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Research Article

On $(\eta, \gamma)_{(f,g)}$-Contractions in Extended $\mathfrak{b}$-Metric Spaces

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In this paper, we give a concept of $(\eta, \gamma)_{(f,g)}$-contraction in the setting of expanded $\mathfrak{b}$–metric spaces and discuss the existence and uniqueness of a common fixed point. Introduced results generalize well-known fixed point theorems on contraction conditions and in the given spaces.

1. Introduction and Preliminaries

The tremendous applications of fixed point theory had always inspired the growth of this domain. In 1922, Banach formulated his most simple but very natural result which is now popularly referred to as the Banach contraction principle. In the course of the last several decades, this principle has been extended and generalized in many directions with several applications in many branches. Employing simulation functions, Khojasteh et al. [1] initiated the idea of $\mathcal{Z}$-contractions and utilized the same to cover the varied types of nonlinear contractions in the existing literature. Later, Argoubi et al. [2] and Roldán-López-de-Hierro et al. [3] independently sharpened the notion of simulation functions and also proved some coincidences and common fixed point results. Very recently, Lopez et al. [4] introduced the notion of $\mathcal{R}$-contractions in order to extend several nonlinear contractions such as $\mathcal{Z}$-contractions, manageable contractions, and Meir-Keeler contractions. Indeed, $\mathcal{R}$-contractions are associated with R-functions that satisfy two independent conditions involving two sequences of nonnegative real numbers. Soon, inspired by $\mathcal{R}$-contractions, Shahzad et al. [5] introduced the notion of (η, γ)-contractions which remains an extension of $(\mathcal{R}, \gamma)$-contractions given in [6] by Roldán-López-de-Hierro and Shahzad wherein the authors proved very interesting results.

Czerwik [7] established a successful generalization of the metric space concept by introducing the notion of $\mathfrak{b}$-metric space. Following this, a number of authors have introduced respective interesting theorems in $\mathfrak{b}$-metric, (see [8–14]). Newly, Kamran et al. [15] inspired by the concept of b-metric space, they introduced the concept of extended $\mathfrak{b}$ space and also developed some fixed point theorems for self-mappings defined in such spaces. Their results extend/generalize many of the results already available in the literature. In this paper, we shall define a general contraction condition with the help of some auxiliary functions and investigate the existence and uniqueness of a fixed point for such mappings in the frame of $\mathfrak{b}$-metric space.

Definition 1 (see [7]). Let X be a nonempty set and let $\mathfrak{d} : X \times X \longrightarrow [0,\infty)$ satisfy the following for all $u, v, w \in X$.

(i) $\mathfrak{d}(u, v) = 0 \Leftrightarrow u = v$

(ii) $\mathfrak{d}(u, v) = \mathfrak{d}(v, u)$

(iii) $\mathfrak{d}(u, w) \le \mathfrak{b}[\mathfrak{d}(u, v) + \mathfrak{d}(v, w)]$, where $\mathfrak{b} \ge 1$

The pair $(X, \mathfrak{d})$ is called a $\mathfrak{b}$-metric space; when $\mathfrak{b} = 1$, the $\mathfrak{b}$-metric space becomes a usual metric space.

Example 2 (see [8, 9]). Let $X = \mathbb{R}^2$. Then, the functional $\mathfrak{d} : X \times X \longrightarrow [0,\infty)$ defined by

$$\mathfrak{d}((u_1, v_1), (u_2, v_2)) := \begin{cases} |u_1 - u_2| + |v_1 - v_2|, & (u_1, v_1), (u_2, v_2) \in [0, 1) \times [0, 1) \\ |u_1 - u_2| + |v_1 - v_2|, & (u_1, v_1), (u_2, v_2) \in (1,\infty) \times (1,\infty) \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

is a $\mathfrak{b}$-metric space on X with $\mathfrak{b} = 2$.

Example 3 (see [16]). The space $L^p[0, 1]$ (where $0 < p < 1$) of all real functions $u(t), t \in [0, 1]$ such that $\int_0^1 |u(t)|^p dt < \infty$, together with the functional

$$\delta(u, v) = \left[\int_0^1 |u(t) - v(t)|^p dt\right]^{1/p}, \forall u, v \in L^p[0, 1], \tag{2}$$

is a $\mathfrak{b}$-metric space with $\mathfrak{b} = 2^{1/p}$.

Example 4 (see [17]). Let $X = \{a, b, c\}$ and $\mathfrak{d} : X \times X \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned} &\mathfrak{d}(a, b) = \mathfrak{d}(b, a) = \mathfrak{d}(a, c) = \mathfrak{d}(c, a) = b, \\ &\mathfrak{d}(b, c) = \mathfrak{d}(c, b) = \alpha \ge c, \\ &\mathfrak{d}(a, a) = \mathfrak{d}(b, b) = \mathfrak{d}(c, c) = a. \end{aligned} \tag{3}$$

Then,

$$\mathfrak{d}(u, v) \le \frac{\alpha}{2}[\mathfrak{d}(u, w) + \mathfrak{d}(w, v)]. \tag{4}$$

Therefore, $(X, \mathfrak{d})$ is a $\mathfrak{b}$-metric space for all $u, v, w \in X$. If $\alpha > c$, then the ordinary triangle inequality does not hold, and $(X, \mathfrak{d})$ is not a metric space.

Definition 5 (see [15]). Let X be a nonempty set and $\psi : X \times X \longrightarrow [1,\infty)$, and let $\mathfrak{d}_\psi : X \times X \longrightarrow [0,\infty)$ satisfy:

(i) $\mathfrak{d}_\psi(u, v) = 0 \Leftrightarrow u = v$

(ii) $\mathfrak{d}_\psi(u, v) = \mathfrak{d}_\psi(v, u)$

(iii) $\mathfrak{d}_\psi(u, w) \le \psi(u, w)[\mathfrak{d}_\psi(u, v) + \mathfrak{d}_\psi(v, w)]$

The pair $(X, \mathfrak{d}_\psi)$ is called an extended $\mathfrak{b}$-metric space. If $\psi(u, v) = \mathfrak{b}$, for $\mathfrak{b} \ge 1$, then we reduce to Definition 1.

Example 6 (see [17]). Let $X = \{a, b, c\} \cup \mathbb{R}_0^+$ and $\mathfrak{d} : X \times X \longrightarrow [0,\infty)$ be defined by

(i) If $u, v \in \mathbb{R}_0^+$, then $\mathfrak{d}_\psi(u, v) = |u - v|^2$,

(ii) If $u \in \{a, b, c\}$ and $v \in \mathbb{R}_0^+$, then $\mathfrak{d}_\psi(u, v) = \mathfrak{d}_\psi(v, u) = 1$ and $\mathfrak{d}_\psi(u, u) = 0$

(iii) If $u, v \in \{a, b, c\}$, such that

$$\mathfrak{d}_\psi(a, b) = 1, \mathfrak{d}_\psi(a, c) = \frac{1}{2} \text{ and } \mathfrak{d}_\psi(b, c) = 2, \tag{5}$$

with $\mathfrak{d}_\psi(u, v) = \mathfrak{d}_\psi(v, u)$ and $\mathfrak{d}_\psi(u, u) = 0$.

Notice that $\mathfrak{d}$ is not a metric space since $\mathfrak{d}_\psi(b, c) > \mathfrak{d}_\psi(b, a) + \mathfrak{d}_\psi(a, c)$. However, it is easy to see that $\mathfrak{d}$ is an extended $\mathfrak{b}$-metric space for $\psi : X \times X \longrightarrow [1,\infty)$, where

$$\psi(u, v) := \begin{cases} \frac{4}{3} & \text{if } u, v \in \{a, b, c\}, \\ 2 & \text{if } u, v \in \mathbb{R}_0^+, \\ 1 & \text{if } (u, v) \text{ or } (v, u) \in \{a, b, c\} \times \mathbb{R}_0^+. \end{cases} \tag{6}$$

Example 7 (see [15]). Let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a, b]$, let $\psi : X \times X \longrightarrow [1,\infty)$ where $\psi(u, v) = |u(t)| + |v(t)| + 2$, and note that X is a complete extended $\mathfrak{b}$-metric space by considering

$$\mathfrak{d}_\psi(u, v) = \sup_{t\in[a,b]} |u(t) - v(t)|^2. \tag{7}$$

In this context, wonderful theorems established by the authors in extended $\mathfrak{b}$-metric space, for examples, Fahed et al. and Swapna and Phaneendra [18, 19], got some new fixed point results in an extended $\mathfrak{b}$-metric space. Also, Ullah et al. [20] proved fixed point theorems in complex-valued extended $\mathfrak{b}$-metric spaces. In this bearing, Mitrović et al. [21] established new results in extended $\mathfrak{b}$-metric space, in follows that we recollect some fundamental notions, for example, convergence, the notion of the Cauchy sequence, and completeness in an extended $\mathfrak{b}$- metric space.

Definition 8 (see [22]). Let $(X, \mathfrak{d}_\psi)$ be an extended $\mathfrak{b}$-metric space, and then

(i) A sequence $u_n \in X$ is said to converge to $u_0 \in X$ if, $\forall \varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\mathfrak{d}_\psi(u_n, u_0) < \varepsilon, \forall n \geq N$. We write $\lim_{n \longrightarrow \infty} u_n = u_0$

(ii) A sequence u_n in X is said to be Cauchy if, $\forall \varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\mathfrak{d}_\psi(u_n, u_m) < \varepsilon$, $\forall n, m \geq N$

Definition 9 (see [15]). An extended $\mathfrak{b}$ -metric space $(X, \mathfrak{d}_\psi)$ is complete if every Cauchy sequence in X is convergent.

Lemma 10 (see [22]). *Let $(X, \mathfrak{d}_\psi)$ be a complete extended $\mathfrak{b}$-metric space. If $\mathfrak{d}_\psi$ is continuous map, then every convergent sequence in X has a unique limit.*

Theorem 11 (see [15]). *Suppose $(X, \mathfrak{d}_\psi)$ is an extended $\mathfrak{b}$-metric space such that $\mathfrak{d}_\psi$ is a continuous mapping. Suppose $T : X \longrightarrow X$, it fulfills*

$$\mathfrak{d}_\psi(Tu, Tv) \leq \eta \mathfrak{d}_\psi(u, v), \forall u, v \in X, \tag{8}$$

where $\eta \in [0, 1)$ is such that, for each $u_0 \in X$, we have $\lim_{n,m \longrightarrow \infty} \psi(u_n, u_m) < 1/\eta$. Here, $T^n u_0 = u_n, n = 1, 2, \cdots$ Then, T has exactly one fixed point u_0, moreover $\forall v \in X, T^n v \longrightarrow u_0$.

For our objectives, we recall the definition of orbital admissible maps introduced by Popescu [23].

Definition 12. Let S be a self-map on X and $\eta : X \times X \longrightarrow [0,\infty)$. We say that S is an η-orbital admissible if for all u, $v \in X$, we have

$$\eta(u, Su) \geq 1 \Rightarrow \eta\left(Su, S^2 u\right) \geq 1. \tag{9}$$

Remark 13 (see [23]). Every η-admissible mapping is an α -orbital admissible mapping.

Definition 14 (see [24]). For a nonempty set X, suppose $T : X \longrightarrow X$ and $\eta : X \times X \longrightarrow [0,\infty)$ are mappings. One says that self-mapping T on X is η -admissible if for $u, v \in X$, one has

$$\eta(u, v) \geq 1 \Rightarrow \eta(Tu, Tv) \geq 1. \tag{10}$$

Definition 15 (see [17]). Let Φ be the family of functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions

(i) ϑ is nondecreasing

(ii) $\vartheta(\chi) < \chi, \chi > 0$

2. Main results

Definition 16. Let $(X, \mathfrak{d})$ be an extended $\mathfrak{b}$-metric space and let $\eta : X \times X \longrightarrow [0,\infty)$ and $\psi : X \times X \longrightarrow [1,\infty)$ such that

$$\eta(u, v)\mathfrak{d}_\psi(Tu, Tv) \leq \psi(f(u, v)), \tag{11}$$

where

$$f(u, v) = \sup \left\{ \begin{array}{c} \boxtimes_\psi(Su, Sv), \boxtimes_\psi(Tu, Su), \boxtimes_\psi(Sv, Tv), \\ \dfrac{\boxtimes_\psi(Su, Tu)\boxtimes_\psi(Sv, Tv)}{\boxtimes_\psi(Su, Sv)}, \dfrac{\boxtimes_\psi(Su, Tv) + \boxtimes_\psi(Sv, Tu)}{2 \sup \left[\psi(Sv, Tu), \psi(Su, Tv)\right]} \end{array} \right\}. \tag{12}$$

Then, S and T are $(\eta, \gamma)_f$-contraction for all $u, v \in X$, where $\psi \in \Psi$.

The headmost principal result of this paper is as follows:

Theorem 17. *Let $(X, \mathfrak{d})$ be a complete extended $\mathfrak{b}$ - metric space, and let $S, T : X \longrightarrow X$ be an $(\eta, \gamma)_f$-contraction mappings. Let*

$$\lim_{n,m \longrightarrow \infty} \sup \frac{\vartheta^{n+1}(r)}{\vartheta^n(r)} \psi(u_n, u_m) < 1, \tag{13}$$

for all $u_0 \in X, r > 0$ where $T^n u_{n-n} = Su_n = u_n, n \in \mathbb{N}$. Assume also that

(i) S and T are η-orbital admissible

(ii) There exists $\mathfrak{w} \in X$ such that $\eta(S\mathfrak{w}, T\mathfrak{w}) \geq 1$

(iii) S and T are continuous

Then, S and T possess a unique coincidence fixed point u_0; that is, $Su = Tu = u_0$.

Proof. By a supposition, for some $u_0 \in X$, we have $u_n = Su_n = T^n u_0, \forall n \in \mathbb{N}$. Suppose that $u_{n_0} = Tu_{n_0 - 1} = Su_{n_0}$. Let that $u_n \neq u_{n+1}, \forall n \in \mathbb{N}$. Since T and S are η-admissible, we get

$$\eta(u_0, u_1) = \eta(Su_0, Tu_0) \geq 1 \Rightarrow \eta(Su_1, Tu_1) = \eta(u_1, u_2) \geq 1. \tag{14}$$

Repeatedly, we obtain for all $n \in \mathbb{N}$

$$\eta(u_n, u_{n+1}) \geq 1. \tag{15}$$

On regard of (15) and (11), we get

$$\mathfrak{d}_\psi(u_n, u_{n+1}) = \mathfrak{d}_\psi(Su_n, Su_{n+1}) = \mathfrak{d}_\psi(Tu_{n-1}, Tu_n) \leq \vartheta(f(u_{n-1}, u_n)), \tag{16}$$

where

$$\begin{aligned} f(u_{n-1}, u_n) &= \sup \left\{ \begin{array}{c} \mathfrak{d}_\psi(Su_{n-1}, Su_n), \mathfrak{d}_\psi(Tu_{n-1}, Su_{n-1}), \mathfrak{d}_\psi(Su_n, Tu_n), \\ \dfrac{\mathfrak{d}_\psi(Su_{n-1}, Tu_{n-1})\mathfrak{d}_\psi(Su_n, Tu_n)}{\mathfrak{d}_\psi(Su_{n-1}, Su_n)}, \dfrac{\mathfrak{d}_\psi(Su_{n-1}, Tu_n) + \mathfrak{d}_\psi(Su_n, Tu_{n-1})}{2 \sup [\psi(Su_n, Tu_{n-1}), \psi(Su_{n-1}, Tu_n)]} \end{array} \right\} \\ &= \sup \left\{ \begin{array}{c} \mathfrak{d}_\psi(u_{n-1}, u_n), \mathfrak{d}_\psi(u_n, u_{n-1}), \mathfrak{d}_\psi(u_n, u_{n+1}), \\ \dfrac{\mathfrak{d}_\psi(u_{n-1}, u_n)\mathfrak{d}_\psi(u_n, u_{n+1})}{\mathfrak{d}_\psi(u_{n-1}, u_n)}, \dfrac{\mathfrak{d}_\psi(u_{n-1}, u_{n+1}) + \mathfrak{d}_\psi(u_n, u_n)}{2 \sup [\psi(u_n, u_n), \psi(u_{n-1}, u_{n+1})]} \end{array} \right\} \\ &= \sup \left\{ \mathfrak{d}_\psi(u_n, u_{n-1}), \mathfrak{d}_\psi(u_n, u_{n+1}), \dfrac{\mathfrak{d}_\psi(u_{n-1}, u_{n+1})}{2 \sup [\psi(u_n, u_n), \psi(u_{n-1}, u_{n+1})]} \right\} \\ &\leq \sup \left\{ \mathfrak{d}_\psi(u_n, u_{n-1}), \mathfrak{d}_\psi(u_n, u_{n+1}), \dfrac{\mathfrak{d}_\psi(u_{n-1}, u_n) + \mathfrak{d}_\psi(u_n, u_{n+1})}{2} \right\} \\ &= \sup \left\{ \mathfrak{d}_\psi(u_n, u_{n-1}), \mathfrak{d}_\psi(u_n, u_{n+1}) \right\}. \end{aligned} \tag{17}$$

Now, if $f(u_{n-1}, u_n) = \mathfrak{d}_\psi(u_n, u_{n+1})$ for some $n \in \mathbb{N}$, thus

$$\mathfrak{d}_\psi(u_n, u_{n+1}) \leq \vartheta(\mathfrak{d}_\psi(u_n, u_{n+1})) < \mathfrak{d}_\psi(u_n, u_{n+1}), \tag{18}$$

which is a contradiction. On the other hand, if $f(u_{n-1}, u_n) = \mathfrak{d}_\psi(u_n, u_{n-1})$, then for all $n \geq 1$, we have

$$\mathfrak{d}_\psi(u_n, u_{n+1}) \leq \vartheta(\mathfrak{d}_\psi(u_n, u_{n-1})) < \mathfrak{d}_\psi(u_n, u_{n-1}). \tag{19}$$

Sequentially for all $n \in \mathbb{N}$, we get

$$\mathfrak{d}_\psi(u_n, u_{n+1}) \leq \vartheta^n(\mathfrak{d}_\psi(u_0, u_1)). \tag{20}$$

Thus, there exists $k \geq 0$ such that

$$\mathfrak{d}_\psi(u_n, u_{n+1}) = k, \text{ when } n \longrightarrow \infty. \tag{21}$$

Taking $n \longrightarrow \infty$ to inequality (19), we obtain

$$k \leq \vartheta(k) \Rightarrow k = 0. \tag{22}$$

Therefore, when $n \longrightarrow \infty$, we get

$$\mathfrak{d}_\psi(u_n, u_{n+1}) = 0. \tag{23}$$

We will show that $\{u_n\}$ is a Cauchy sequence, as follows:

$$\begin{aligned} \mathfrak{d}_\psi(u_n, u_{n+m}) &\leq \psi(u_n, u_{n+m})\left[\mathfrak{d}_\psi(u_n, u_{n+1}) + \mathfrak{d}_\psi(u_{n+1}, u_{n+m})\right] \\ &\leq \psi(u_n, u_{n+m})\mathfrak{d}_\psi(u_n, u_{n+1}) \\ &\quad + \psi(u_n, u_{n+m})\mathfrak{d}_\psi(u_{n+1}, u_{n+m}). \end{aligned} \tag{24}$$

Also,

$$\begin{aligned} \mathfrak{d}_\psi(u_{n+1}, u_{n+m}) &\leq \psi(u_{n+1}, u_{n+m})\left[\mathfrak{d}_\psi(u_{n+1}, u_{n+2}) + \mathfrak{d}_\psi(u_{n+2}, u_{n+m})\right] \\ &\leq \psi(u_{n+1}, u_{n+m})\mathfrak{d}_\psi(u_{n+1}, u_{n+2}) \\ &\quad + \psi(u_{n+1}, u_{n+m})\mathfrak{d}_\psi(u_{n+2}, u_{n+m}). \end{aligned} \tag{25}$$

And so, until the inequality (24) reaches to

$$\begin{aligned} \mathfrak{d}_\psi(u_n, u_{n+m}) &\leq \psi(u_n, u_{n+m})\left[\mathfrak{d}_\psi(u_n, u_{n+1}) + \mathfrak{d}_\psi(u_{n+1}, u_{n+m})\right] \\ &\leq \psi(u_1, u_{n+m})\psi(u_2, u_{n+m}) \cdots \psi(u_n, u_{n+m})\vartheta^n(\mathfrak{d}_\psi(u_0, u_1)) \\ &\quad + \psi(u_1, u_{n+m})\psi(u_2, u_{n+m}) \cdots \psi(u_n, u_{n+m})\psi \\ &\quad \cdot (u_{n+1}, u_{n+m})\vartheta^{n+1}(\mathfrak{d}_\psi(u_0, u_1)) \\ &\quad + \cdots + \cdots + \psi(u_1, u_{n+m})\psi(u_2, u_{n+m}) \cdots \psi \\ &\quad \cdot (u_{n+m-1}, u_{n+m})\vartheta^{n+m-1}(\mathfrak{d}_\psi(u_0, u_1)) \\ &= \sum_{i=n}^{n+m-1} \vartheta^i(\mathfrak{d}_\psi(u_0, u_1)) \prod_{j=1}^{i} \psi(u_j, u_{n+m}) = \mathfrak{b}_{n+m-1}, \end{aligned} \tag{26}$$

we conclude that

$$\mathfrak{d}_\psi(u_n, u_{n+m}) < \mathfrak{b}_{n+m-1} - \mathfrak{b}_n. \tag{27}$$

The series

$$\sum_{i=1}^{\infty} \vartheta^i(\mathfrak{d}_\psi(u_0, u_1)) \prod_{j=1}^{i} \psi(u_j, u_{n+m}). \tag{28}$$

Suppose $\vartheta^n(\mathfrak{d}_\psi(u_0, u_1))\prod_{j=1}^n \psi(u_j, u_{n+m}) = \mathscr{B}_n$, then

$$\begin{aligned} &\frac{\vartheta^{n+1}(\mathfrak{d}_\psi(u_0, u_1))\prod_{j=1}^{n+1}\psi(u_j, u_{n+m})}{\vartheta^{n}(\mathfrak{d}_\psi(u_0, u_1))\prod_{j=1}^{n}\psi(u_j, u_{n+m})} = \frac{\mathscr{B}_{n+1}}{\mathscr{B}_n}, \\ &\frac{\vartheta^{n+1}(\mathfrak{d}_\psi(u_0, u_1))}{\vartheta^{n}(\mathfrak{d}_\psi(u_0, u_1))}\psi(u_{n+1}, u_{n+m}) = \frac{\mathscr{B}_{n+1}}{\mathscr{B}_n} < 1, \end{aligned} \tag{29}$$

by (13) where $n, m \longrightarrow \infty$. Then, (28) converges by [25]. As result, in perspective of (27), we have

$$\mathfrak{d}_\psi(u_n, u_{n+m}) = 0, asn, m \longrightarrow \infty. \tag{30}$$

Then, $\{u_n\}$ is a Cauchy sequence, and since $(X, \mathfrak{d})$ is a complete extended quasimetric space, there exists $\omega \in X$ such that

$$\mathfrak{d}_\psi(u_n, \omega) = 0, as\, n \longrightarrow \infty. \tag{31}$$

By condition (ii), we obtain

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \mathfrak{d}_\psi(Tu_n, T\omega) = \lim_{n\longrightarrow\infty} \mathfrak{d}_\psi(u_{n+1}, T\omega) = \mathfrak{d}_\psi(\omega, T\omega) = 0, \\ &\lim_{n\longrightarrow\infty} \mathfrak{d}_\psi(Su_n, S\omega) = \lim_{n\longrightarrow\infty} \mathfrak{d}_\psi(u_n, T\omega) = \mathfrak{d}_\psi(\omega, S\omega) = 0. \end{aligned} \tag{32}$$

Hence, we deduce that $T\omega = S\omega = \omega$. Furthermore, let $\omega_* \in X$ such that $T\omega_* = S\omega_* = \omega_*$ where $\omega \neq \omega_*$. So, by (12), we obtain

$$\mathfrak{d}_\psi(S\omega, S\omega_*) = \mathfrak{d}_\psi(T\omega, T\omega_*) \leq \vartheta(f(\omega, \omega_*)), \tag{33}$$

where

$$\begin{aligned} &f(\omega, \omega_*) \\ &= \sup\left\{\begin{array}{c} \mathfrak{d}_\psi(S\omega, S\omega_*), \mathfrak{d}_\psi(T\omega, S\omega), \mathfrak{d}_\psi(S\omega_*, T\omega_*), \\ \dfrac{\mathfrak{d}_\psi(S\omega, T\omega)\mathfrak{d}_\psi(S\omega_*, T\omega_*)}{\mathfrak{d}_\psi(S\omega, S\omega_*)}, \dfrac{\mathfrak{d}_\psi(S\omega, T\omega_*) + \mathfrak{d}_\psi(S\omega_*, T\omega)}{2\sup[\psi(S\omega_*, T\omega), \psi(S\omega, T\omega_*)]} \end{array}\right\} \\ &= \sup\left\{\mathfrak{d}_\psi(\omega, \omega_*), \frac{\mathfrak{d}_\psi(\omega, \omega_*)}{\sup[\psi(\omega_*, \omega), \psi(\omega, \omega_*)]}\right\} = \mathfrak{d}_\psi(\omega, \omega_*). \end{aligned} \tag{34}$$

Then,

$$\mathfrak{d}_\psi(S\omega, S\omega_*) = \mathfrak{d}_\psi(T\omega, T\omega_*) = \mathfrak{d}_\psi(\omega, \omega_*) \leq \vartheta(\mathfrak{d}_\psi(\omega, \omega_*)). \tag{35}$$

Therefore, $\mathfrak{d}_\psi(\omega, \omega_*) = 0$; thus, $\omega = \omega_*$. Hence, T and S possess a unique coincidence fixed point in X. □

To mitigation the continuity case on the given self-mappings, we will modify Definition 16 as follows:

Definition 18. Let $(X, \mathfrak{d})$ be an extended quasimetric space, and we say that S and T are $(\eta, \gamma)_g$ -contraction such that $\eta : X \times X \longrightarrow [0,\infty)$ and $\psi : X \times X \longrightarrow [1,\infty)$ if for all u, $v \in X$ fulfilled

$$\eta(u, v)\mathfrak{d}_\psi(Tu, Tv) \leq \vartheta(g(u, v)), \psi \in \Psi, \tag{36}$$

and

$$g(u, v) = \sup\left\{\frac{\mathfrak{d}_\psi(Su, Sv)}{2}, \frac{\mathfrak{d}_\psi(Su, Tu) + \mathfrak{d}_\psi(Su, Tv)}{2}, \frac{\mathfrak{d}_\psi(Su, Tv) + \mathfrak{d}_\psi(Su, Tu)}{2\sup[\psi(u, Tu), \psi(u, Su)]}\right\}. \tag{37}$$

By remove continuity of the given mappings, we get the following major result.

Theorem 19. *Let $(X, \mathfrak{d})$ be a complete extended quasimetric space, and let $S, T : X \longrightarrow X$ be $(\eta, \gamma)_g$-contraction mappings. Let (13) and conditions (i) and (ii) of Theorem 17 be satisfied. Assume also that*

(iii) If $\{u_n\}$ is a sequence in X such that $\eta(u_n, u_{n+1}) \geq 1, \forall n$ and $u_n \longrightarrow u \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{u_{nk}\} \subset \{u_n\}$ such that $\eta(u_n, u_{nk}) \geq 1, \forall k$

Then, S and T possess a coincidence fixed point u_0, that is, $Su = Tu = u_0$.

Proof. By inequality (11) and condition (iii) in Theorem 19, there exists $\{u_{nk}\} \subset \{u_n\}$ such that $\eta(u_{nk}, u) \geq 1, \forall k$. Applying inequality (11), we obtain that

$$\mathfrak{d}_\psi(u_{nk+1}, Tu) = \mathfrak{d}_\psi(Tu_{nk}, Tu) \leq \eta(u_{nk}, u)\mathfrak{d}_\psi(Tu_{nk}, Tu) \leq \vartheta(g(u_{nk}, u)). \tag{38}$$

And

$$\mathfrak{d}_\psi(u_{nk}, Su) = \mathfrak{d}_\psi(Su_{nk}, Su) \leq \eta(u_{nk}, u)\mathfrak{d}_\psi(Su_{nk}, Su) \leq \vartheta(g(u_{nk}, u)). \tag{39}$$

Also, we have

$$g(u_{nk},u)=\sup\left\{\frac{\mathfrak{d}_\psi(Su_{nk},Su)}{2},\frac{\mathfrak{d}_\psi(Su_{nk},Tu_{nk})+\mathfrak{d}_\psi(Su_{nk},Tu)}{2},\frac{\mathfrak{d}_\psi(Su_{nk},Tu_{nk})+\mathfrak{d}_\psi(Su_{nk},Tu)}{\sup\left[\psi(u_{nk},Tu_{nk}),\psi(u_{nk},Su_{nk})\right]}\right\}$$
$$=\sup\left\{\frac{\mathfrak{d}_\psi(u,Su)}{2},\frac{\mathfrak{d}_\psi(u,Tu)}{2},\frac{\mathfrak{d}_\psi(u,Tu)}{\sup\left[\psi(Su,u),\psi(u,Tu)\right]}\right\}. \tag{40}$$

Then,

$$g(u_{nk},u)=\sup\left\{\frac{\mathfrak{d}_\psi(u,Su)}{2},\frac{\mathfrak{d}_\psi(u,Tu)}{2}\right\}. \tag{41}$$

When

$$g(u_{nk},u)=\frac{\mathfrak{d}_\psi(u,Su)}{2}, \tag{42}$$

then neither $\mathfrak{d}_\psi(u,Su)>0$ nor $\mathfrak{d}_\psi(u,Su)=0$ for some $u\in X$ when $k\longrightarrow\infty$. If $\mathfrak{d}_\psi(u,Tu)>0$, then by inequality (39), we obtain $g(u_{nk},u)>0$. Thus, from Definition 18, we get

$$\psi(g(u_{nk},u))<g(u_{nk},u). \tag{43}$$

Also, by inequalities (38) and (42), we have

$$\mathfrak{d}_\psi(Su_{nk},Su)\le\eta(u_{nk},u)\mathfrak{d}_\psi(Su_{nk},Su)\le\vartheta(g(u_{nk},u))<g(u_{nk},u)=\frac{\mathfrak{d}_\psi(u,Su)}{2}. \tag{44}$$

Thus,

$$2\mathfrak{d}_\psi(u,Su)\le\mathfrak{d}_\psi(u,Su), ask\longrightarrow\infty, \tag{45}$$

Which is an ambivalence. Therefore, $\mathfrak{d}_\psi(u,Su)=0$ and $u=Su$. Likewise, we can get that $u=Tu$. Hence, u is a common fixed point for S and T in X, that is, $u=Su=Tu$.

Let $u_1,u_2\in X$ be two common fixed points of S and T such that $u_1\ne u_2$ then by (38) and (39), we get

$$\begin{aligned}\mathfrak{d}_\psi(u_1,u_2)&=\mathfrak{d}_\psi(Tu_1,Tu_2)\\&\le\eta(u_1,u_2)\mathfrak{d}_\psi(Tu_1,Tu_2)\\&\le\vartheta(g(u_1,u_2)),\end{aligned} \tag{46}$$

where

$$g(u_1,u_2)=\sup\left\{\frac{\mathfrak{d}_\psi(Su_1,Su_2)}{2},\frac{\mathfrak{d}_\psi(Su_1,Tu_1)+\mathfrak{d}_\psi(Su_1,Tu_2)}{2},\frac{\mathfrak{d}_\psi(Su_1,Tu_1)+\mathfrak{d}_\psi(Su_1,Tu_2)}{\sup\left[\psi(u_1,Tu_1),\psi(u_1,Su_1)\right]}\right\}$$
$$=\sup\left\{\frac{\mathfrak{d}_\psi(u_1,u_2)}{2},\frac{\mathfrak{d}_\psi(u_1,u_2)}{\sup\left[\psi(u_1,u_1)\right]}\right\}=\frac{\mathfrak{d}_\psi(u_1,u_2)}{2}. \tag{47}$$

Now, if $\mathfrak{d}_\psi(u_1,u_2)>0$, then $\vartheta(g(u_1,u_2))<(g(u_1,u_2))$, which implies by (46) that

$$2d_\psi(u_1,u_2)<\mathfrak{d}_\psi(u_1,u_2), \tag{48}$$

but this is a contradiction. Hence, $\mathfrak{d}_\psi(u_1,u_2)=0$, i.e., $u_1=u_2$. This proves the uniqueness of the common fixed point of given mappings. □

Example 20. Suppose that $\mathscr{G}_1\cup\mathscr{G}_2\cup\mathscr{G}_3=X$, where $\mathscr{G}_1=(-\infty,0]$, $\mathscr{G}_2=(1/2,1/3,1/4,1/5)$ and $\mathscr{G}_3=[1,2]$. Consider the extended $\mathfrak{b}$-metric space on X as follows:

$$\begin{gathered}\mathfrak{d}_\psi\left(\frac{1}{2},\frac{1}{3}\right)=\mathfrak{d}_\psi\left(\frac{1}{4},\frac{1}{5}\right)=\frac{3}{10},\\\mathfrak{d}_\psi\left(\frac{1}{2},\frac{1}{5}\right)=\mathfrak{d}_\psi\left(\frac{1}{3},\frac{1}{4}\right)=\frac{2}{10},\\\mathfrak{d}_\psi\left(\frac{1}{2},\frac{1}{4}\right)=\mathfrak{d}_\psi\left(\frac{1}{5},\frac{1}{3}\right)=\frac{6}{10},\\\mathfrak{d}_\psi\left(\frac{1}{2},\frac{1}{2}\right)=\mathfrak{d}_\psi\left(\frac{1}{3},\frac{1}{3}\right)=\cdots=0,\\\mathfrak{d}_\psi(u,v)=|u-v|\in\mathscr{G}_3, else.\end{gathered} \tag{49}$$

It is apparent that the triangle inequality on $\mathscr{G}_1$ is not fulfilled. Actually,

$$\frac{6}{10} = \mathfrak{d}_\psi\left(\frac{1}{2}, \frac{1}{4}\right) \geq \mathfrak{d}_\psi\left(\frac{1}{2}, \frac{1}{3}\right) + \mathfrak{d}_\psi\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{5}{10}. \tag{50}$$

Observes that the condition (iii) in Definition 5 is satisfied. Suppose that $f, g : X \longrightarrow X$ are defined as

$$Tu = \begin{cases} 0, \text{if} u \in \mathcal{G}_1, \\ \frac{1}{4}, \text{if} u \in \mathcal{G}_2, \\ \frac{1}{2}, \text{if} u \in \mathcal{G}_3, \end{cases} \tag{51}$$
$$Su = u, \forall u \in X.$$

Taking

$$\eta(u, v) = \begin{cases} 1, & \text{if } u, v \in \mathcal{G}_2 \cup \mathcal{G}_3 \\ 0, & \text{if else,} \end{cases} \tag{52}$$

T and S are an (η, γ)-contractive mappings with $\gamma(t) = t/2, t \in [0,\infty)$. Furthermore, there exists $u_0 \in X$ such that $\eta(Tu_0, Su_0) \geq 1$.

Actually, we have for $u = 1/2 \in X$.

$$\eta\left(T\left(\frac{1}{2}\right), S\left(\frac{1}{2}\right)\right) = \eta\left(\frac{1}{4}, \frac{1}{2}\right) = 1. \tag{53}$$

Now, suppose that $u, v \in X$ with $\eta(u, v) \geq 1$. It implies that $u, v \in \mathcal{G}_2 \cup \mathcal{G}_3$.

By Definition 1, we have $\eta(Tu, Sv) \geq 1$, and then T and S are η-admissible mappings. Also, T and S are clearly not continuous mappings.

Otherwise, if $u_n \subset X$ such that, $\eta(u_n, u_{n+1}) \geq 1$, then $u_n \in \mathcal{G}_2 \cup \mathcal{G}_3$ for all n. Assume the sequence $\{u_n\}$ is considered iteratively as $fu_{n-1} = u_n$ for all n. Considering the arbitrary point $u_0 \in X$, it is located in either $\mathcal{G}_2$ or $\mathcal{G}_3$; so, we have two cases:

In case $u_0 \in \mathcal{G}_2$, thus $\{u_n\}$ is constant sequence and $u_n \longrightarrow 1/4 \in \mathcal{G}_2$. Then, for all n, we have $\eta(u_n, u_{n+1}) \geq 1$, which implies that $\eta(u_n, 1/4) = \eta(u_{n+1}, 1/4) \geq 1$.

In case $u_0 \in \mathcal{G}_3$, thus $\{u_n\}$ is constant sequence and $u_n \longrightarrow 1/2 \in \mathcal{G}_2$. Then, for all n, we have $\eta(u_n, u_{n+1}) \geq 1$, which implies that $\eta(u_n, 1/2) = \eta(u_{n+1}, 1/2) \geq 1$.

Consequently, T and S fulfill the conditions of Theorem 19; hence, T and S have a unique common fixed point on X, which is $u = 1/4$.

3. Conclusion

By replacing $f(u, v)$ or $g(u, v)$ with a proper one, we can conclude several results from the showed prime result in this paper on different sides. For example, we can obtain results in this frame of periodic contractions and partially ordered spaces.

Data Availability

No data was used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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