gamma(s): I\longrightarrow\mathbb{E}_1^4$ be a null curve with the arc length s and the curvature

$$k_2=\frac{1}{s^2},\quad k_3=-\frac{2}{s^2}, \quad (55)$$

and then, $\gamma(s)$ is a Darboux helix whose fixed direction is given by

$$V=\left(-\frac{1}{s^2}T-\frac{1}{s}N+B_1-\frac{1}{s}B_2\right)\frac{1}{s}. \quad (56)$$

4. Darboux Helix and Planes of Pseudo null Curve

When the Frenet frame $\{T, N, B_1, B_2\}$ of a nongeodesic pseudonull curve makes an instantaneous helix motion in $\mathbb{E}_1^4$, there exists an axis of the frame's rotation. The direction of such axis is given by the vector

$$D_1=N, D_2=k_2T, D_3=T, D_4=-k_2N-k_1B_2, \quad (57)$$

and we call them the Darboux vectors for the pseudo null curves in $\mathbb{E}_1^4$. The Darboux vectors satisfy the Darboux equations

$$\begin{aligned}T'&=D_1\times D_2\times T,\\ N'&=D_3\times D_4\times N,\\ B_{1'}&=D_4\times D_3\times B_1,\\ B_{2'}&=D_2\times D_1\times B_2.\end{aligned} \quad (58)$$

From (58), we know that Frenet vectors T and B_2 rotate around the D_1D_2 plane, and Frenet vectors N and B_1 rotate around the D_3D_4 plane. We find that the D_1D_2 plane and D_3D_4 plane play the role of Darboux vector in three-dimensional space. We also note that D_1 and D_3 are Frenet vectors of the pseudonull curve, $\{D_1, D_2, D_3, D_4\}$ is linearly independent, and D_4 is orthogonal to D_2 and D_3. We are going to use the subspace spanned by $\{D_2, D_4\}$ and $\{D_3, D_4\}$ to represent D_2D_4 plane and D_3D_4 plane, respectively.

Similar to Section 3, we discuss the situation when the curve $\gamma(s)$ is in D_2D_4 and D_3D_4 planes.

Theorem 9. *Let $\gamma(s): I\longrightarrow\mathbb{E}_1^4$ be a pseudonull curve with parameterized by arc length s. k_1, k_2 are the curvature functions of the pseudonull curve $\gamma(s)$. If $\gamma(s)$ lies in D_2D_4 plane, then the curvature functions k_1, k_2 satisfy*

$$c_2k_{2''}+c_1k_2k_{2'}+k_2=0, \quad (59)$$

and in addition, the curve $\gamma(s)$ can be expressed as

$$\gamma(s)=c_1D_2+\frac{c_2}{k_2}D_4, \quad (60)$$

where c_1 and c_2 are nonzero constants.

Proof. We may assume that

$$\gamma(s) = \lambda D_2 + \mu D_4, \tag{61}$$

and we take the derivative of (61) according to s, and we obtain

$$\begin{aligned} T = \gamma' &= \left(\mu k_1 - (\lambda k_2)'\right) T - (\mu k_2)' N \\ &+ \lambda' B_1 - \left(\lambda k_1 + (\mu k_1)'\right) B_2. \end{aligned} \tag{62}$$

Hence,

$$\begin{cases} u_1 k_1 - (\lambda k_2)' = 1, \\ (\mu k_2)' = 0, \\ \lambda' = 0, \\ \lambda k_1 + (\mu k_1)' = 0. \end{cases} \tag{63}$$

From the second and the third equations of (63), we get

$$\lambda = c_1, \mu k_2 = c_2, \tag{64}$$

where c_1 and c_2 are nonzero constants.

Then, the curve $\gamma(s)$ can be denoted as

$$\gamma(s) = c_1 D_2 + \frac{c_2}{k_2} D_4. \tag{65}$$

Substituting (64) into the fourth equation of (63), we have

$$c_2 k_1 = c_1 k_2 k_{2'} + k_2. \tag{66}$$

From (66) and the first equation of (63), we get

$$c_2 k_{2''} + c_1 k_2 k_{2'} + k_2 = 0. \tag{67}$$

This ends the proof. □

Theorem 10. *Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudonull curve with parameterized by arc length s. k_1, k_2 are the curvature functions of the pseudonull curve $\gamma(s)$. If the $\gamma(s)$ lies in $D_3 D_4$ plane, then the curvature functions k_1, k_2 satisfy*

$$\frac{k_2}{k_1} = \frac{(1 - c_1) s^2}{2 c_1} + \frac{c_2 s}{c_1} + \frac{c_3}{c_1}, \tag{68}$$

and in addition, the curve can be expressed as

$$\gamma(s) = (c_2 + (1 - c_1) s) D_3 + \frac{c_1}{k_1} D_4, \tag{69}$$

where c_1, c_2, c_3 are constants.

Proof. Assume that

$$\gamma(s) = \lambda D_3 + \mu D_4. \tag{70}$$

Differentiating equation (70) with respect to s, we have

$$T = \gamma' = \left(\mu k_1 + \lambda'\right) T - \left(\lambda - (\mu k_2)'\right) N - (\mu k_1)' B_2. \tag{71}$$

So we obtain the system of differential equations

$$\begin{cases} \mu k_1 + \lambda' = 1, \\ \lambda - (\mu k_2)' = 0, \\ (\mu k_1)' = 0. \end{cases} \tag{72}$$

From the first and the third equations of (72), we get

$$\lambda = (1 - c_1) s + c_2, \tag{73}$$

$$\mu k_1 = c_1. \tag{74}$$

Then, the curve $\gamma(s)$ can be denoted as

$$\alpha(s) = ((1 - c_1) s + c_2) D_3 + \frac{c_1}{k_1} D_4. \tag{75}$$

Substituting (73) and (74) into the second equation of (72), we have

$$\mu k_2 = \frac{1}{2} (1 - c_1) s^2 + c_2 s + c_3. \tag{76}$$

From equations (74) and (76), we can obtain

$$\frac{k_2}{k_1} = \frac{(1 - c_1) s^2}{2 c_1} + \frac{c_2 s}{c_1} + \frac{c_3}{c_1}, \tag{77}$$

where c_1, c_2, c_3 are constants. □

Corollary 11. *In particular, when $c_1 = 1/1$, $c_2 = c_3 = 0$, we have $k_2/k_1 = s^2$. Let $k_1 = 3s/(s^2 + 1)^2$ and $k_2 = 3s^3/(s^2 + 1)^2$. Then, the curve $\gamma(s)$ can be expressed as*

$$\gamma'' = \frac{3s\sqrt{s^2 + 1}}{2(s^2 + 1)^3} \left(2s, s^2, 1 - s^2 - \frac{s^4}{4}, 1 + s^2 + \frac{s^4}{4}\right). \tag{78}$$

Definition 12. Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudo null curve with parameterized by arc length s. If there exists a fixed direction $V \neq 0$ such that

$$\langle D_2, V \rangle = a, a \in \mathbb{R}, \tag{79}$$

then the pseudo null curve $\gamma(s)$ is called the pseudo null Darboux helix, and the fixed direction V is called an axis of the pseudo null Darboux helix.

Theorem 13. *Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudonull curve with parameterized by arc length s. If $\gamma(s)$ is a pseudonull Darboux* *helix in $\mathbb{E}_1^4$ whose fixed direction V satisfies*

$$\langle D_2, V\rangle = a, a \in \mathbb{R}, \tag{80}$$

then V is given by

$$V = u_1 T - \frac{k_{2'}}{k_1} u_1 N + (u_1 k_2 + a) B_1 + u_{1'} B_2, \tag{81}$$

and the curvature functions k_1, k_2 satisfy

$$u_{1''} - k_1 k_2 u_1 = a k_1, \tag{82}$$

where

$$\xi = \frac{1 + k_2^2 - (k_{2'}/k_1)'}{k_{2'}/k_1}, \tag{83}$$

and $a, b \in \mathbb{R}, k_1 \neq 0, k_2 \neq const.$

Proof. Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudo null Darboux helix with parameterized by arc length s. Then, for a fixed direction V satisfying

$$\langle D_2, V\rangle = a, a \in \mathbb{R}, \tag{84}$$

we can assume

$$V = u_1 T + u_2 N + u_3 B_1 + u_4 B_2. \tag{85}$$

By using (84), we can obtain

$$\langle D_2, V\rangle = u_3 - u_1 k_2 = a, \tag{86}$$

$$\langle D_{2'}, V\rangle = -k_1 u_2 - k_{2'} u_1 = 0. \tag{87}$$

Taking the derivative of equation (86) according to s, we obtain

$$u_{3'} = (k_2 u_1)'. \tag{88}$$

Differentiating equation (85) and using the Frenet equation (7), we have

$$\begin{cases} u_{1'} - u_4 = 0, \\ u_1 + u_{2'} + k_2 u_3 = 0, \\ k_1 u_2 + u_{3'} - k_2 u_4 = 0, \\ u_{4'} - k_1 u_3 = 0. \end{cases} \tag{89}$$

Substituting (86) and (87) into the second equation of (89), we can obtain

$$u_{1'} - \frac{1 + k_2^2 - (k_{2'}/k_1)'}{k_{2'}/k_1} u_1 = a k_2. \tag{90}$$

By (90), we can obtain

$$u_1 = e^{\int \xi ds}\left(a k_2 e^{-\int \xi ds} + b\right), \tag{91}$$

where

$$\xi = \frac{1 + k_2^2 - (k_{2'}/k_1)'}{k_{2'}/k_1}. \tag{92}$$

From (87), (88), and the first equation of (89), we have

$$\begin{aligned} u_2 &= -\frac{k_{2'}}{k_1} u_1 = -\frac{k_{2'}}{k_1} e^{\int \xi ds}\left(a k_2 e^{-\int \xi ds} + b\right), \\ u_3 &= u_1 k_2 + a = k_2 e^{\int \xi ds}\left(a k_2 e^{-\int \xi ds} + b\right) + a, \\ u_4 &= u_{1'} = c_1 k_2 + \xi e^{\int \xi ds}\left(a k_2 e^{-\int \xi ds} + b\right). \end{aligned} \tag{93}$$

Thus,

$$V = u_1 T - \frac{k_{2'}}{k_1} u_1 N + (u_1 k_2 + a) B_1 + u_{1'} B_2. \tag{94}$$

From the fourth equation of (89), the relationship between k_1 and k_2 can be expressed as

$$u_{1''} - k_1 k_2 u_1 = a k_1, a \in \mathbb{R}. \tag{95}$$

This ends the proof. □

Corollary 14. *In particular, when $a = 0, b = 1$, we have*

$$V = \left(T - \frac{k_{2'}}{k_1} N + k_2 B_1 + \xi B_2\right) e^{\int \xi ds}, \tag{96}$$

and the curvature functions k_2, k_3 satisfy

$$\xi' + \xi^2 - k_1 k_2 = 0, \tag{97}$$

where

$$\xi = \frac{1 + k_2^2 - (k_{2'}/k_1)'}{k_{2'}/k_1}. \tag{98}$$

Some examples of pseudo null Darboux helix in $\mathbb{E}_1^4$ are given below.

Example 4. Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudo null curve, and s is the pseudoarc length. The curvature function k_1, k_2 satisfies

$$k_1 = \frac{1}{s(s^2+1)\sqrt{s^2+1}}, \quad k_2 = \frac{s}{\sqrt{s^2+1}}, \tag{99}$$

and then, $\gamma(s)$ is a Darboux helix whose fixed direction is given by

$$V = \sqrt{s^2+1}T - s\sqrt{s^2+1}N + sB_1 + \frac{s}{\sqrt{s^2+1}}B_2. \quad (100)$$

Example 5. Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudo null curve with the arc length s and the curvature

$$k_1 = \frac{2\sec^2 s}{\sqrt{2\sec^2 s - 1}}, k_2 = \sqrt{2\sec^2 s - 1}, \quad (101)$$

and then, $\gamma(s)$ is a Darboux helix whose fixed direction is given by

$$V = \tan sT - \tan^2 sN + \tan s\sqrt{2\sec^2 s - 1}B_1 + \sec^2 sB_2. \quad (102)$$

Example 6. Let $\gamma(s): I \longrightarrow \mathbb{E}_1^4$ be a pseudo null curve with the arc length s and the curvature

$$k_1 = \frac{1}{\sqrt{2e^s - 1}}, k_2 = \sqrt{2e^s - 1}, \quad (103)$$

and then, $\gamma(s)$ is a Darboux helix whose fixed direction is given by

$$V = e^s T - e^{2s}N + e^s\sqrt{2e^s - 1}B_1 + e^s B_2. \quad (104)$$

5. Conclusion

In this paper, we discuss some new space curves and planes in four-dimensional space-time and give characterizations of them in terms of the curvature functions. Before this study, most researchers studied Darboux vector in three-dimensional space and four-dimensional Euclidean space. In this paper, the Darboux vector fields in three-dimensional space are extended to four-dimensional space-time by mathematical method. By defining Darboux vector fields in four-dimensional space-time in the form of vector products, we find that the Frenet vectors rotate around a plane spanned by two new vector fields, and this plane plays the role that the Darboux vector plays in three-dimensional space. This paper gives a new description of Darboux vector in four-dimensional space-time, which promotes the further development of angular momentum vector in physics and geometry.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors have made the same contribution.

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