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Research Article

On Compact Trans-Sasakian Manifolds

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We study 3-dimensional compact and simply connected trans-Sasakian manifolds and find necessary and sufficient conditions under which these manifolds are homothetic to Sasakian manifolds. The first two results deal with finding necessary and sufficient conditions on a compact and simply connected trans-Sasakian manifold to be homothetic to an Einstein Sasakian manifold and in the third result deals with finding necessary and sufficient condition on a compact and simply connected trans-Sasakian manifold to be homothetic to a Sasakian manifold.

1. Introduction

It is well known that for an almost contact metric manifold (M, F, ζ, η, g) (cf. [1]), the product $\bar{M} = M \times R$ has an almost complex structure J, which with product metric $\bar{g}$ makes $(\bar{M}, \bar{g})$ an almost Hermitian manifold. The properties of the almost Hermitian manifold $(\bar{M}, J, \bar{g})$ control the properties of the almost contact metric manifold (M, F, ζ, η, g) and provide several structures on M such as a Sasakian structure and a quasi-Sasakian structure (cf. [1–3]). There are known sixteen different types of structures on $(\bar{M}, J, \bar{g})$ (cf. [4]), and using the structure in the class $\mathscr{W}_4$ on $(\bar{M}, J, g)$, a structure $(F, \zeta, \eta, g, \alpha, \beta)$ was introduced on M, which is called trans-Sasakian structure (cf. [5]), that generalizes Sasakian structure, Kenmotsu structure, and cosymplectic structure on a contact metric manifold (cf. [2, 3]), where α and β being the real functions defined on M.

Recall that a trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ is called a trans-Sasakain manifold of type (α, β), and trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$, and $(0, \beta)$ are called a cosymplectic, a α-Sasakian, and a β-Kenmotsu manifolds, respectively. It is on account of a result proved in [6] that a trans-Sasakian manifold of dimension five or greater than five reduces to a cosymplectic manifold, a α-Sasakian manifold, or a β-Kenmotsu manifold, so there is an emphasis on studying three-dimensional trans-Sasakian manifolds.

Among other questions, finding conditions under which a compact 3-dimensional trans-Sasakian manifold (M, F, ζ, η, g) is homothetic to a Sasakian manifold is of prime importance. The geometry of 3-dimensional trans-Sasakian manifold is also important owing to Thurston's conjecture (cf. [7]), and fetching conditions on a 3-dimensional trans-Sasakian manifold (M, F, ζ, η, g) in matching it among Thurston's eight geometries becomes more interesting. It is worth noting that in Thurston's eight geometries, the first place is occupied by the spherical geometry S^3.

In ([8–13]), the authors have studied compact 3-dimensional trans-Sasakian manifolds with some suitable restrictions on functions α and β appearing in the definition of a trans-Sasakian manifold for getting conditions under which a trans-Sasakian manifold is homothetic to a Sasakian manifold. In particular, it is known that a 3-dimensional compact simply connected trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ satisfying Poisson equations $\Delta\alpha = \beta$ and $\Delta\alpha = \alpha^2\beta$, respectively, is necessarily homothetic to a Sasakian manifold (cf. [10]).

An interesting work on 3-dimensional trans-Sasakian manifolds is found in [14, 15], where the authors have considered other aspects in Thurston's eight geometries. In [10], it is asked whether the function β on a 3-dimensional compact trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ satisfying $\operatorname{grad}\beta = \zeta(\beta)\zeta$ necessitates the trans-Sasakian manifold to be homothetic to a Sasakian manifold. In [15], it is shown that this question has negative answer.

Einstein Sasakian manifolds are very important due to their geometric importance (cf. [16]). In this paper, in our first two results, we find necessary and sufficient conditions on a compact simply connected 3-dimensional trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ to be homothetic to an Einstein Sasakian manifold, and in the third, we find a necessary and sufficient condition on a compact simply connected 3-dimensional trans-Sasakian $(M, F, \zeta, \eta, g, \alpha, \beta)$ to be homothetic to a Sasakian manifold.

In the first result, we consider a compact and simply connected trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ of positive constant scalar curvature τ, the function β satisfying Fischer-Marsden equation shows that the functions α and β are related to τ by the inequality $\beta(\alpha^2 - \beta^2 - \tau/4) \geq 0$, and the Ricci operator Q satisfying Codazzi-type equation with respect to vector field ζ necessarily implies that $(M, F, \zeta, \eta, g, \alpha, \beta)$ is homothetic to an Einstein Sasakian manifold. In the second result, we show that a compact simply connected trans-Sasakian manifold with function α constant along the integral curves of ζ, scalar curvature τ satisfying the inequality $\alpha(6\alpha^2 - \tau) \geq 0$, and the Ricci operator Q satisfying Codazzi-type equation with respect to vector field ζ necessarily imply that $(M, F, \zeta, \eta, g, \alpha, \beta)$ is homothetic to an Einstein Sasakian manifold. Finally, in the last result, we show that on a compact and simply connected trans-Sasakian manifold, the function β satisfies the differential inequality $\zeta(\beta^2) \leq -2\beta^3$, and vector fields $(\nabla Q)(\operatorname{grad}\alpha, \zeta), \zeta$ are orthogonal, which necessarily imply that $(M, F, \zeta, \eta, g, \alpha, \beta)$ is homothetic to a Sasakian manifold, where the covariant derivative $(\nabla Q)(U, \zeta) = \nabla_U Q\zeta - Q(\nabla_U \zeta)$ for a smooth vector field U on M.

2. Preliminaries

Let (M, F, ζ, η, g) be an almost contact metric manifold $\dim M = 3$, where F being a $(1,1)$-tensor field, ζ a unit vector field, and η smooth 1-form dual to ζ with respect to the Riemannian metric g satisfying

$$F^2 = -I + \eta \otimes \zeta, F(\zeta) = 0, \eta \circ F = 0, g(FU_1, FU_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2), \quad U_1, U_2 \in \Gamma(TM), \tag{1}$$

where $\Gamma(TM)$ is the space of smooth sections of the tangent bundle TM (cf. [1]). If there exist functions α and β on an almost contact metric manifold (M, F, ζ, η, g) satisfying

$$(\nabla F)(U_1, U_2) = \alpha(g(U_1, U_2)\zeta - \eta(U_2)U_1) + \beta(g(FU_1, U_2)\zeta - \eta(U_2)FU_1), \tag{2}$$

then $(M, F, \zeta, \eta, g, \alpha, \beta)$ is said to be a trans-Sasakian manifold, where $(\nabla F)(U_1, U_2) = \nabla_{U_1} FU_2 - F(\nabla_{U_1} U_2)$, $U_1, U_2 \in \Gamma(TM)$, and ∇ is the Levi-Civita connection with respect to the metric g (cf. [8–15]). Using equations (1) and (2), it follows that

$$\nabla_U \zeta = -\alpha F(U) + \beta(U - \eta(U)\zeta), \quad U \in \Gamma(TM). \tag{3}$$

Using the Ricci tensor Ric of a Riemannian manifold (M, g), the Ricci operator Q is defined by $Ric(U_1, U_2) = g(QU_1, U_2)$ and $U_1, U_2 \in \Gamma(TM)$. We have the following for a 3-dimensional trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$:

$$\zeta(\alpha) = -2\alpha\beta, \tag{4}$$

$$Q\zeta = F(\operatorname{grad}\alpha) - \operatorname{grad}\beta + 2\left(\alpha^2 - \beta^2\right)\zeta - \zeta(\beta)\zeta. \tag{5}$$

Note that equation (3) implies

$$\operatorname{div}\zeta = 2\beta, \tag{6}$$

and using this equation together with equation (4), we have

$$\operatorname{div}\left(\alpha^k\zeta\right) = k\alpha^{k-1}\zeta(\alpha) + \alpha^k \operatorname{div}\zeta = -2k\alpha^k\beta + 2\alpha^k\beta = -2(k-1)\alpha^k\beta. \tag{7}$$

Thus, on compact 3-dimensional trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$, using equation (6) and the above equation, we have

$$\int_M \beta = 0, \qquad \int_M \alpha^k\beta = 0 \text{for} k \neq 1. \tag{8}$$

Now, we state the following result of Okumura.

Theorem 1. *[17] Let (M, g) be a Riemannian manifold. If M admits a Killing vector field ζ of constant length satisfying*

$$c^2\left(\nabla_{U_1}\nabla_{U_2}\zeta - \nabla_{\nabla_{U_1}U_2}\zeta\right) = g(U_2, \zeta)U_1 - g(U_1, U_2)\zeta \tag{9}$$

for nonzero constant c and any vector fields U_1 and U_2, then M is homothetic to a Sasakian manifold.

For a smooth function h on the Riemannian manifold (M, g), then the operator A_h defined by

$$A_h(U) = \nabla_U \operatorname{grad} h, \quad U \in \Gamma(TM) \tag{10}$$

is called the Hessian operator of h, and it is a symmetric operator. Moreover, the Hessian $\operatorname{Hess}(h)$ of h is defined by

$$\operatorname{Hess}(h)(U_1, U_2) = g(A_h(U_1), U_2), \quad U_1, U_2 \in \Gamma(TM). \tag{11}$$

The Laplace operator Δ on (M, g) is defined by $\Delta h = \operatorname{div}(\operatorname{grad} h)$, and we also have

$$\Delta h = trA_h. \tag{12}$$

Fischer-Marsden differential equation on a Riemannian manifold (M, g) is (cf. [18])

$$(\Delta h)g + h\mathrm{Ric} = \mathrm{Hess}(h). \tag{13}$$

3. Trans-Sasakian Manifolds Homothetic to Einstein Sasakian Manifolds

In this section, we find necessary and sufficient conditions for a compact and simply connected 3-dimensional trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ to be homothetic to an Einstein Sasakian manifold.

Theorem 2. *A compact and simply connected 3-dimensional trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ with positive constant scalar curvature τ and the function β a solution of Fischer-Marsden equation satisfying*

$$\beta\left(\alpha^2 - \beta^2 - \frac{\tau}{4}\right) \geq 0 \tag{14}$$

is homothetic to an Einstein Sasakian manifold of positive scalar curvature, if and only if, the Ricci operator Q satisfies

$$(\nabla Q)(U, \zeta) = (\nabla Q)(\zeta, U), \quad U \in \Gamma(TM). \tag{15}$$

Proof. Suppose $(M, F, \zeta, \eta, g, \alpha, \beta)$ is a compact simply connected 3-dimensional trans-Sasakian manifold satisfying the hypothesis. Then, equation (13) gives

$$(\Delta\beta)g + \beta\mathrm{Ric} = \mathrm{Hess}(\beta), \tag{16}$$

and taking trace in above equation and using equation (12), we have

$$\Delta\beta = -\frac{\tau}{2}\beta. \tag{17}$$

□

Note that by equation (3), we have $\nabla_\zeta \zeta = 0$, and therefore, $\mathrm{Hess}(\beta)(\zeta, \zeta) = \zeta\zeta(\beta)$. Using this equation and equation (17) in equation (16), we get

$$-\frac{\tau}{2}\beta + \beta\mathrm{Ric}(\zeta, \zeta) = \zeta\zeta(\beta). \tag{18}$$

Now, using equation (5), we have $\mathrm{Ric}(\zeta, \zeta) = 2(\alpha^2 - \beta^2 - \zeta(\beta))$. Thus, the above equation becomes

$$-\frac{\tau}{2}\beta + 2\beta\left(\alpha^2 - \beta^2 - \zeta(\beta)\right) = \zeta\zeta(\beta). \tag{19}$$

Using equation (6), we have $\operatorname{div}(\zeta(\beta)\zeta) = \zeta\zeta(\beta) + 2\beta\zeta(\beta)$, and inserting it in the above equation, we conclude

$$-\frac{\tau}{2}\beta + 2\beta\left(\alpha^2 - \beta^2\right) = \operatorname{div}(\zeta(\beta)\zeta). \tag{20}$$

Integrating the above equation, we get

$$\int_M \beta\left(\alpha^2 - \beta^2 - \frac{\tau}{4}\right) = 0. \tag{21}$$

Using the inequality in the statement, we conclude

$$\beta\left(\alpha^2 - \beta^2 - \frac{\tau}{4}\right) = 0. \tag{22}$$

Since M is simply connected, it is connected, and therefore equation (22) implies either (i) $\beta = 0$ or (ii) $\alpha^2 - \beta^2 - \tau/4 = 0$. Suppose (ii) holds, then as τ is a constant, we get $\zeta(\alpha^2) = \zeta(\beta^2)$, which in view of equation (4) implies $\beta\zeta(\beta) = -2\alpha^2\beta$; that is, $3\beta^2\zeta(\beta) = -6\alpha^2\beta^2$. Thus, we have

$$\zeta\left(\beta^3\right) = -6\alpha^2\beta^2. \tag{23}$$

Using equation (6), we have $\operatorname{div}(\beta^3\zeta) = \zeta(\beta^3) + 2\beta^4$, and inserting it in above equation, we get

$$\operatorname{div}\left(\beta^3\zeta\right) = 2\beta^2\left(\beta^2 - 3\alpha^2\right). \tag{24}$$

Integrating the above equation, we get

$$\int_M \beta^2\left(3\alpha^2 - \beta^2\right) = 0. \tag{25}$$

Now, using (ii) in above integral, we have

$$\int_M \beta^2\left(2\alpha^2 + \frac{\tau}{4}\right) = 0, \tag{26}$$

and since the scalar curvature $\tau > 0$, through above integral, we conclude that $\beta = 0$. Thus, using equations (2), (3), (4), and (5), take the forms

$$\begin{gathered} (\nabla F)(U_1, U_2) = \alpha(g(U_1, U_2)\zeta - \eta(U_2)U_1), \\ \nabla_U \zeta = -\alpha F U, \end{gathered} \tag{27}$$

$$\begin{gathered} \zeta(\alpha) = 0, \\ Q\zeta = F(\operatorname{grad} \alpha) + 2\alpha^2\zeta. \end{gathered} \tag{28}$$

Taking the covariant derivative in the second equation of equation (28), we get

$$(\nabla Q)(U, \zeta) + Q(\nabla_U \zeta) = (\nabla F)(U, \operatorname{grad} \alpha) + F(A_\alpha U) + 4\alpha U(\alpha)\zeta + 2\alpha^2\nabla_U \zeta, \tag{29}$$

and using equation (27) in above equation, we arrive at

$$(\nabla Q)(U,\zeta)-\alpha Q(FU)=5\alpha U(\alpha)\zeta+F(A_\alpha U)-2\alpha^3 FU,\quad U\in\Gamma(TM). \tag{30}$$

Now, using the Codazzi equation type condition on Q in the hypothesis, we get

$$(\nabla Q)(\zeta,U)-\alpha Q(FU)=5\alpha U(\alpha)\zeta+F(A_\alpha U)-2\alpha^3 FU,\quad U\in\Gamma(TM). \tag{31}$$

Using the second equation in equation (27), we compute the Lie derivative of g with respect to ζ to conclude

$$(\pounds_\zeta g)(U_1,U_2)=-\alpha g(FU_1,U_2)-\alpha g(FU_2,U_1)=0, \tag{32}$$

that is, ζ is a Killing vector field and that the flow of ζ consists of isometries of the Riemannian manifold M. Thus, we have

$$(\pounds_\zeta Q)(U)=0,\quad U\in\Gamma(TM), \tag{33}$$

and using equation (27), we conclude

$$(\nabla Q)(\zeta,U)=\alpha Q(FU)-\alpha F(QU),\quad U\in\Gamma(TM). \tag{34}$$

Combining the above equation with equation (31), we have

$$-\alpha F(QU)=5\alpha U(\alpha)\zeta+F(A_\alpha U)-2\alpha^3 FU,\quad U\in\Gamma(TM). \tag{35}$$

Taking the inner product with ζ in above equation, we conclude

$$\alpha U(\alpha)=0,\quad U\in\Gamma(TM). \tag{36}$$

We claim that M being simply connected, $\alpha\neq 0$; for if $\alpha=0$, then by equation (27), we see that ζ is parallel and that η is closed, which implies η is exact; that is, $\eta=df$ for a smooth function f on M. This implies $\zeta=\mathrm{gradf}$, and M being compact, there is a point $q\in M$ such that $(\mathrm{gradf})(q)=0$, and we get $\zeta(q)=0$, a contradiction to the fact that ζ is a unit vector field. Hence, $\alpha\neq 0$, and equation (36) implies $U(\alpha)=0$, $U\in\Gamma(TM)$; that is, α is a nonzero constant.

Now, equation (28) gives $Q(\zeta)=2\alpha^2\zeta$, and taking the covariant derivative in this equation yields

$$(\nabla Q)(U,\zeta)-\alpha Q(FU)=-2\alpha^3 FU. \tag{37}$$

Using the condition in the hypothesis and equation (34) with $\alpha\neq 0$, in above equation, we get

$$-F(QU)=-2\alpha^2 FU,\quad U\in\Gamma(TM). \tag{38}$$

Operating F on above equation while using equation (1) and $Q(\zeta)=2\alpha^2\zeta$, we conclude

$$QU=2\alpha^2 U,\quad U\in\Gamma(TM). \tag{39}$$

This proves that M is an Einstein manifold. Finally, using equation (27), with α a nonzero constant, we compute

$$\begin{aligned}\nabla_{U_1}\nabla_{U_2}\zeta-\nabla_{\nabla_{U_1}U_2}\zeta&=-\alpha\left(\nabla_{U_1}FU_2-F\left(\nabla_{U_1}U_2\right)\right)\\&=-\alpha(\nabla F)(U_1,U_2)=\alpha^2\left(g(U_2,\zeta)U_1-g(U_1,U_2)\zeta\right).\end{aligned} \tag{40}$$

Hence, by Theorem 1, we conclude that M is homothetic to a compact simply connected Einstein Sasakian manifold of positive scalar curvature. The converse is trivial.

Theorem 3. *A compact and simply connected 3-dimensional trans-Sasakian manifold $(M,F,\zeta,\eta,g,\alpha,\beta)$ satisfying $\zeta(\alpha)=0$ and the scalar curvature τ satisfying*

$$\alpha\left(6\alpha^2-\tau\right)\geq 0 \tag{41}$$

is homothetic to an Einstein Sasakian manifold, if and only if, the Ricci operator Q satisfies

$$(\nabla Q)(U,\zeta)=(\nabla Q)(\zeta,U),\quad U\in\Gamma(TM). \tag{42}$$

Proof. Suppose $(M,F,\zeta,\eta,g,\alpha,\beta)$ is a compact and simply connected 3-dimensional trans-Sasakian manifold satisfying the hypothesis. Then, using equation (4) and the condition $\zeta(\alpha)=0$, we get $\alpha\beta=0$. However, if $\alpha=0$, then equation (3) implies that η is closed, and as seen in the proof of Theorem 2, we get a contradiction owing to simply connectedness of M. Thus, $\alpha\neq 0$, and on connected M, equation $\alpha\beta=0$ implies that $\beta=0$. Therefore, equations (27) and (28) hold. Now, using equation (28), we have $Q\zeta=F(\mathrm{grad}\,\alpha)+2\alpha^2\zeta$, which gives

$$-\mathrm{grad}\,\alpha=F(Q\zeta). \tag{43}$$

□

Taking covariant derivative in above equation, we have

$$-A_\alpha U=(\nabla F)(U,Q\zeta)+F(\nabla Q)(U,\zeta)+FQ(\nabla_U\zeta),\quad U\in\Gamma(TM), \tag{44}$$

and using equation (27), we get

$$-A_\alpha U=\alpha(\mathrm{Ric}(U,\zeta)\zeta-\mathrm{Ric}(\zeta,\zeta)U)+F(\nabla Q)(U,\zeta)-\alpha FQFU,\quad U\in\Gamma(TM). \tag{45}$$

Using condition in the hypothesis, we have

$$-A_\alpha U=\alpha(Ric(U,\zeta)\zeta-Ric(\zeta,\zeta)U)+F(\nabla Q)(\zeta,U)-\alpha FQFU,\quad U\in\Gamma(TM). \tag{46}$$

Also, equation (27) implies that ζ is a Killing vector field, and therefore, using its outcome equation (34) as well as equation (28), we conclude

$$-A_\alpha U = \alpha\left(\mathrm{Ric}(U,\zeta)\zeta - 2\alpha^2 U\right) + F(\alpha Q(FU) - \alpha F(QU)) - \alpha FQFU, \quad U \in \Gamma(TM). \tag{47}$$

That is, on using equation (1), we have

$$A_\alpha U = \alpha\left(2\alpha^2 U - QU\right), \quad U \in \Gamma(TM). \tag{48}$$

Thus, we have

$$\Delta\alpha = \alpha\left(6\alpha^2 - \tau\right), \tag{49}$$

and integrating the above equation, we get

$$\int_M \alpha\left(6\alpha^2 - \tau\right) = 0. \tag{50}$$

Using the condition in the hypothesis, we have $\alpha(6\alpha^2 - \tau) = 0$, and as $\alpha \neq 0$, we conclude $\tau = 6\alpha^2$. Consequently, equation (49) implies that $\Delta\alpha = 0$, and M being compact, we conclude that α is a constant. Thus, α being nonzero constant, equation (48) gives

$$QU = 2\alpha^2 U, \quad U \in \Gamma(TM). \tag{51}$$

This proves that M is an Einstein manifold, and using equation (27), we see that the unit Killing vector field ζ satisfies

$$\nabla_{U_1}\nabla_{U_2}\zeta - \nabla_{\nabla_{U_1}U_2}\zeta = \alpha^2(g(U_2,\zeta)U_1 - g(U_1,U_2)\zeta), \quad U_1, U_2 \in \Gamma(TM). \tag{52}$$

Hence, in view of Theorem 1, we conclude that M is homothetic to an Einstein Sasakian manifold. The converse is trivial.

Theorem 4. *A compact and simply connected 3-dimensional trans-Sasakian manifold $(M, F, \zeta, \eta, g, \alpha, \beta)$ satisfying*

$$\zeta\left(\beta^2\right) \leq -2\beta^3 \tag{53}$$

is homothetic to a Sasakian manifold, if and only if, the vector fields $(\nabla Q)(\mathrm{grad}\ \alpha, \zeta)$ and ζ are orthogonal.

Proof. Suppose $(M, F, \zeta, \eta, g, \alpha, \beta)$ is a compact and simply connected 3-dimensional trans-Sasakian manifold satisfying the hypothesis. Then, using equation (5), we have

$$\mathrm{grad}\ \beta = F(\mathrm{grad}\ \alpha) - Q\zeta + 2\left(\alpha^2 - \beta^2\right)\zeta - \zeta(\beta)\zeta, \tag{54}$$

and taking covariant derivative in above equation with respect to $U \in \Gamma(TM)$, while using equation (3), we get

$$\begin{aligned} A_\beta U = {} & (\nabla F)(U, grad\alpha) + FA_\alpha U - (\nabla Q)(U,\zeta) - Q(-\alpha FU + \beta U - \beta\eta(U)\zeta) \\ & + 2U\left(\alpha^2 - \beta^2\right)\zeta + 2\left(\alpha^2 - \beta^2\right)(-\alpha FU + \beta U - \beta\eta(U)\zeta) \\ & - U\zeta(\beta)\zeta - \zeta(\beta)(-\alpha FU + \beta U - \beta\eta(U)\zeta). \end{aligned} \tag{55}$$

□

Taking trace in above equation and noting the following outcome of equation (2),

$$\sum_{i=1}^{3}(\nabla F)(e_i, e_i) = 2\alpha\zeta, \tag{56}$$

for a local orthonormal frame $\{e_1, e_2, e_3\}$ on M, we get

$$\begin{aligned} \Delta\beta = {} & -2\alpha\zeta(\alpha) - \frac{1}{2}\zeta(\tau) - \beta\tau + \beta Ric(\zeta,\zeta) + 2\zeta\left(\alpha^2 - \beta^2\right) \\ & + 4\beta\left(\alpha^2 - \beta^2\right) - \zeta\zeta(\beta) - 2\beta\zeta(\beta), \end{aligned} \tag{57}$$

where we have used $TrFA_\alpha = 0$, $TrQF = 0$, $TrF = 0$, and well known formula

$$\sum_{i=1}^{3}(\nabla Q)(e_i, e_i) = \frac{1}{2}\ \mathrm{grad}\ \tau. \tag{58}$$

Now, using equation (4) and $\mathrm{Ric}(\zeta,\zeta) = 2(\alpha^2 - \beta^2 - \zeta(\beta))$ in equation (57), we have

$$\Delta\beta = -\frac{1}{2}\zeta(\tau) - \beta\tau - 6\beta^3 + 10\alpha^2\beta - 8\beta\zeta(\beta) - \zeta\zeta(\beta), \tag{59}$$

that is,

$$\Delta\beta = -\frac{1}{2}\zeta(\tau) - \beta\tau - 6\beta^3 + 10\alpha^2\beta - \zeta\left(\zeta(\beta) + 4\beta^2\right). \tag{60}$$

Note that on using equation (6), we have $1/2\ \mathrm{div}\ (\tau\zeta) = 1/2\zeta(\tau) + \beta\tau$ and

$$\mathrm{div}\ \left(\left(\zeta(\beta) + 4\beta^2\right)\zeta\right) = \zeta\left(\zeta(\beta) + 4\beta^2\right) + 2\beta\left(\zeta(\beta) + 4\beta^2\right). \tag{61}$$

Inserting these equations in equation (60), we arrive at

$$\Delta\beta = -\frac{1}{2}\ \mathrm{div}\ (\tau\zeta) - 6\beta^3 + 10\alpha^2\beta + 2\beta\left(\zeta(\beta) + 4\beta^2\right) - \mathrm{div}\ \left(\left(\zeta(\beta) + 4\beta^2\right)\zeta\right), \tag{62}$$

and integrating the above equation while keeping in mind equation (8), we get

$$\int_M \left(\zeta\left(\beta^2\right) + 2\beta^3\right) = 0. \tag{63}$$

Using the condition in the hypothesis, we conclude $\zeta($

$\beta^2) = -2\beta^3$ and $\beta\zeta(\beta) = -\beta^3$; that is, $3\beta^2\zeta(\beta) = -3\beta^4$. We get $\zeta(\beta^3) = -3\beta^4$, which in view of equation (6) implies

$$\text{div}\left(\beta^3\zeta\right) = -\beta^4. \tag{64}$$

Integrating the above equation yields $\beta = 0$. Thus, equations (27) and (28) are now available to us. Taking covariant derivative in second equation of equation (28) with respect to $U \in \Gamma(TM)$, we get

$$(\nabla Q)(U, \zeta) - \alpha Q(FU) = (\nabla F)(U, \text{grad}\ \alpha) + FA_\alpha U + 4\alpha U(\alpha)\zeta - 2\alpha^3 FU, \tag{65}$$

where we have used the second equation in equation (27). Now, using equation (27) and noting that $\zeta(\alpha) = 0$ in above equation, we conclude

$$(\nabla Q)(U, \zeta) - \alpha Q(FU) = 5\alpha U(\alpha)\zeta + FA_\alpha U - 2\alpha^3 FU. \tag{66}$$

Taking the inner product with ζ in above equation, we get

$$5\alpha U(\alpha) = g((\nabla Q)(U, \zeta), \zeta) - \alpha g(FU, Q\zeta). \tag{67}$$

Now, using equation (28), we have

$$g(FU, Q\zeta) = g\left(FU, F(\text{grad}\ \alpha) + 2\alpha^2\zeta\right) = U(\alpha), \tag{68}$$

and inserting it in equation (67), we get

$$6\alpha U(\alpha) = g((\nabla Q)(U, \zeta), \zeta), \quad U \in \Gamma(TM). \tag{69}$$

Taking $U = \text{grad}\ \alpha$ in above equation and using the condition in the hypothesis, we conclude

$$\alpha\|\text{grad}\ \alpha\|^2 = 0. \tag{70}$$

Note that M being compact and simply connected, α is not allowed to be zero. Hence, the above equation implies that α is nonzero constant. Thus, we have by virtue of equation (27) that

$$\nabla_{U_1}\nabla_{U_2}\zeta - \nabla_{\nabla_{U_1}U_2}\zeta = \alpha^2(g(U_2, \zeta)U_1 - g(U_1, U_2)\zeta), \quad U_1, U_2 \in \Gamma(TM). \tag{71}$$

This proves that M is homothetic to a Sasakian manifold. The converse is trivial.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] D. E. Blair, "Contact manifolds," in *Contact Manifolds in Riemannian Geometry*, vol. 509, Springer, Berlin, Heidelberg, 1976.

[2] D. E. Blair and J. A. Oubiña, "Conformal and related changes of metric on the product of two almost contact metric manifolds," *Publicacions Matemàtiques*, vol. 34, no. 1, pp. 199–207, 1990.

[3] A. Fujimoto and H. Muto, "On cosymplectic manifolds," *Tenso*, vol. 28, pp. 43–52, 1974.

[4] A. Gray and L. M. Hervella, "The sixteen classes of almost Hermitian manifolds and their linear invariants," *Annali Di Matematica Pura Ed Applicata*, vol. 123, no. 1, pp. 35–58, 1980.

[5] J. A. Oubiña, "New classes of almost contact metric structures," *Universitatis Debreceniensis*, vol. 32, no. 3-4, pp. 187–193, 1985.

[6] J. C. Marrero, "The local structure of trans-Sasakian manifolds," *Annali di Matematica Pura ed Applicata*, vol. 162, no. 1, pp. 77–86, 1992.

[7] W. P. Thurston, "Three dimensional manifolds, Kleinian groups and hyperbolic geometry," *Bulletin (New Series) of The American Mathematical Society*, vol. 6, no. 3, pp. 357–382, 1982.

[8] S. Deshmukh and M. M. Tripathi, "A note on trans-Sasakian manifolds," *Mathematica Slovaca*, vol. 63, no. 6, pp. 1361–1370, 2013.

[9] S. Deshmukh, U. C. De, and F. Al-Solamy, "Trans-Sasakian manifolds homothetic to Sasakian manifolds," *Universitatis Debreceniensis*, vol. 88, no. 3-4, pp. 439–448, 2016.

[10] S. Deshmukh, "Trans-Sasakian manifolds homothetic to Sasakian manifolds," *Mediterranean Journal of Mathematics*, vol. 13, no. 5, pp. 2951–2958, 2016.

[11] S. Deshmukh and F. al-Solamy, "A note on compact trans-Sasakian manifolds," *Mediterranean Journal of Mathematics*, vol. 13, no. 4, pp. 2099–2104, 2016.

[12] S. Deshmukh and U. C. De, "A note on trans-Sasakian manifolds," *Universitatis Debreceniensis*, vol. 92, no. 1-2, pp. 159–169, 2018.

[13] S. Deshmukh, A. Ishan, O. Belova, and S. B. Al-Shaikh, "Some conditions on trans-Sasakian manifolds to be homothetic to Sasakian manifolds," *Mathematics*, vol. 9, no. 16, p. 1887, 2021.

[14] W. Wang and X. Liu, "Ricci tensors on trans-Sasakian 3-manifolds," *Univerzitet u Nišu*, vol. 32, no. 12, pp. 4365–4374, 2018.

[15] Y. Wang and W. Wang, "A remark on trans-Sasakian 3-manifolds," *Revista de La Union Mat. Argentina*, vol. 60, no. 1, pp. 257–264, 2019.

[16] C. Boyer and K. Galicki, *Sasakian Geometry; Oxford Mathematical Monographs*, Oxford University Press, Oxford, NY, USA, 2007.

[17] M. Okumura, "Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures," *Tohoku Mathematical Journal, Second Series*, vol. 16, no. 3, pp. 270–284, 1964.

[18] A. E. Fischer and J. E. Marsden, "Manifolds of Riemannian metrics with prescribed scalar curvature," *Bulletin of the American Mathematical Society*, vol. 80, no. 3, pp. 479–485, 1974.