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Research Article

Growth Series of the Braid Monoid MB_5 in Band Generators

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Growth series is an important invariant associated with group or monoid which classifies all the words of group or monoid. Therefore, the growth series of braid monoids and Hecke algebras in Artin's generators is presented in many scholarly published articles. The growth series of braid monoids MB_3 and MB_4 in band generators is known. In this work, we compute the complete presentation of braid monoid MB_5 in band generators by solving all the ambiguities of MB_5. The words on the left-hand of each relation are reducible words, and the words on the right-hand side are canonical words. We partially find the growth series ($Q_*^{(5)}$) of reducible words. Then, we construct a linear system for canonical words of MB_5 in band presentation and compute the corresponding growth series. We also find the growth rate of growth series of MB_5 in band generators.

1. Introduction

The growth series also known as Hilbert series is an important invariant in the study of modern geometry. In physics, growth series have recently become a power full tool in high energy theory, appearing, for example in the study of Bogomol'nyi-Prasad-Sommerfield operators of supersymmetric gauge theories [1, 2]; supersymmetric quantum chromodynamics [3, 4], and instanton moduli space [5, 6]. In [7], Hilbert series was used to construct an operator basis in 1/m expansion of a theory with a nonrelativistic heavy fermion in an electromagnetic (NRQED) or color gauge field (NRQCD/HQET).

The braid group B_{n+1} admits the presentation given by Artin [8].

$$B_{n+1} = \left\langle x_1, \cdots, x_n \middle| \begin{array}{ll} x_i x_j = x_j x_i, & \text{if } |i-j| \geq 2 \\ x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i, & \text{if } 1 \leq i \leq n-1 \end{array} \right\rangle. \tag{1}$$

The braid group B_{n+1} admits other presentations such as Sergiescu graph-presentation and Birman-Ko-Lee presentation or band presentation. The last presentation is given by

$$B_{n+1} = \left\langle a_{ts}, n \geq t > s \geq 1 \middle| \begin{array}{l} a_{ts} a_{rq} = a_{rq} a_{ts}, (t-r)(s-r)(s-q)(t-q) > 0 \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr}, n \geq t > s > r \geq 1 \end{array} \right\rangle. \tag{2}$$

Growth series of braid monoid MB_3 and MB_4 is computed in [9]. In [10], growth series of the finite dimensional Hecke algebra is presented. Growth series for graded S-module was computed by Haider in [11]. The growth series of binomial edge ideals was computed by Kumar and Sarkar in [12]. In [13], growth series of the graded algebra of real regular functions on the symplectic quotient associated to an SU_2-module is computed and given an explicit expression for the first nonzero coefficient of the Laurent expansion of the growth series at $t = 1$. Growth series and the coefficient of Laurent expansion of growth series of special linear group of 2×2 matrices are computed in [14]. In [15], Saito proved that the growth functions associated with Artin monoids of finite type are rational functions whose nominator is 1 and the denominator is polynomial $N_n(t)$ having distinct roots. Growth series of symplectic quotients by 2-torus is computed in [16]. In [17], the Laurent coefficients of the growth series of a Gorenstein algebra is presented. In the growth series $1/q_n(t)$ of MB_n (for $n = 3, 4, 5, 6$) for Artin generators, the degrees of the polynomials $q_n(t)$ are 3, 4, 10, 15, respectively (for detail see [18]). Universal upper bound for the growth of Artin monoid is computed in [19]. Growth series of braid monoid MB_3 and MB_4 in band generators is computed in [9]. The degrees of polynomial $q_n(t)$ in case of band generators are 2 and 3 (for $n = 3$ and 4). In this paper, we compute the growth series of braid monoid MB_5 using band presentation, and we see that the degree of the polynomial $q_n(t)$ is 4. We note that growth rate of MB_5 in band generators is much slower than that of Artin generators.

2. Materials and Methods

In MB_5, we fix a total order $a_{21} < a_{31} < a_{32} < \cdots < a_{n(n-1)}$ on the generators. In the monoid, the relation $\alpha = \beta$ will be written as $\alpha < \beta$ in the length-lexicographic order. Let $\alpha_1 = uw$ and $\alpha_2 = wv$; then the word of the form uwv is said to be ambiguity (for detail see [20]). If $\alpha_1 v = u\alpha_2$ as a relation as well as in the length-lexicographic order, then, we say that the ambiguity is solvable. A presentation is complete if and only if all the ambiguities are solvable (for detail see [21, 22]). Corresponding to the relations $\alpha = \beta$, the changes $\gamma\alpha\delta \longrightarrow \gamma\beta\delta$ give a rewriting system. A presentation will be called a complete presentation if and only if all the ambiguities are solvable.

In a complete presentation (or in the general presentation) of MB_{n+1} if a word W contains α, then W is called a reducible word and we denote it by B_*^n in general. If word W does not contain α, then W is called canonical word or canonical form. Let U and V be nonempty words; then, the word $Ua_{ij}V$ will be denoted as $a_{ij} \times {}_{ij}a_{ij}V$.

Definition 1 (see [23]). Let G be a finitely generated group and S be a finite set of generators of G. Then, the word length $l_s(g)$ of an element $g \in G$ is the smallest integer n for which there exist $s_1, \cdots, s_n \in SUS^{-1}$ such that $g = s_1, \cdots, s_n$.

Definition 2 (see [23]). Let G be a finitely generated group and S be a finite set of generators of G. Then, the growth function of the pair (G, S) associates to an integer $k \geq 0$ the number $a(k)$ of the element $g \in G$ such that $l_s(g) = k$, and the corresponding growth series is given by $P_G(t) = \sum_{k=0}^{\infty} a(k)t^k$.

In 2008, Bokut [22] gave the Gröbner-shirshov basis (GSB) of B_{n+1} in band generators. The notion of this basis is in [14, 20, 24–27] under different names: complete presentation, presentations with solvable ambiguities, Gröbner-shirshov basis, rewriting system, and so on. In [28], we proved the subset of GSB of B_{n+1} given by Bokut [26] is a GSB of MB_{n+1}. Using the notation (used in [28]) (t, s) for generator a_{ts} and $V_{[t,s]}$ or $W_{[t,s]}$ for the words in a_{kl} such that $t \geq k > l \geq s$, we have.

Theorem 3 (see [28]). *A GSB of braid monoid MB_{n+1} consist of following relations:*

$$
\begin{gathered}
(k,l)(i,j) = (i,j)(k,l)\; k > l > i > j, \\
(k,l)V_{[i-1,l]}(i,j) = (i,j)V_{[j-1,l]}(k,l) k > i > j > l, \\
(t_3,t_2)(t_2,t_1) = (t_2,t_1)(t_3,t_1), \\
(t_3,t_1)V_{[t_2-1,l]}(t_3,t_2) = (t_2,t_1)(t_3,t_1)V_{[t_2-1,l]}, \\
(t,s)V_{[t_2-1,l]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_3,t_2)(t,s)V_{[t_2-1,l]}(t_2,t_1)W'_{[t_3-1,t_1]}, \\
(t_3,s)V_{[t_2-1,l]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_2,s)(t_3,s)V_{[t_2-1,l]}(t_2,t_1)W'_{[t_3-1,t_1]}, \\
\text{For } t_3 > t_2 > t_1, t > t_3, t_2 > s \text{ and } W_{[t_3-1,t_1]}(t_3,t_1) = (t_3,t_1)W'_{[t_3-1,t_1]},
\end{gathered} \tag{3}
$$

where $W'_{[t_3-1,t_1]} = W_{[t_3-1,t_1]}|(p,q) \longrightarrow (p,q)$, *if* $q \neq t_1$; $(p,t_1) \longrightarrow (t_3,p)$.

Proposition 4 (see [9]). *Solution of linear system for canonical words of braid monoid MB_4 is given by*

$$
\begin{aligned}
P_{21}^{(4)} &= \frac{t}{(1-t)(1-5t+5t^2)}, \\
P_{31}^{(4)} &= \frac{t}{1-5t+5t^2}, \\
P_{32}^{(4)} &= \frac{t}{1-5t+5t^2}, \\
P_{41}^{(4)} &= \frac{t(1-2t)}{1-5t+5t^2}, \\
P_{42}^{(4)} &= \frac{t-t^2}{1-5t+5t^2}, \\
P_{43}^{(4)} &= \frac{t(1-2t)}{1-5t+5t^2}.
\end{aligned} \tag{4}
$$

And the growth series of MB_4 in band generators is given as

$$
P_M^{(4)}(t) = \frac{1}{(1-t)(1-5t+5t^2)}. \tag{5}
$$

3. Results and Discussion

In this paper, we compute the growth series of MB_5 (in band presentation). From Equation (2), we have the following band-presentation of MB_5:

$$
\left\langle a_{54}, a_{53}, a_{52}, a_{51}, a_{43}, a_{42}, a_{41}, a_{32}, a_{31}, a_{21} \middle| R_1^{(3)}, R_2^{(3)}, R_i^{(4)}, R_j^{(5)}, \quad i = 3, \cdots, 10, j = 11, \cdots, 30 \right\rangle, \tag{6}
$$

where

$$\begin{aligned}
R_1^{(3)} &: a_{31}a_{32} = a_{21}a_{31}, R_2^{(3)} : a_{32}a_{21} = a_{21}a_{31}, R_3^{(4)} \\
&: a_{41}a_{32} = a_{32}a_{41}, R_4^{(4)} : a_{41}a_{42} = a_{21}a_{41}, R_5^{(4)} \\
&: a_{41}a_{43} = a_{31}a_{41}, R_6^{(4)} : a_{42}a_{21} = a_{21}a_{41}, R_7^{(4)} \\
&: a_{42}a_{43} = a_{32}a_{42}, R_8^{(4)} : a_{43}a_{21} = a_{21}a_{43}, R_9^{(4)} \\
&: a_{43}a_{31} = a_{31}a_{41}, R_{10}^{(4)} : a_{43}a_{32} = a_{32}a_{42}, R_{11}^{(5)} \\
&: a_{51}a_{32} = a_{32}a_{51}, R_{12}^{(5)} : a_{51}a_{42} = a_{42}a_{51}, R_{13}^{(5)} \\
&: a_{51}a_{43} = a_{43}a_{51}, R_{14}^{(5)} : a_{51}a_{52} = a_{21}a_{51}, R_{15}^{(5)} \\
&: a_{51}a_{53} = a_{31}a_{51}, R_{16}^{(5)} : a_{51}a_{54} = a_{41}a_{51}, R_{17}^{(5)} \\
&: a_{52}a_{43} = a_{43}a_{52}, R_{18}^{(5)} : a_{52}a_{21} = a_{21}a_{51}, R_{19}^{(5)} \\
&: a_{52}a_{53} = a_{32}a_{52}, R_{20}^{(5)} : a_{52}a_{54} = a_{42}a_{52}, R_{21}^{(5)} \\
&: a_{53}a_{21} = a_{21}a_{53}, R_{22}^{(5)} : a_{53}a_{31} = a_{31}a_{51}, R_{23}^{(5)} \\
&: a_{53}a_{32} = a_{32}a_{52}, R_{24}^{(5)} : a_{53}a_{54} = a_{43}a_{53}, R_{25}^{(5)} \\
&: a_{54}a_{21} = a_{21}a_{54}, R_{26}^{(5)} : a_{54}a_{31} = a_{31}a_{54}, R_{27}^{(5)} \\
&: a_{54}a_{32} = a_{32}a_{54}, R_{28}^{(5)} : a_{54}a_{41} = a_{41}a_{51}, R_{29}^{(5)} \\
&: a_{54}a_{42} = a_{42}a_{52}, R_{30}^{(5)} : a_{54}a_{43} = a_{43}a_{53}
\end{aligned} \tag{7}$$

are given basic relations.

For the braid monoid MB_5, we have given another form of Theorem 3 that is directly used to compute the growth series of MB_5. This form is obtained by solving all the ambiguities in the band presentation of MB_5.

Proposition 5. *A complete presentation of MB_5 for band presentation is given by*

$$\left\langle a_{54}, a_{53}, a_{52}, a_{51}, a_{43}, a_{42}, a_{41}, a_{32}, a_{31}, a_{21} \middle| R_1^{(3)}, R_2^{(3)}, R_{84}^{(3)}, R_i^{(4)}, R_j^{(4)}, R_k^{(5)}, \right. \\ \left. i = 3, \cdots, 10, j = 76, \cdots, 83, k = 11, \cdots, 75 \right\rangle, \tag{8}$$

where the new relations $R_{31}^{(5)}, \dots, R_{84}^{(3)}$ are given as follows

$$R_{31}^{(5)} : a_{51}a_{21}^n a_{31} = a_{32}a_{51}a_{21}a_{32}^{n-1},$$

$$R_{32}^{(5)} : a_{51}a_{21}^n a_{41} = a_{42}a_{51}a_{21}a_{42}^{n-1},$$

$$R_{33}^{(5)} : a_{51}a_{21}^n a_{32} W(32) a_{41} = a_{42}a_{51}a_{21}a_{42}^{n-1} W'(32),$$

$$R_{34}^{(5)} : a_{51}a_{21}^n a_{32}^r a_{42} = a_{43}a_{51}a_{21}^n a_{32}a_{43}^{r-1},$$

$$R_{35}^{(5)} : a_{51}a_{21}^n a_{43} = a_{43}a_{51}a_{21}^n,$$

$$R_{36}^{(5)} : a_{51}a_{21}^n a_{51} = a_{21}a_{51}a_{21}a_{52}^{n-1},$$

$$R_{37}^{(5)} : a_{51}a_{21}^n a_{32} W(32) a_{51} = a_{21}a_{51}a_{21}a_{52}^{n-1} W'(32),$$

$$R_{38}^{(5)} : a_{51}a_{21}^n a_{42} V(42) a_{51} = a_{21}a_{51}a_{21}a_{52}^{n-1} V'(42),$$

$$\begin{aligned} R_{39}^{(5)} &: a_{51}a_{21}^n a_{32} W(32) a_{43} V(43) a_{51} \\ &= a_{21}a_{51}a_{21}a_{52}^{n-1} W'(32, 21) V'(43), \end{aligned}$$

$$\begin{aligned} R_{40}^{(5)} &: a_{51}a_{21}^n a_{32}^r a_{31} W(31) a_{42} V(42, 21) a_{51} \\ &= a_{21}a_{51}a_{21}a_{52}^{n-1} a_{32}^r W'(31) V'(42), \end{aligned}$$

$$R_{41}^{(5)} : a_{51}a_{21}^n a_{32}^r a_{52} = a_{31}a_{51}a_{21}^n a_{32}a_{53}^{r-1},$$

$$\begin{aligned} R_{42}^{(5)} &: a_{51}a_{21}^n a_{32}^r a_{43} V(43, 42, 32) a_{52} \\ &= a_{31}a_{51}a_{21}^n a_{32}a_{53}^{r-1} V'(43, 42, 32), \end{aligned}$$

$$\begin{aligned} R_{43}^{(5)} &: a_{51}a_{21}^n a_{32}^r a_{31} W(31) a_{42} V(42, 43, 32) a_{52} \\ &= a_{41}a_{51}a_{21}^n a_{32}^r W'(31) V'(42, 43, 32), \end{aligned}$$

$$R_{44}^{(5)} : a_{51}a_{21}^n a_{42} V(42, 43, 32) a_{52} = a_{41}a_{51}a_{21}^n V'(42, 43, 32),$$

$$R_{45}^{(5)} : a_{51}a_{21}^n a_{53} = a_{31}a_{51}a_{21}^n,$$

$$R_{46}^{(5)} : a_{51}a_{21}^n a_{32} W(32) a_{43}^r a_{53} = a_{41}a_{51}a_{21}^n W'(32) a_{43}^r,$$

$$R_{47}^{(5)} : a_{51}a_{21}^n a_{54} = a_{41}a_{51}a_{21}^n,$$

$$R_{48}^{(5)} : a_{51}a_{21}^n a_{32} W(32) a_{54} = a_{41}a_{51}a_{21}^n W'(32),$$

$$R_{49}^{(5)} : a_{51}a_{31} W(31) a_{41} = a_{43}a_{51}a_{31} W'(31),$$

$$R_{50}^{(5)} : a_{51}a_{31} W(31) a_{51} = a_{31}a_{51}a_{31} W'(31),$$

$$R_{51}^{(5)} : a_{51}a_{31} W(31) a_{42} V(42) a_{51} = a_{31}a_{51}a_{31} W'(31) V'(42),$$

$$R_{52}^{(5)} : a_{51}a_{31} W(31) a_{43} V(43) a_{51} = a_{31}a_{51}a_{31} W'(31) V'(43),$$

$$\begin{aligned} R_{53}^{(5)} &: a_{51}a_{31} W(31) a_{42} V(42, 43, 32) a_{52} \\ &= a_{41}a_{51}a_{31} W'(31) V'(42, 43, 32), \end{aligned}$$

$$R_{54}^{(5)} : a_{51}a_{31} W(31) a_{43}^n a_{53} = a_{41}a_{51}a_{31} W'(31) a_{43}a_{54}^{n-1},$$

$$R_{55}^{(5)} : a_{51}a_{31} W(31) a_{54} = a_{41}a_{51}a_{31} W'(31),$$

$$R_{56}^{(5)} : a_{51}a_{41} V(41) a_{51} = a_{41}a_{51}a_{41} V'(41),$$

$$R_{57}^{(5)} : a_{52}a_{31} W(31) a_{41} = a_{43}a_{52}a_{31} W'(31),$$

$$R_{58}^{(5)} : a_{52}a_{31} W(31) a_{51} = a_{32}a_{52}a_{31} W'(31),$$

$$R_{59}^{(5)} : a_{52}a_{31} W(31) a_{42} V(42,) a_{51} = a_{32}a_{52}a_{31} W'(31) V'(42),$$

$$R_{60}^{(5)} : a_{52}a_{31} W(31) a_{43} V(42) a_{51} = a_{32}a_{52}a_{31} W'(31) V'(43),$$

$$R_{61}^{(5)} : a_{52}a_{31}W(31)a_{42}V(42,43,32)a_{52} = a_{42}a_{52}a_{31}W'(31)V'(42,43,32),$$

$$R_{62}^{(5)} : a_{52}a_{31}W(31)a_{43}^{n}a_{53} = a_{42}a_{52}a_{31}W'(31)a_{43}a_{54}^{n-1},$$

$$R_{63}^{(5)} : a_{52}a_{31}W(31)a_{54} = a_{42}a_{52}a_{31}W'(31),$$

$$R_{64}^{(5)} : a_{52}a_{32}^{n}a_{42} = a_{43}a_{52}a_{32}a_{43}^{n-1},$$

$$R_{65}^{(5)} : a_{52}a_{32}W(32)a_{41}V(41)a_{51} = a_{42}a_{52}a_{32}W'(32)V'(41),$$

$$R_{66}^{(5)} : a_{52}a_{32}^{n}a_{52} = a_{32}a_{52}a_{32}a_{53}^{n-1},$$

$$R_{67}^{(5)} : a_{52}a_{32}^{n}a_{31}W(31)a_{42}V(42,43,32)a_{52} = a_{42}a_{52}a_{32}^{n}W'(31)V'(42,43,32),$$

$$R_{68}^{(5)} : a_{52}a_{32}^{n}a_{43}V(43,42,32)a_{52} = a_{32}a_{52}a_{32}a_{53}^{n-1}V'(43,42,32),$$

$$R_{69}^{(5)} : a_{52}a_{32}W(32)a_{43}^{n}a_{53} = a_{42}a_{52}a_{32}W'(32)a_{43}a_{54}^{n-1},$$

$$R_{70}^{(5)} : a_{52}a_{32}W(32)a_{54} = a_{42}a_{52}a_{32}W'(32),$$

$$R_{71}^{(5)} : a_{52}a_{41}V(41)a_{51} = a_{42}a_{52}a_{41}V'(41),$$

$$R_{72}^{(5)} : a_{52}a_{42}V(42,43,32)a_{52} = a_{42}a_{52}a_{42}V'(42,43,32),$$

$$R_{73}^{(5)} : a_{53}a_{41}V(41)a_{51} = a_{43}a_{53}a_{41}V'(41),$$

$$R_{74}^{(5)} : a_{53}a_{42}V(42,43,32)a_{53} = a_{43}a_{53}a_{42}V'(42,43,32),$$

$$R_{75}^{(5)} : a_{53}a_{43}^{n}a_{53} = a_{43}a_{53}a_{43}a_{54}^{n-1},$$

$$R_{76}^{(4)} : a_{41}a_{21}^{n}a_{31} = a_{32}a_{41}a_{21}a_{32}^{n-1},$$

$$R_{77}^{(4)} : a_{41}a_{21}^{n}a_{41} = a_{21}a_{41}a_{21}a_{42}^{n-1},$$

$$R_{78}^{(4)} : a_{41}a_{21}^{n}a_{32}^{r}a_{42} = a_{31}a_{41}a_{21}^{n}a_{32}a_{43}^{r-1},$$

$$R_{79}^{(4)} : a_{41}a_{21}^{n}a_{43} = a_{31}a_{41}a_{21}^{n},$$

$$R_{80}^{(4)} : a_{41}a_{21}a_{32}W(32)a_{41} = a_{21}a_{41}a_{21}a_{32}W'(32),$$

$$R_{81}^{(4)} : a_{41}a_{21}a_{31}W(31)a_{41} = a_{31}a_{41}a_{31}W'(31),$$

$$R_{82}^{(4)} : a_{42}a_{31}W(31)a_{41} = a_{32}a_{42}a_{31}W'(31),$$

$$R_{83}^{(4)} : a_{42}a_{32}^{n}a_{42} = a_{32}a_{42}a_{32}a_{43}^{n-1},$$

$$R_{84}^{(3)} : a_{31}a_{21}^{n}a_{31} = a_{21}a_{31}a_{21}a_{32}^{n-1}, \tag{9}$$

where n and r are positive integers, $W(3k)$ a canonical word in MB_3 starting with a_{3k}, $(k = 1, 2)$ and

(i) $W'(3k) = W(3k)$: $a_{32} \longrightarrow a_{32}, a_{21} \longrightarrow a_{41}, a_{31} \longrightarrow a_{43}$, $V(4l)$ is a canonical word in MB_4 starting with a_{4l}, $(l = 1, 2, 3)$

(ii) $V'(3l) = V(3l)$:- $a_{41} \longrightarrow a_{54}, a_{42} \longrightarrow a_{42}, a_{43} \longrightarrow a_{53}, a_{32} \longrightarrow a_{32}, a_{21} \longrightarrow a_{41}, a_{31} \longrightarrow a_{43}$ and $V(4m,32) = \sum A_{3(m-1)}^{(3)}$, $m = 2, 3$ (as mentioned in Theorem 3)

Proof. We denote the ambiguity formed by left sides of the relation R_i and R_j in MB_5 by R_i-$R_j = swt$ (say). If in the ambiguity $z = swt$, $L(z) = (sw)t$ and $R(z) = s(wt)$ are different lexicographically, then we get a new relation in the complete presentation of MB_5, and if $L(z) = (sw)t$ and $R(z) = s(wt)$ are reduced to an identical word, then we say the ambiguity is solvable and no new relation is formed. The above relations are formed by solving the ambiguities involving basic relations and new relations.

For an ambiguity $R_{11}^{(5)} - R_{2}^{(3)} = a_{51}a_{32}a_{21} = w_1$(say), we have

$$R(w_1) = a_{51}\underline{a_{32}a_{21}} = a_{51}a_{21}a_{31}, L(w_1) = \underline{a_{51}a_{32}}\,a_{21} = a_{32}a_{51}a_{21}. \tag{10}$$

Hence, we have a new relation $R_{w_1}^{(5)} : a_{51}a_{21}a_{31} = a_{32}a_{51}a_{21}$. Again, by solving new ambiguity $R_{w_1}^{(5)} - R_{1}^{(3)} = a_{51}a_{21}a_{31}a_{32} = w_2$(say), we have

$$R(w_2) = a_{51}a_{21}\underline{a_{31}a_{32}} = a_{51}a_{21}^{2}a_{31}, L(w_2) = \underline{a_{51}a_{21}a_{31}}\,a_{32} = a_{32}a_{51}a_{21}a_{32}. \tag{11}$$

Hence, we have another relation $R_{w_2}^{(5)} : a_{51}a_{21}^{2}a_{31} = a_{32}a_{51}a_{21}a_{32}$. Now by solving ambiguity $R_{w_2}^{(5)} - R_{1}^{(3)} = a_{51}a_{21}^{2}a_{31}a_{32} = w_3$(say), we have

$$R(w_3) = a_{51}a_{21}^{2}\underline{a_{31}a_{32}} = a_{51}a_{21}^{3}a_{31}, L(w_2) = \underline{a_{51}a_{21}^{2}a_{31}}\,a_{32} = a_{32}a_{51}a_{21}a_{32}^{2}. \tag{12}$$

Hence, we have a relation $R_{w_2}^{(5)} : a_{51}a_{21}^{3}a_{31} = a_{32}a_{51}a_{21}a_{32}^{2}$. By continuing the same process, we have the general relation

$$R_{31}^{(5)} : a_{51}a_{21}^{n}a_{31} = a_{32}a_{51}a_{21}a_{32}^{n-1}, \quad n \geq 1. \tag{13}$$

For an ambiguity $R_{12}^{(5)} - R_{6}^{(4)} = a_{51}a_{42}a_{21} = w_4$ (say), we have

$$R(w_4) = a_{51}\underline{a_{42}a_{21}} = a_{51}a_{21}a_{41}, L(w_4) = \underline{a_{51}a_{42}}\, a_{21} = a_{42}a_{51}a_{21}. \tag{14}$$

Hence, we have a new relation $R^{(5)}_{w_4}: a_{51}a_{21}a_{41} = a_{42}a_{51}a_{21}$. Again, by solving new ambiguity $R^{(5)}_{w_4} - R^{(4)}_4 = a_{51}a_{21}a_{41}a_{42} = w_5$ (say), we have

$$\begin{aligned} R(w_5) &= a_{51}a_{21}\underline{a_{41}a_{42}} = a_{51}a_{21}^2a_{41}, L(w_5) = \underline{a_{51}a_{21}a_{41}}\, a_{42} \\ &= a_{42}a_{51}a_{21}a_{42}, \end{aligned} \tag{15}$$

which give another relation $R^{(5)}_{w_5}: a_{51}a_{21}^2a_{41} = a_{42}a_{51}a_{21}a_{42}$. Now by solving ambiguity $R^{(5)}_{w_5} - R^{(3)}_4 = a_{51}a_{21}^2a_{41}a_{42} = w_6$ (say), we have

$$\begin{aligned} R(w_6) &= a_{51}a_{21}^2\underline{a_{41}a_{42}} = a_{51}a_{21}^3a_{41}, L(w_6) = \underline{a_{51}a_{21}^2a_{41}}\, a_{42} \\ &= a_{42}a_{51}a_{21}a_{42}^2. \end{aligned} \tag{16}$$

Hence, $R^{(5)}_{w_6}: a_{51}a_{21}^3a_{41} = a_{42}a_{51}a_{21}a_{42}^2$. By continuing the same process, we have the general relation

$$R^{(5)}_{32}: a_{51}a_{21}^n a_{41} = a_{42}a_{51}a_{21}a_{42}^{n-1}, \quad n \geq 1. \tag{17}$$

For an ambiguity $R^{(5)}_{12} - R^{(4)}_7 = a_{51}a_{42}a_{43} = w_7$ (say), we have

$$\begin{aligned} R(w_7) &= a_{51}\underline{a_{42}a_{43}} = \underline{a_{51}a_{32}}\, a_{42} = a_{32}\underline{a_{51}a_{42}} = a_{32}a_{42}a_{51}, \\ L(w_7) &= \underline{a_{51}a_{42}}\, a_{43} = a_{42}\underline{a_{51}a_{43}} = \underline{a_{42}a_{43}}\, a_{51} = a_{32}a_{42}a_{51}, \end{aligned} \tag{18}$$

where $L(w_7) = R(w_7)$ are reduced to identical word, so ambiguity is solvable and no new relation is formed. Using a similar procedure, we obtained all above new relations in complete presentation of MB_5. Hence the proof is omitted. □

As defined above, A^m_* denotes the set of canonical words and B^m_* the set of reducible words in MB_{n+1} in general. In particular, $B^{(5)}_{5i:kl:mn:pq;rs}$, where $1 \leq i \leq 3$ denote the set of reducible words starting with $a_{5i}a_{kl}$ and ending on a_{rs}. In this notation a_{rs} is a generator in MB_d: $3 \leq d \leq 5$, pq denotes the canonical word (possibly empty) in MB_{d-1} starting with a_{pq}, mn denotes the canonical word (possibly empty) in MB_{d-2} starting with a_{mn}, and kl denotes the canonical word (possibly empty) in MB_{d-3} starting with a_{kl}. We will denote the empty word by ϕ. We denote the set all reducible words starting with $a_{5i}a_{kl}$ and ending on a_{rs} by $B^{(5)}_{5i.kl;rs}$. Hence, $B^{(5)}_{5i:kl:mn:pq;rs}$ is a subset of $B^{(5)}_{5i.kl;rs}$.

We are using other notions as follows:

(i) We denote the set $\{a_{21}, a_{21}^2, a_{21}^3, \cdots\}$ by $A^{(2)}_{21}$

(ii) $A^{(n)}_{ij}$ denotes the set of canonical words starting with a_{ij} in MB_n

(iii) $\sum^r A^{(n)}_{ij}$ denotes the set of all the word in $A^{(n)}_{ij}$ such that the index of each generator is increased by r. Hence, $|\sum^r A^{(n)}_{ij}| = |A^{(n)}_{ij}|$, i.e., the cardinality remains unchanged. In particular for the set $A^{(2)}_{21} = \{a_{21}, a_{21}^2, a_{21}^3, \cdots\}$, we have $\sum A^{(2)}_{21} = \{a_{32}, a_{32}^2, a_{32}^3, \cdots\}$

(iv) $A^{(n)}_{nj.kl}$ denotes the set of canonical words starting with $a_{nj}a_{kl}$ in MB_n

(v) The growth series of $B^{(m)}_*$, $A^{(m)}_*$ and MB_5 is denoted by $Q^{(m)}_*$, $P^{(m)}_*$ and $P^{(5)}_M(t)$, respectively

Note that growth series of $A^{(2)}_{21}$ is $P^{(2)}_{21} = t/(1-t)$.

Proposition 6. *The following equalities hold for the reducible words in MB_5:*

$$Q^{(5)}_{51.21;31} = \frac{t^3}{1-t}.$$

$$Q^{(5)}_{51.21;41} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{51.21;42} = \frac{t^4}{(1-t)^2},$$

$$Q^{(5)}_{51.21;43} = \frac{t^3}{1-t},$$

$$Q^{(5)}_{51.21;51} = \frac{t^3-2t^4}{1-5t+5t^2},$$

$$Q^{(5)}_{51.21;52} = \frac{t^4\left(2-6t+5t^2\right)}{(1-t)^2(1-2t)^2},$$

$$Q^{(5)}_{51.21;53} = \frac{t^3\left(1-3t+3t^2\right)}{(1-t)^2(1-2t)},$$

$$Q^{(5)}_{51.21;54} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{51.31;41} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{51.31;51} = \frac{t^3-t^4}{1-5t+5t^2},$$

$$Q^{(5)}_{51.31;52} = \frac{t^4}{(1-t)^2},$$

$$Q^{(5)}_{51.31;53} = \frac{t^4}{(1-t)(1-2t)},$$

$$Q^{(5)}_{51.31;54} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{51.41;51} = \frac{t^3(1-2t)}{1-5t+5t^2},$$

$$Q^{(5)}_{52.31;41} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{52.31;51} = \frac{t^3-t^4}{1-5t+5t^2},$$

$$Q^{(5)}_{52.31;52} = \frac{t^4}{(1-t)^2},$$

$$Q^{(5)}_{52.31;53} = \frac{t^4}{(1-t)(1-2t)},$$

$$Q^{(5)}_{52.31;54} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{52.32;42} = \frac{t^3}{1-t},$$

$$Q^{(5)}_{52.32;51} = \frac{t^4}{1-5t+5t^2},$$

$$Q^{(5)}_{52.32;52} = \frac{t^3\left(1-3t+3t^2\right)}{(1-t)(1-2t)^2},$$

$$Q^{(5)}_{52.32;53} = \frac{t^4}{(1-t)(1-2t)},$$

$$Q^{(5)}_{52.32;54} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{52.41;51} = \frac{t^3(1-2t)}{1-5t+5t^2},$$

$$Q^{(5)}_{52.42;52} = \frac{t^3}{1-2t},$$

$$Q^{(5)}_{53.41;51} = \frac{t^3(1-2t)}{1-5t+5t^2},$$

$$Q^{(5)}_{53.42;52} = \frac{t^3}{1-2t}.$$

$$Q^{(5)}_{53.43;53} = \frac{t^3}{1-t}. \tag{19}$$

Proof. Using simple decomposition of words and $\bigsqcup$ denotes the disjoint union of sets then, we have

(1) Since $B^{(5)}_{51.21;31} = B^{(5)}_{51:21;31} = \{a_{51}a^n_{21}a_{31}\} = \{a_{51}\} \times A^{(2)}_{21} \times \{a_{31}\}$, hence , $Q^{(5)}_{51.21;31} = (t^3/1-t)$

(2) $B^{(5)}_{51.21;41} = B^{(5)}_{51:21:\phi;41} \bigsqcup B^{(5)}_{51:21:32;41}$
$= \{a_{51}a^n_{21}a_{41}\} \bigsqcup \{a_{51}a^n_{21}a_{32}V(32)a_{41}\}$
$= [\{a_{51}\} \times A^{(2)}_{21} \times \{a_{41}\}] \bigsqcup [\{a_{51}\} \times A^{(2)}_{21} \times A^{(3)}_{32} \times \{a_{41}\}]$
implies $Q^{(5)}_{51.21;41} = t^3/(1-2t)$

(3) $B^{(5)}_{51.21;42} = B^{(5)}_{51:21:32;42} = \{a_{51}a^n_{21}a^r_{32}a_{42}\} = \{a_{51}\} \times A^{(2)}_{21} \times \sum A^{(2)}_{21} \times \{a_{42}\}$ implies $Q^{(5)}_{51.21;42} = t^4/(1-t)^2$

Using similar procedure, we obtained all above $Q^{(5)}_*$ of MB_5. Hence, the proof is omitted. □

Next, we construct linear system for canonical form in MB_5.

Proposition 7. *The following equalities hold for the canonical words in MB_5:*

$$P^{(5)}_{21} = \frac{t}{(1-t)(1-5t+5t^2)}\left(1+\sum_{i=1}^{4} P^{(5)}_{5i}\right),$$

$$P^{(5)}_{31} = \frac{t}{1-5t+5t^2}\left(1+\sum_{i=1}^{4} P^{(5)}_{5i}\right),$$

$$P^{(5)}_{32} = \frac{t}{1-5t+5t^2}\left(1+\sum_{i=1}^{4} P^{(5)}_{5i}\right),$$

$$P^{(5)}_{41} = \frac{t(1-2t)}{1-5t+5t^2}\left(1+\sum_{i=1}^{4} P^{(5)}_{5i}\right),$$

$$P^{(5)}_{42} = \frac{t-t^2}{1-5t+5t^2}\left(1+\sum_{i=1}^{4} P^{(5)}_{5i}\right),$$

$$P^{(5)}_{43} = \frac{t(1-2t)}{1-5t+5t^2}\left(1+\sum_{i=1}^{4} P^{(5)}_{5i}\right),$$

$$P^{(5)}_{51} = t + tP^{(5)}_{51} + \sum_{i=2}^{4} P^{(5)}_{51.i1},$$

$$P^{(5)}_{52} = t + t\sum_{i=1}^{2} P^{(5)}_{5i} + \sum_{i=1}^{2} P^{(5)}_{52.3i} + \sum_{i=1}^{2} P^{(5)}_{52.4i},$$

$$P^{(5)}_{53} = t + t\sum_{i=1}^{3} P^{(5)}_{5i} + \sum_{i=1}^{3} P^{5}_{53.4i},$$

$$P^{(5)}_{54} = t + t\sum_{i=1}^{4} P^{(5)}_{5i},$$

$$P^{(5)}_{51.21} = tP^{(5)}_{21} - \frac{t^2}{1-t}P^{(5)}_{31} - \frac{t^2}{1-2t}P^{(5)}_{41} - \frac{t^3}{(1-t)^2}P^{(5)}_{42} - \frac{t^2}{1-t}P^{(5)}_{43} - \frac{t^2-2t^3}{1-5t+5t^2}P^{(5)}_{51} - \frac{t^3\left(2-6t+5t^2\right)}{(1-t)^2(1-2t)^2}P^{(5)}_{52} - \frac{t^2\left(1-3t+3t^2\right)}{(1-t)^2(1-2t)}P^{(5)}_{53} - \frac{t^2}{1-2t}P^{(5)}_{54},$$

$$P^{(5)}_{51.31} = tP^{(5)}_{31} - \frac{t^2}{1-2t}P^{(5)}_{41} - \frac{t^2-t^3}{(1-5t+5t^2)}P^{(5)}_{51} - \frac{t^3}{(1-2t)^2}P^{(5)}_{52} - \frac{t^3}{(1-t)(1-2t)}P^{(5)}_{53} - \frac{t^2}{1-2t}P^{(5)}_{54},$$

$$P^{(5)}_{51.41} = tP^{(5)}_{41} - \frac{t^2(1-2t)}{1-5t+5t^2}P^{(5)}_{51},$$

$$P^{(5)}_{52.31} = tP^{(5)}_{31} - \frac{t^2}{1-2t}P^{(5)}_{41} - \frac{t^2-t^3}{1-5t+5t^2}P^{(5)}_{51} - \frac{t^3}{(1-2t)^2}P^{(5)}_{52} - \frac{t^3}{(1-t)(1-2t)}P^{(5)}_{53} - \frac{t^2}{1-2t}P^{(5)}_{54},$$

$$P^{(5)}_{52.32} = tP^{(5)}_{32} - \frac{t^2}{1-t}P^{(5)}_{42} - \frac{t^3}{1-5t+5t^2}P^{(5)}_{51} - \frac{t^2\left(1-3t+3t^2\right)}{(1-t)(1-2t)^2}P^{(5)}_{52} - \frac{t^3}{(1-t)(1-2t)}P^{(5)}_{53} - \frac{t^2}{1-2t}P^{(5)}_{54},$$

$$P^{(5)}_{52.41} = tP^{(5)}_{41} - \frac{t^2(1-2t)}{1-5t+5t^2}P^{(5)}_{51},$$

$$P^{(5)}_{52.42} = tP^{(5)}_{42} - \frac{t^2}{1-2t}P^{(5)}_{52},$$

$$P^{(5)}_{53.41} = tP^{(5)}_{41} - \frac{t^2(1-2t)}{1-5t+5t^2}P^{(5)}_{51},$$

$$P^{(5)}_{53.42} = tP^{(5)}_{42} - \frac{t^2}{1-2t}P^{(5)}_{52},$$

$$P^{(5)}_{53.43} = tP^{(5)}_{43} - \frac{t^2}{1-t}P^{(5)}_{53}. \qquad (20)$$

Proof. We compute the growth series inductively. By using decomposition of words, we have

(1) $A^{(5)}_{21} = A^{(4)}_{21}\bigsqcup[A^{(4)}_{21} \times (A^{(5)}_{51}\bigsqcup A^{(5)}_{52}\bigsqcup A^{(5)}_{53}\bigsqcup A^{(5)}_{54})]$ gives $P^{(5)}_{21} = t/((1-t)(1-5t+5t^2))(1+\sum_{i=1}^{4}P^{(5)}_{5i})$

(2) $A^{(5)}_{31} = A^{(4)}_{31}\bigsqcup[A^{(4)}_{31} \times (A^{(5)}_{51}\bigsqcup A^{(5)}_{52}\bigsqcup A^{(5)}_{53}\bigsqcup A^{(5)}_{54})]$ implies $P^{(5)}_{31} = 1/(1-5t+5t^2)(1+\sum_{i=1}^{4}P^{(5)}_{5i})$

(3) $A^{(5)}_{32} = A^{(4)}_{32}\bigsqcup[A^{(4)}_{32} \times (A^{(5)}_{51}\bigsqcup A^{(5)}_{52}\bigsqcup A^{(5)}_{53}\bigsqcup A^{(5)}_{54})]$ implies $P^{(5)}_{32} = 1/(1-5t+5t^2)(1+\sum_{i=1}^{4}P^{(5)}_{5i})$

(4) $A^{(5)}_{41} = A^{(4)}_{41}\bigsqcup[A^{(4)}_{41} \times (A^{(5)}_{51}\bigsqcup A^{5}_{52}\bigsqcup A^{5}_{53}\bigsqcup A^{5}_{54})]$ implies $P^{(5)}_{41} = (t(1-2t))/(1-5t+5t^2)(1+\sum_{i=1}^{4}P^{(5)}_{5i})$

(5) $A^{(5)}_{42} = A^{(4)}_{42}\bigsqcup[A^{(4)}_{42} \times (A^{(5)}_{51}\bigsqcup A^{5}_{52}\bigsqcup A^{5}_{53}\bigsqcup A^{5}_{54})]$ implies $P^{(5)}_{42} = (t-t^2)/(1-5t+5t^2)(1+\sum_{i=1}^{4}P^{(5)}_{5i})$

(6) $A^{(5)}_{43} = A^{(4)}_{43}\bigsqcup[A^{(4)}_{43} \times (A^{(5)}_{51}\bigsqcup A^{5}_{52}\bigsqcup A^{5}_{53}\bigsqcup A^{5}_{54})]$ implies $P^{(5)}_{43} = (t(1-2t))/(1-5t+5t^2)(1+\sum_{i=1}^{4}P^{(5)}_{5i})$

The set $A^{(5)}_{5i}$ consists of all the words starting with the generator a_{5i}. Therefore, the set $\{a_{5i}\} \times A^{5}_{5i}$ is subset of $A^{(5)}_{5i}$ consisting of all the words starting with a^2_{5i}. We apply this concept in the proof of items (7), (8), (9), and (10).

(7) The set $A^{(5)}_{51}$ is disjoint union of its subsets $\{a_{51}\}$, $\{a_{51}\} \times A^{(5)}_{51}, A^{5}_{51.21}, A^{(5)}_{51.31}, A^{(5)}_{51.41}$, i.e $A^{(5)}_{51} = \{a_{51}\}\bigsqcup(\{a_{51}\} \times A^{(5)}_{51})\bigsqcup A^{(5)}_{51.21}\bigsqcup A^{(5)}_{51.31}\bigsqcup A^{(5)}_{51.41}$. Therefore, we have $P^{(5)}_{51} = t + tP^{(5)}_{51} + \sum_{i=2}^{4}P^{(5)}_{51.i1}$

Similarly, we have

(8) $A^{(5)}_{52} = \{a_{52}\}\bigsqcup(\{a_{52}\} \times A^{(5)}_{52})\bigsqcup(\{a_{52}\} \times A^{(5)}_{51})\bigsqcup A^{(5)}_{52.31}\bigsqcup A^{(5)}_{52.32}\bigsqcup A^{(5)}_{52.41}\bigsqcup A^{(5)}_{52.42}$ implies $P^{(5)}_{52} = t + t\sum_{i=1}^{2}P^{(5)}_{5i} + \sum_{i=1}^{2}P^{(5)}_{52.3i} + \sum_{i=1}^{2}P^{(5)}_{52.4i}$

(9) $A^{(5)}_{53} = \{a_{53}\}\bigsqcup(\{a_{53}\} \times A^{(5)}_{53})\bigsqcup(\{a_{53}\} \times A^{(5)}_{52})\bigsqcup(\{a_{53}\} \times A^{(5)}_{51})\bigsqcup A^{(5)}_{53.41}\bigsqcup A^{(5)}_{53.42}\bigsqcup A^{(5)}_{53.43}$ implies $P^{(5)}_{53} = t + t\sum_{i=1}^{3}P^{(5)}_{5i} + \sum_{i=1}^{3}P^{(5)}_{53.4i}$

(10) $A^{(5)}_{54} = \{a_{54}\}\bigsqcup(\{a_{54}\} \times A^{(5)}_{54})\bigsqcup(\{a_{54}\} \times A^{(5)}_{53})\bigsqcup(\{a_{54}\} \times A^{(5)}_{52})\bigsqcup(\{a_{54}\} \times A^{(5)}_{51})$ implies $P^{(5)}_{54} = t + t\sum_{i=1}^{4}P^{(5)}_{5i}$

(11) $A^{(5)}_{51.21} = \{a_{51}\} \times A^{(5)}_{21}[(B^{(5)}_{51.21;31} \times {}_{31}A^{(5)}_{31})\bigsqcup(B^{(5)}_{51.21;41} \times {}_{41}A^{(5)}_{41})\bigsqcup(B^{(5)}_{51.21;42} \times {}_{42}A^{(5)}_{42})\bigsqcup(B^{(5)}_{51.21;43} \times {}_{43}A^{(5)}_{43})\bigsqcup(B^{(5)}_{51.21;51} \times {}_{51}A^{(5)}_{51})\bigsqcup(B^{(5)}_{51.21;52} \times {}_{52}A^{(5)}_{52})\bigsqcup(B^{(5)}_{51.21;53} \times {}_{53}A^{(5)}_{53})\bigsqcup(B^{(5)}_{51.21;54} \times {}_{54}A^{(5)}_{54})]$ implies

$$P^{(5)}_{51.21} = tP^{(5)}_{21} - \frac{t^2}{1-t}P^{(5)}_{31} - \frac{t^2}{1-2t}P^{(5)}_{41} - \frac{t^3}{(1-t)^2}P^{(5)}_{42} - \frac{t^2}{1-t}P^{(5)}_{43} - \frac{t^2-2t^3}{1-5t+5t^2}P^{(5)}_{51} - \frac{t^3\left(2-6t+5t^2\right)}{(1-t)^2(1-2t)^2}P^{(5)}_{52} - \frac{t^2\left(1-3t+3t^2\right)}{(1-t)^2(1-2t)}P^{(5)}_{53} - \frac{t^2}{1-2t}P^{(5)}_{54} \qquad (21)$$

(12) $A^{(5)}_{51.31} = \{a_{51}\} \times A^{(5)}_{31}/[B^{(5)}_{51.31;41} \times {}_{41}A^{(5)}_{41}\bigsqcup(B^{(5)}_{51.31;51} \times {}_{51}A^{(5)}_{51})\bigsqcup(B^{(5)}_{51.31;52} \times {}_{52}A^{(5)}_{52})\bigsqcup(B^{(5)}_{51.31;53} \times {}_{53}A^{(5)}_{53})\bigsqcup(B^{(5)}_{51.31;54} \times {}_{54}A^{(5)}_{54})$ implies

$$P_{51.31}^{(5)} = tP_{31}^{(5)} - \frac{t^2}{1-2t}P_{41}^{(5)} - \frac{t^2-t^3}{1-5t+5t^2}P_{51}^{(5)} - \frac{t^3}{(1-2t)^2}P_{52}^{(5)} - \frac{t^3}{(1-t)(1-2t)}P_{53}^{(5)} - \frac{t^2}{1-2t}P_{54}^{(5)} \tag{22}$$

(13) $A_{51.41}^{(5)} = \{a_{51}\} \times A_{41}^{(5)} \setminus [(B_{51.41;51}^{(5)} \times {}_{51}A_{51}^{(5)})]$ implies $P_{51.41}^{(5)} = tP_{41}^{(5)} - ((t^2(1-2t))/(1-5t+5t^2))P_{51}^{(5)}$

(14) $A_{52.31}^{(5)} = \{a_{52}\} \times A_{31}^{(5)} [B_{52.31;41}^{(5)} \times {}_{41}A_{41}^{(5)} \bigsqcup (B_{52.31;51}^{(5)} \times {}_{51}A_{51}^{(5)}) \bigsqcup (B_{52.31;52}^{(5)} \times {}_{52}A_{52}^{(5)}) \bigsqcup (B_{52.31;53}^{(5)} \times {}_{53}A_{53}^{(5)}) \bigsqcup (B_{52.31;54}^{(5)} \times {}_{54}A_{54}^{(5)})]$ implies

$$P_{52.31}^{(5)} = tP_{31}^{(5)} - \frac{t^2}{1-2t}P_{41}^{(5)} - \frac{t^2-t^3}{1-5t+5t^2}P_{51}^{(5)} - \frac{t^3}{(1-2t)^2}P_{52}^{(5)} - \frac{t^3}{(1-t)(1-2t)}P_{53}^{(5)} - \frac{t^2}{1-2t}P_{54}^{(5)} \tag{23}$$

(15) $A_{52.32}^{(5)} = \{a_{52}\} \times A_{32}^{(5)} [B_{52.32;42}^{(5)} \times {}_{42}A_{42}^{(5)} \bigsqcup (B_{52.32;51}^{(5)} \times {}_{51}A_{51}^{(5)}) \bigsqcup (B_{52.32;52}^{(5)} \times {}_{52}A_{52}^{(5)}) \bigsqcup (B_{52.32;53}^{(5)} \times {}_{53}A_{53}^{(5)}) \bigsqcup (B_{52.32;54}^{(5)} \times {}_{54}A_{54}^{(5)})$ implies

$$P_{52.32}^{(5)} = tP_{32}^{(5)} - \frac{t^2}{1-t}P_{42}^{(5)} - \frac{t^3}{(1-5t+5t^2)}P_{51}^{(5)} - \frac{t^2(1-3t+3t^2)}{(1-t)(1-2t)^2}P_{52}^{(5)} - \frac{t^3}{(1-t)(1-2t)}P_{53}^{(5)} - \frac{t^2}{1-2t}P_{54}^{(5)} \tag{24}$$

(16) $A_{52.41}^{(5)} = \{a_{52}\} \times A_{41}^{(5)} \setminus [(B_{52.41;51}^{(5)} \times {}_{51}A_{51}^{(5)})]$ implies $P_{52.41}^{(5)} = tP_{41}^{(5)} - ((t^2(1-2t))/(1-5t+5t^2))P_{51}^{(5)}$

(17) $A_{52.42}^{(5)} = \{a_{52}\} \times A_{42}^{(5)} \setminus [(B_{52.42;52}^{(5)} \times {}_{52}A_{52}^{(5)})]$ implies $P_{52.42}^{(5)} = tP_{42}^{(5)} - (t^2/(1-2t))P_{52}^{(5)}$

(18) $A_{53.41}^{(5)} = \{a_{53}\} \times A_{41}^{(5)} \setminus [(B_{53.41;51}^{(5)} \times {}_{51}A_{51}^{(5)})]$ gives $P_{53.41}^{(5)} = tP_{41}^{(5)} - ((t^2(1-2t))/(1-5t+5t^2))P_{51}^{(5)}$

(19) $A_{53.42}^{(5)} = \{a_{53}\} \times A_{42}^{(5)} \setminus [(B_{53.42;52}^{(5)} \times {}_{52}A_{52}^{(5)})]$ implies $P_{52.42}^{(5)} = tP_{42}^{(5)} - (t^2/(1-2t))P_{52}^{(5)}$

(20) $A_{53.43}^{(5)} = \{a_{53}\} \times A_{43}^{(5)} \setminus [(B_{53.43;53}^{(5)} \times {}_{53}A_{53}^{(5)})]$ implies $P_{53.43}^{(5)} = tP_{43}^{(5)} - (t^2/(1-t))P_{53}^{(5)}$

□

Theorem 8. *The Hilbert series of the braid monoid MB_5 in band generators is given by*

$$P_M^{(5)}(t) = \frac{1}{(1-t)(1-2t)(1-7t+7t^2)}. \tag{25}$$

Proof. Solving the system of linear equations constructed in Proposition 7, we get

$$\begin{aligned}
P_{21}^{(5)} &= \frac{t}{(1-t)(1-2t)(1-7t+7t^2)},\\
P_{31}^{(5)} &= \frac{t}{(1-2t)(1-7t+7t^2)},\\
P_{32}^{(5)} &= \frac{t}{(1-2t)(1-7t+7t^2)},\\
P_{41}^{(5)} &= \frac{t}{1-7t+7t^2},\\
P_{42}^{(5)} &= \frac{t}{1-7t+7t^2},\\
P_{43}^{(5)} &= \frac{t}{1-7t+7t^2},\\
P_{51}^{(5)} &= \frac{t-5t^2+5t^3}{(1-2t)(1-7t+7t^2)},\\
P_{52}^{(5)} &= \frac{t-t^2}{1-7t+7t^2},\\
P_{53}^{(5)} &= \frac{t-t^2}{1-7t+7t^2},\\
P_{54}^{(5)} &= \frac{t-5t^2+5t^3}{(1-2t)(1-7t+7t^2)}.
\end{aligned} \tag{26}$$

Therefore, the Hilbert series of the braid monoid MB_5 is given by

$$\begin{aligned}
P_M^{(5)}(t) &= 1 + P_{21}^{(5)} + \sum_{i=1}^{2} P_{3i}^{(5)} + \sum_{i=1}^{3} P_{4i}^{(5)} + \sum_{i=1}^{4} P_{5i}^{(5)}\\
&= \frac{1}{(1-t)(1-2t)(1-7t+7t^2)}.
\end{aligned} \tag{27}$$

□

Remark 9. By partial fractions of growth series of MB_5 we have $1/((1-t)(1-2t)(1-7t+7t^2)) = -(1/(1-t)) - (8/(3(1-2t))) - (7/(3(1-((7-\sqrt{21})/2)t))) + (7/(3(1-((7+\sqrt{21})/2)t)))$. The only term that contributes in approximation of the series is $7/(3(1-((7+\sqrt{21})/2)t))$ and $7/(3(1-((7+\sqrt{21})/2)t)) = (7/3)(1+((7+\sqrt{21})/2)t + ((7+\sqrt{21})/2)^2t^2+\cdots)$. Therefore, the growth function is $a_k^{(5)} = (7/3)((7+\sqrt{21})/2)^k$, and hence, the growth rate of MB_5 is $(7+\sqrt{21})/2$ (approximately equal to 5.791).

4. Conclusion

From band presentation of braid monoid MB_5 we solve all the ambiguities and get new and more interesting relations $R_{31}^{(5)}$ to $R_{84}^{(5)}$ which are given in Proposition 5. When no more ambiguity is remaining to be solved, then these new relations with the basics relations are complete presentation of braid monoid MB_5 in band generators. The words on the left-hand side of these new relations are reducible words, and the words on the right-hand side are canonical words. In Proposition 6, we partially find the growth series $(Q_*^{(5)})$ of reducible words. We also construct a linear system for canonical words. The most important outcome of our work is the growth series of braid monoid MB_5 in band generators which is given in Theorem 8. These results are very interesting and useful for mathematicians. Using these results, one can find the growth series of higher order braid monoids and can generalize the results for MB_n.

Data Availability

The datasets collected and analyzed during the current work are available from the corresponding author on request. The corresponding author had full access to all data in the study and takes responsibility for the integrity of the data and the accuracy of data analyzed.

Conflicts of Interest

All authors have no conflicts of interests.

References

[1] S. Benvenuti, B. Feng, A. Hanany, and Y. H. He, "Counting BPS operators in gauge theories: quivers, syzygies and plathystics," *Journal of High Energy Physics*, vol. 2007, p. 050, 2007.

[2] B. Feng, A. Hanany, and Y. H. He, "Counting gauge invariants: theories: the plethystic program," *Journal of High Energy Physics*, vol. 2007, p. 090, 2007.

[3] J. Gray, Y. H. He, A. Hanany, N. Mekareeya, and V. Jejjala, "SQCD: a geometric apercu," *Journal of High Energy Physics*, vol. 5, pp. 99–99, 2008.

[4] N. Jokela, M. Jarvinen, and E. K. Vakkuri, "New results for the SQCD Hilbert series," *Journal of High Energy Physics*, vol. 2012, no. 3, p. 048, 2012.

[5] S. Benvenuti, A. Hanany, and N. Mekareeya, "The Hilbert series of the one-instanton moduli space," *Journal of High Energy Physics*, vol. 6, pp. 100–140, 2010.

[6] A. Hanany, N. Mekareeya, and S. S. Razamat, "Hilbert series for moduli spaces of two instantons," *Journal of High Energy Physics*, vol. 2013, no. 1, p. 070, 2013.

[7] A. Kobach and S. Pal, "Hilbert series and operator basis for *NRQED and NRQCD/HQET*," *Physics Letters B*, vol. 772, pp. 225–231, 2017.

[8] E. Artin, "Theory of braids," *Annals of Mathematics*, vol. 48, no. 1, pp. 101–126, 1947.

[9] Z. Iqbal and S. Yousaf, "Hilbert series of the braid monoid MB_{4} in band generators," *Turkish Journal of Mathematics*, vol. 38, pp. 977–984, 2014.

[10] Z. Iqbal, "Hilbert series of the finite dimensional generalized Heckealgebras," *Turkish Journal of Mathematics*, vol. 39, pp. 698–705, 2015.

[11] A. Haider, "A way of computing the Hilbert series," *Algebra and Discrete Mathematics.*, vol. 1, pp. 35–38, 2018.

[12] A. Kumar and R. Sarkar, "Hilbert series of binomial edge ideals," *Communications in Algebra*, vol. 47, no. 9, pp. 3830–3841, 2019.

[13] C. H. Herbig, D. Herden, and C. Seaton, "Hilbert series associated to symplectic quotients by SU2," *International Journal of Algebra and Computation*, vol. 30, no. 7, pp. 1323–1357, 2020.

[14] C. P. Pinto, C. H. Herbig, D. Herden, and C. Seaton, "The Hilbert series of SL_2 *-Invariants*," *Communications in Contemporary Mathematics*, vol. 22, no. 7, pp. 132–139, 2021.

[15] K. Saito, "Growth functions associated with Artin monoids of finite type," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 10, pp. 179–183, 2008.

[16] C. H. Herbig, D. Herden, and C. Seaton, "Hilbert series of symplectic quotients by the 2-torus," *Collectanea Mathematica*, vol. 74, 2022.

[17] C. H. Herbig, D. Herden, and C. Seaton, "The Laurent coefficients of the Hilbert series of a Gorenstein algebra," *Experimental Mathematics*, vol. 30, no. 1, pp. 56–75, 2021.

[18] Z. Iqbal, S. Yousaf, and S. Noureen, "Growth rate of the braid monoids MB_{n+1}; $n \leq 5$," in *The 5th International Conference On Research And Education In Mathematics: ICREM5. AIP Conference Proceedings*, vol. 1450, pp. 346–350, Bandung, Indonesia, 2012.

[19] B. Berceanu and Z. Iqbal, "Universal upper bound for the growth of Artin monoids," *Communications in Algebra*, vol. 43, no. 5, pp. 1967–1982, 2015.

[20] G. Bergman, "The diamond lemma for ring theory," *Advances in Mathematics*, vol. 29, no. 2, pp. 178–218, 1978.

[21] J. Birman, *Braids Links and Mapping-Class Groups. Ann Math Stud*, Princeton University Press, Princeton, USA, 1947.

[22] L. A. Bokut, "Grobner-Shirshov basis for the Braid group in the Birman-Ko-Lee generators," *Journal of Algebra*, vol. 321, no. 2, pp. 361–376, 2009.

[23] P. D. Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press, Chicago, IL, USA, 2000.

[24] D. J. Anick, "On the homology of associative algebras," *Transactions of the American Mathematical Society*, vol. 296, no. 2, pp. 641–659, 1986.

[25] K. B. Brown, "The geometry of rewriting systems: a proof of Anick-Groves-Squeir theorem," in *Algorithms and classification in combinatorial group theory*, vol. 23, pp. 137–164, MSRI Publications, Springer, NY, 1992.

[26] P. M. Cohn, *Further Algebra and Applications*, Springer, India, 2004.

[27] V. A. Ufnarovskij, "Combinatorial and asymptotic methods in algebra," in *Encyclopaedia of Mathematical Sciences*, vol. 57, Springer, Berlin, Germany, 1995.

[28] U. Ali, Z. Iqbal, and S. Nazir, "Canonical forms and Infimum of positive braids," *Algebra Colloquium*, vol. 18, no. spec01, pp. 1007–1016, 2011.