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Research Article

Relativistic Wave Propagation in Anisotropic Two-Component Magnetohydrodynamics Plasmas

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This paper investigates the instabilities and characteristics of relativistic linear waves of two-component plasma by assuming the plasma to be in-viscid, homogeneous, collision-less, and magnetized. To do this, by taking moments of the relativistic Vlasov equation, the basic equations of the two-component relativistic an-isotropic plasma are derived. The linearized equations are analyzed for small perturbation under the assumption of the plasma which is initially at rest. After we derived the dispersion relations, different wave modes and instabilities are discussed analytically and numerically presented as well.

1. Introduction

Recently, the wave propagation of the relativistic plasma has been gained great attention in a variety of astrophysical problems. There are several astrophysical [1] and magnetospheric situations where different plasma components which are coupled by the large-scale ambient magnetic field coexist. The plasma component can be described as a concatenation of magneto-hydrodynamic (MHD) fluid components [2]. As presented in [3], the study of MHD theory is devoted to describing a compressional pulsation with a large azimuthal number. In this situation, the plasma system may be appropriately described as the concatenation of two MHD fluid components [4]. This concatenated modeling of a composite plasma system has been extended to study the effect of pressure anisotropy in the nonrelativistic [5, 6] as well as relativistic [2, 7] domains. For instance, Rogava et al. [8] used nonmodal analysis to investigate linear waves in a cold electron-positron plasma and found that the velocity shear induces mode coupling and transient growth of the compressional and shear $Alfve'n$ perturbations. In [9], the propagation of linear and nonlinear electrostatic waves is investigated in a magnetized anisotropic electron-positron-ion (e-p-i) plasma with superthermal electrons and positrons being studied. Chagelishvili et al. [10] moreover utilized nonmodal investigation to think about the impact of plane Couette stream on the solidness of direct waves in a two-temperature plasma. Also, Goodman [11] considered the impact of classical transport coefficients, and variety of stream speed in a round and hollow symmetric electrically driven steady-state MHD show for input parameters which may mimic combination plasma in a tokamak. Vladimirov and Ilin [12] utilized the vitality rule to explore the soundness of a perfect, in-compressible consistent MHD stream and found adequate conditions for the stream to be steady. Furthermore, Goldberger et al. (hereafter called CGL) [13] also derived a set of equations in which the ambient magnetic field simulates the role of collisions to diagonalize the thermal pressure tenser into having two components, one in the direction of a magnetic field and the other perpendicular to it. Notwithstanding the extent of thermal anisotropy which may exist in a particular situation, recently, many researchers have studied pressure anisotropy in different plasma wave situations using the CGL approximation [9, 14, 15] and investigated different properties of linear and nonlinear plasma waves.

On the other hand, the plasma waves not only studied in the classical approach; many researchers also have studied waves in quantum plasmas using the quantum hydrodynamic (QHD) model [16, 17] which has received much interest due to their applications in a variety of physical systems. Abdikian and Mahmood [18] investigated the electrostatic solitary waves in a relativistically quantum magnetized plasma in the presence of stationary ions and the external magnetic field. In addition to these waves, the nonlinear propagation of small amplitude magnetosonic solitons and their chaotic motions in quantum plasma with degenerate inertialess spin-up electrons, spin-down electrons, and classical inertial ions [19] studied.

The framework of relativistic MHD wave equations has been done by Lichnerowicz [20], and the model is extended to include the pressure anisotropy by Cissoko [21] who proposed an energy closure similar to the double-adiabatic or CGL laws, $d(p_\perp/\rho B)/dt$ and $d(p_\parallel B^2/\rho^3)/dt$ (where $p_\parallel$ and $p_\perp$ are the parallel and perpendicular components of the pressure relative the ambient magnetic field, respectively [9]), and ρ is the plasma fluid density, for the case of nonrelativistic plasma in a strong magnetic field [13]. As presented in [22], Chou and Hau have studied relativistic MHD plasma in the presence of pressure anisotropic slow waves in a single component. The relativistic anisotropic MHD model was then applied to obtain the solutions for pulsar wind with the special limit of $p_\parallel \neq 0$ and $p_\perp = 0$. In those studies, the relativistic effect is due to the bulk flow but not the thermal velocity of particles. The well-known linear MHD wave dispersion relations have been examined by Gedalin [23]. Tsikarishvili and Rogava [24] have done the relativistic and ultrarelativistic gyrotropic plasma, respectively. Kalra and Gebretsadkan [25] have further examined the characteristics of linear MHD waves and instabilities in several physical situations, and they discussed the implications of the dispersion relations for any physical situation where the magnetic field is strong. Moreover, Kumar and Kalra [2] also investigated the propagation of waves and instabilities in a plasma model which consists of the concatenation of two magnetohydrodynamic (MHD) fluids, one of which is relativistic and has anisotropic pressure components given by double adiabatic equations derived by Gedalin [23], while the other one has pressure components given by generalized polytropic laws.

Dougherty [26] has studied wave propagation in hot plasma by treating the dynamics of the medium by kinetic theory [27, 28] and combines the theory with Maxwell's equations [29]. For the case of nonrelativistic two-component magnetohydrodynamics plasma, Kalra and Ghildyal [4] investigated the low-frequency plane waves supported by a medium containing a thermal plasma of isotropic pressure and a suprathermal collisionless plasma having anisotropic pressure. They also analyzed the low-frequency waves in a plasma model that is made up of two thermally anisotropic MHD components using wave-front diagrams [5]. The question of stability has been completely ignored so far in the plasma models based on two MHD components. Kumar and Singh [30] investigated the role of relative motion between the fluid components of a plasma model, which is simulated by concatenation of two anisotropic MHD fluids, on the propagation of low-frequency waves and instabilities. A significant outcome of such studies based on a two-component MHD is that it allows an additional magnetosonic mode, which is the suprathermal mode, besides the usual slow, fast, and $Alfve'n$ MHD modes. These modes basically originate due to the interaction between the two fluids of the plasma system and are the fastest mode of propagation. These studies also admitted two-population plasma systems were undertaken by considering to have isotropic pressure. However, two-population plasma is embedded in a strong magnetic field, the magnetic field suppresses the equilibration of pressures parallel and perpendicular to itself.

The studies were undertaken so far assuming that either one component of plasma or both the components of plasma have isotropic pressure or nonrelativistic properties. However, there are wide claims of astrophysical systems where an-isotropic relativistic effects are important. But, to the best of our knowledge, not much attention seems to be given to both relativistic and anisotropic two-component MHD waves. This motivated us to do the present research which addresses this issue.

In this work, we devote the two-component plasma for the case of relativistic and anisotropic pressure to study the characteristics of relativistic linear waves and instabilities of two-component plasma in the inviscid, homogeneous, collisionless, and magnetized fluid. The well-known wave modes such as slow, fast, and $Alfve'n$ modes of a single MHD fluid is persist and is compared with our numerical results as well. In addition, a new fourth mode, the fastest of all modes, referred to as the suprathermal mode appears due to the coupling between the two components of the plasma which displays a rich variety of behavior depending upon the numerical values of the input parameters. From the results, we observe that the phase speeds of the wave modes of the relativistic two-component plasmas are found to be slower than their nonrelativistic counterpart due to the relativistic corrections and the pressure anisotropies.

The layout of the paper is as follows. In the following section, we review the derivation of basic equations of relativistic anisotropic magnetohydrodynamics; hereafter, we call it RAM, the two-component relativistic anisotropic plasmas are shown in the framework of the Vlasov equation. The linearization and dispersion relations of RAM equations are discussed in Section 3. In Section 4, the numerical results of different polar plots of the phase speeds of the wave modes are presented and analyzed. This section also covers the discussion and conclusions of the work.

2. Derivation of Basic Equations of RAM

In this section, by employing the velocity moments, the Vlasov equation (where c is the speed of light, f is the distribution function, $F^{\alpha\beta}$ is the energy-stress tensor, and u_β is four vector velocity),

$$c u^\alpha \frac{\partial f}{\partial x^\alpha} + \frac{e}{m} F^{\alpha\beta} u_\beta \frac{\partial f}{\partial u^\alpha} = 0. \tag{1}$$

The differential operator $\partial/\partial x^{\alpha}$ transforms like a four-vector and is denoted by ∂_{α}. The Latin letters $k, l, m, \cdots$ represent space indices 1, 2, 3, and the Greek letters $\alpha, \beta, \mu, \cdots$ represent space time and take the values 1 to 4. The fundamental relativistic MHD equations are constructed under consideration of the conservation laws of energy and momentum of the system. Taking the different moments of the Vlasov equation, we consider an unbounded and nondissipative plasma that consists of two components denoted by subscripts 1 and 2, each described by relativistic anisotropic magneto hydrodynamics; hereafter, we call it RAM.

The plasma is permeated by a uniform magnetic field, which, on account of the idealized Ohm's law

$$E + \left(\frac{1}{c}\right) v \times B = 0, \tag{2}$$

where E is the electric field and B is the magnetic field. These two constraints are anisotropic components to move with the same velocity perpendicular to the direction of the magnetic field. It does not restrict the component of velocity along the direction of the magnetic field. The equations assumed for the two-fluid plasma system consist of two sets of equations, the full Maxwell's equations. Each fluid is characterized by its continuity, momentum, and energy equations. We also assume that there are no collisions between particles so there is no viscosity, thermal conduction, or resistivity.

We use a two-model system that interstellar environment has been simulated to be a concatenation of two anisotropic MHD fluids [5, 30] consisting of a background plasma (fluid1) and cosmic ray fluid (fluid2); further assumption for the simplification of calculations that these two fluids have a values s where $s = 1$ and $s = 2$ represents *fluid1* and *fluid2*, respectively. So throughout this paper, the quantities with suffix s take the values 1 and 2 for *fluid1* and *fluid2*. Using these assumptions, we rewrite the continuity, Maxwell, and energy-momentum equations [7, 25] of RAM for the two fluids which can be written as:

$$\frac{\partial J_s^{\alpha}}{\partial x^{\alpha}} = 0, \text{the continuity equation}, \tag{3}$$

$$\frac{\partial^* F_s^{\alpha\beta}}{\partial x^{\alpha}} = 0, \text{the relevant Maxwell}'\text{sequations}, \tag{4}$$

$$\frac{\partial T_s^{\alpha\beta}}{\partial x^{\alpha}} = 0, \text{the energy} - \text{momentum equation}. \tag{5}$$

respectively, where $J_s^{\alpha} = \rho_s u_s^{\alpha}$ and $^*F_s^{\alpha}\beta = b^{\alpha} u_s^{\beta} - b^{\beta} u_s^{\alpha}$. The proper mass density and the specific internal energy density for the two fluids are defined, respectively, as

$$e_s = \rho_s\left(c^2 + \varepsilon_s\right), \text{and}\, \varepsilon_s = \frac{p_{\perp s}}{\rho_s} + \frac{p_{\parallel s}}{2\rho_s}. \tag{6}$$

Equation of state

$$\frac{d}{dt}\left(\frac{p_{\parallel s} B^2}{\rho_s^3}\right) = 0, \tag{7}$$

$$\frac{d}{dt}\left(\frac{p_{\perp s}}{\rho_s B}\right) = 0. \tag{8}$$

The induction equation under the presumption of idealize conductivity Ohm's law together with Faraday's law gives:

$$\frac{\partial B}{\partial t} = \nabla \times (v_1 \times B) = \nabla \times (v_2 \times B), \tag{9}$$

where v_1 and v_2 denote velocities of $fluid_1$ and $fluid_2$, respectively. By using Equations (3)–(8), we derive the linearization and dispersion relation in the following section to analyze the characteristics of relativistic linear waves and instabilities of two-component plasma.

3. Linearization and Dispersion Relations of RAM Equations

In this section, we assume that the physical quantities have been perturbed with time, and the small amounts of the perturbed quantities have a sinusoidal behavior. To be more clear, we use the notation "0" and "1" to identify the unperturbed and perturbed amount, respectively.

Then, the linearized form of the continuity equation read as

$$\frac{\partial}{\partial t}\left(\frac{\rho_1}{\rho_1^0}\right) + \nabla \cdot v_1 = 0, \tag{10}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho_2}{\rho_2^0}\right) + \nabla \cdot v_2 = 0, \tag{11}$$

where $\rho_{1,2}^0$ for the equilibrium and $\rho_{1,2}$ denotes corresponding perturbed densities. The linearized equation of state for the perpendicular and parallel component of pressures are obtained, respectively, as

$$\begin{aligned} p_{\perp s}^1 &= p_{\perp s}^0 \left[\frac{\rho_s^1}{\rho_s^0} + \frac{B_{1z}}{B_0}\right], \\ \frac{p_{\parallel s}^1}{p_{\parallel s}^0} &= 3\frac{\rho_s^1}{\rho_s^0} - 2\frac{B_{1z}}{B_0}. \end{aligned} \tag{12}$$

Similarly, the linearized induction equation become as

$$\frac{\partial B_1}{\partial t} = (B_0 \cdot \nabla) v_s^1 - B_0\left(\nabla \cdot v_s^1\right). \tag{13}$$

where v_s^1 denotes the perturbed fluid velocities for the values of $s = 1$ for fluid one and $s = 2$ for fluid two. It is evident from these equations that one can obtain the two fluid velocity components along the perpendicular direction of magnetic field which are equal.

$$v_{1x} = v_{2x} = v_x \text{ and } v_{1y} = v_{2y} = v_y. \tag{14}$$

However, along the parallel direction of the magnetic field, the two fluids can have different velocities. Because of the assumption that the two fluids are under consideration of constrained to move together by the Lorentz force across the magnetic field, the equation of conservation of momentum in the directions perpendicular to the magnetic field in the linearized form may be written as

$$\begin{aligned}\rho_1^0 R_1^* \frac{\partial}{\partial t}\left(v_{1y}\widehat{e}_x - v_{1x}\widehat{e}_y\right) + \rho_2^0 R_2^* \frac{\partial}{\partial t}\left(v_{2y}\widehat{e}_x - v_{2x}\widehat{e}_y\right) \\ + \left(\widehat{e}_x \frac{\partial}{\partial y} - \widehat{e}_y \frac{\partial}{\partial x}\right)\left(p_{\perp s}^1 + \frac{B_0 B_{1z}}{4\pi}\right) - B_0 \frac{\partial}{\partial z} \\ \times \left(\left(\frac{p_{\perp s}^0 - p_{\parallel s}^0}{B_0^2} + \frac{1}{4\pi}\right)\left(B_{1y}\widehat{e}_x - B_{1x}\widehat{e}_y\right)\right) = 0,\end{aligned} \tag{15}$$

where

$$R_1^* = 1 + \frac{1}{\rho_1^0 c^2}\left(2p_{\perp 1}^0 + \frac{1}{2}p_{\parallel 1}^0 + \frac{B_0^2}{4\pi}\right), \tag{16}$$

and

$$R_2^* = 1 + \frac{1}{\rho_2^0 c^2}\left(2p_{\perp 2}^0 + \frac{1}{2}p_{\parallel 2}^0 + \frac{B_0^2}{4\pi}\right). \tag{17}$$

We can rewrite Equation (15) separately for $\widehat{e}_x$ and $\widehat{e}_y$ components, respectively, as

$$\begin{aligned}\rho_1^0 R_1^* \frac{\partial}{\partial t} v_{1y} + \rho_2^0 R_2^* \frac{\partial}{\partial t} v_{2y} + \frac{\partial}{\partial y} p_{\perp s}^1 - \frac{\partial}{\partial z}\left(\frac{p_{\perp s}^0 - p_{\parallel s}^0}{B_0}\right) B_{1y} + \frac{B_0}{4\pi}\left(\frac{\partial B_{1z}}{\partial y} - \frac{\partial B_{1y}}{\partial z}\right) = 0, \\ \rho_1^0 R_1^* \frac{\partial}{\partial t} v_{1x} + \rho_2^0 R_2^* \frac{\partial}{\partial t} v_{2x} + \frac{\partial}{\partial x} p_{\perp s}^1 - \frac{\partial}{\partial z}\left(\frac{p_{\perp s}^0 - p_{\parallel s}^0}{B_0}\right) B_{1x} + \frac{B_0}{4\pi}\left(\frac{\partial B_{1z}}{\partial x} - \frac{\partial B_{1x}}{\partial z}\right) = 0.\end{aligned} \tag{18}$$

Here, the pressure of the plasma components is given by

$$p_s = \left(p_{\parallel s}\right) - p_{\perp s}\right) nn + p_{\perp s} I, \tag{19}$$

where $p_{\parallel s}, p_{\perp s}$ stands for parallel and perpendicular components of pressures to the direction of a magnetic field, respectively, I represents the unit second order tensor, and n refers a unit vector in the parallel direction of a magnetic field. The linearized equation for the conservation of momentum along the magnetic field may also be written as

$$\frac{1}{c^2}\left(e_{s0} + p_{\parallel s}^0\right)\frac{\partial}{\partial t} v_{sz} - \left(p_{\perp s}^0 - p_{\parallel s}^0\right)\frac{\partial}{\partial z}\left(\frac{B_{1z}}{B_0}\right) + \frac{\partial p_{\parallel s}^1}{\partial z} = 0. \tag{20}$$

Here, we can define a new term for $1/c^2(e_{s0} + p_{\parallel s}^0)$ as

$$\begin{aligned}\frac{1}{c^2}\left(e_{s0} + p_{\parallel s}^0\right) &= \frac{1}{c^2}\left(\rho_s^0\left(c^2 + \varepsilon_s^0\right) + p_{\parallel s}^0\right) \\ &= \rho_s^0\left[1 + \frac{1}{\rho_s^0 c^2}\left(p_{\perp s}^0 + 3p_{\parallel s}^0/2\right)\right] = \rho_s^0 Q_s^*,\end{aligned} \tag{21}$$

where $Q_s^* = 1 + 1/\rho_s^0 c^2 (p_{\perp s}^0 + 3p_{\parallel s}^0/2)$.

Using Equation (20), it can be written for *fluid1* and *fluid2*

$$\rho_1^0 Q_1^* \frac{\partial}{\partial t} v_{1z} - \left(p_{\perp 1}^0 - p_{\parallel 1}^0\right)\frac{\partial}{\partial z}\left(\frac{B_{1z}}{B_0}\right) + \frac{\partial p_{\parallel 1}^1}{\partial z} = 0, \tag{22}$$

and

$$\rho_2^0 Q_2^* \frac{\partial}{\partial t} v_{2z} - \left(p_{\perp 2}^0 - p_{\parallel 2}^0\right)\frac{\partial}{\partial z}\left(\frac{B_{2z}}{B_0}\right) + \frac{\partial p_{\parallel 2}^1}{\partial z} = 0, \tag{23}$$

respectively, where $Q_1^* = 1 + 1/\rho_1^0 c^2 (p_{\perp 1}^0 + 3/2 p_{\parallel 1}^0)$, and $Q_2^* = 1 + 1/\rho_2^0 c^2 (p_{\perp 2}^0 + 3/2 p_{\parallel 2}^0)$. Here, we consider an unbounded homogeneous medium in which any arbitrary perturbation can be Fourier analyzed and written in terms of plane waves of frequency ω and wave vector k. Since the magnetic field has been taken along the z-axis, the wave vector can be taken in the xz plane with out any loss of generality. So, the perturbation quantities is proportional to

$$\sim \exp\left\{i\left(\boldsymbol{\omega} t - k_\perp x - k_\parallel z\right)\right\}, \tag{24}$$

where the wave vector $k(k_\perp, 0, k_\parallel)$ is taken to be real. For the differentiation with respect to time and space, we use the following expressions for

$$\frac{\partial}{\partial t} \longrightarrow i\omega, \nabla \longrightarrow -ik \text{ or } \nabla_{x,y,z} \longrightarrow -i\left(k_\parallel x, 0, k_\parallel z\right). \tag{25}$$

By account, this assumption and the space-time dependence given by Equation (24) is then applied to Equations (10)–(20). Carrying out the usual normal-mode analysis, we derive the desertion relations for $Alfve'n$ mode

$$U^2 = \left\{\frac{\sum_{s=1}^2\left(b_{\perp s}^2 - b_{\parallel s}^2 + 1\right)}{(1+d)R}\right\}\cos^2\theta. \tag{26}$$

In the current problem, the plasma is a two-component MHD fluid, then it needs a careful treatment for using parameters that have similar definition with those used by Gedalin [31] and later modified by Gebretadkan and Kalra [32] for a single-component RAM. Using these parameters, it is convenient to write the dispersion relation for the two-component RAM as follows:

$$\left[\omega^2-\left(k_\parallel^2 V_{RAs}^2+k_\perp^2 V_{Fs}^2\right)\right]\left[\left(\omega^2-k_\parallel^2 V_{S1}^2\right)\left(\omega^2-k_\parallel^2 V_{S2}^2\right)\right] - k_\parallel k_\perp^2\left\{V_{T1}^2\left[V_{T1}^2\left(1-\frac{V_{RA1}^2}{c^2}\right)\left(\omega^2-k_\parallel^2 V_{S2}^2\right)\right]\right\} + k_\parallel k_\perp^2\left\{V_{T2}^2\left[V_{T2}^2\left(1-\frac{V_{RA2}^2}{c^2}\right)\left(\omega^2-k_\parallel^2 V_{S1}^2\right)\right]\right\}=0, \tag{27}$$

and the dispersion relations for two-components relativistic anisotropic MHD plasma written as

$$\prod_{s=1}^{2}\left\{U^2-\frac{3\cos^2\theta b_{\parallel s}^2}{Q_s}\right\}_s \cdot\left[(1+d)RU^2-2\sum_{s=1}^{2} b_{\perp s}^2+\sum_{s=1}^{2}\left(b_{\parallel s}^2+b_{\perp s}^2\right)\cos^2\theta-1\right] - \sin^2\theta\cos^2\theta\times\left[U^2\left(\frac{b_{\perp 1}^4}{Q_1}+\frac{d b_{\perp 2}^4}{Q_2}\right) - \left(b_{\perp 1}^4 b_{\parallel 2}^2+d b_{\perp 2}^4 b_{\parallel 1}^2\right)\frac{3\cos^2\theta}{Q_1 Q_2}\right]=0. \tag{28}$$

In this dispersion relation, the following dimensionless parameters have been introduced

$$\begin{gathered} d=\frac{\rho_2^0}{\rho_1^0}, U=\frac{\omega}{k v_A}=\frac{v}{v_A}, k_\parallel=k\cos\theta, \\ k_\perp=k\sin\theta, k^2=k_\parallel^2+k_\perp^2, \\ b_{\perp s,\parallel s}^2=\frac{p_{\perp s,\parallel s}}{v_A^2\rho_1^0}v_A^2=\frac{B_0^2}{4\pi\rho_1^0}M_A^2=\frac{v_A^2}{(1+d)c^2}, \\ R=1+M_A^2\sum_{s=1}^{2}\left(2b_{\perp s}^2+\frac{1}{2}b_{\parallel s}^2+1\right), \\ Q_s=1+M_A^2(1+d)\left[b_{\perp s}^2+\frac{3}{2}b_{\parallel s}^2\right], \end{gathered} \tag{29}$$

where M_A is $Alfve'n$ speed of the two component plasma normalized to c, V_A is $Alfve'n$ speed of the single-component plasma, and v is phase speed.

4. Result and Discussion

We already factor-out $Alfve'n$ mode solution in Equation (26) and Equation (28), the 8[th]-order dispersion relation of the two population relativistic anisotropic plasma which is mainly addressed in this paper. The dispersion relation obtained in Equation (27) is more general, and we first compare with the well-known results by taking various limits. For instance, the nonrelativistic analog of RAM is found by using the results in Equation (28) by considering that $c \longrightarrow \infty$, then the values of $M_A=0$ implying that $R=Q=1$. So, the dispersion relation for the modified $Alfve'n$ wave Equation (26) reduced to the characteristic equation for the propagation of in anisotropic plasma as presented in

$$V^2=\left\{\frac{\sum_{s=1}^{2}\left(p_{\perp s}^0-p_{\parallel s}\right)+B_0^2/4\pi}{\left(\rho_1^0+\rho_2^0\right)}\right\}\cos^2\theta, \tag{30}$$

and the dispersion relation expressed by Equation (28) also reduced and exactly agrees with the dispersion relation obtained by Ghildyal and Kalra [5]. In the limit of the single-component relativistic anisotropic plasma system (since $d=0$ implies that $\rho_2^0=0$), the dispersion relations of Equation (26) is reduced and exactly fits a characteristic equation for the propagation of hydrodynamic waves in an anisotropic relativistic plasma which is done by Gedalin [23] with the modifications that is including c has been retained to facilitate reducing them to the nonrelativistic limits ($c \longrightarrow \infty$) by Gebretsadkan and Kalra [32], where $V_A=\left(B_0^2/4\pi\rho^0\right)^{1/2}$ denotes the classical $Alfve'n$ speed and $b_\perp^2=p_\perp 0/\rho^0 V_A^2$ and $b_\parallel^2=p_\parallel 0/\rho^0 V_A^2$ are the analogs of sound speeds along and across the magnetic field [23, 32].

4.1. Wave Propagation Parallel to Magnetic Field ($k_\perp=0, k_\parallel=k$). In the special case when the wave vector is parallel to the direction of the magnetic field $\theta=0$, there are no constraints to bind the motion of the two fluids. Each fluid moves independently. Except for the $Alfve'n$ mode where the combined density of the fluids governs its propagation, the other modes are expected to propagate as if one mode is unaffected by the presence of the other. Equations (26) and (27) show that all the four modes propagate; two of the modes have a double point which is a relativistic generalization of the nonrelativistic model [23]. The phase speed ($v=\omega/k$) of these modes is given by

$$V=\pm V_{\mathrm{ARS}}=\pm\sqrt{\frac{\sum_{s=1}^{2}\left(p_{\perp s}^0-p_{\parallel s}^0+B_0^2/4\pi\right)}{\rho_1^0+\rho_2^0+1/c^2\left[\sum_{s=1}^{2}\left(2p_{\perp s}^0 p_{\parallel s}^0/2+B_0^2/4\pi\right)\right]}}. \tag{31}$$

The other three modes propagate with phase speeds

$$\left(V^2-V_{ARS}^2\right)\left(V^2-V_{S1}\right)\left(V^2-V_{S2}\right)=0. \tag{32}$$

From this, we can understand that

$$V_1=\pm\sqrt{\frac{\sum_{s=1}^{2}\left(p_{\perp s}^0-p_{\parallel s}^0+B_0^2/4\pi\right)}{\rho_1^0+\rho_2^0+1/c^2\left[\sum_{s=1}^{2}\left(2p_{\perp s}^0+p_{\parallel s}^0/2+B_0^2/4\pi\right)\right]}}, \tag{33}$$

$$V_2=\pm\left\{\frac{3p_{\parallel 1}^0}{\rho_1^0+\rho_2^0+1/c^2\left[\sum_{s=1}^{2}\left(p_{\perp s}^0+3p_{\parallel s}^0/2\right)\right]}\right\}^{1/2}, \tag{34}$$

$$V_3=\pm\left\{\frac{3p_{\parallel 2}^0}{\rho_1^0+\rho_2^0+1/c^2\left[\sum_{s=1}^{2}\left(p_{\perp s}^0+3p_{\parallel s}^0/2\right)\right]}\right\}^{1/2}. \tag{35}$$

The double mode given by Equations (31) and (33) arises because the $Alfve'n$ mode and the fast mode have identical speeds along the direction of the magnetic field. The other two modes given by Equations (34) and (35) are the suprathermal modes which are the generalization of the corresponding sound speed modes in the nonrelativistic model [5]. They are essentially analog to the sound mode propagating in the relativistic MHD fluids and are not influenced by the magnetic field. The velocity (B) perturbations which are responsible for these modes (34) and (35) are parallel to the direction of propagation. Each fluid component triggers these modes independently. Since the propagation phase speed of the mode depends on the peculiar of MHD fluids, one can choose parameters so that in the parallel direction of a magnetic field, the suprathermal mode may not be the fastest mode. The phase speed of these modes are also reduced due to pressure anisotropies as well as relativistic effects.

The analysis of different wave modes show that only the $Alfve'n$ wave mode can lead to instability if

$$\sum_{s=1}^{2} p_{\parallel s}^{0} > \sum_{s=1}^{2} p_{\perp s}^{0} + \frac{B_0^2}{4\pi}. \tag{36}$$

This is the condition for fire-hose instability for the system of two an-isotropic nonrelativistic concatenated plasmas [5]. which is not affected by relativistic consideration. Since Equation (31) gives a double point, in the nonlinear regime, this mode will grow fastest [33]. However, the relativistic framework affects the growth rate of the phase speed when it is stable reduced.

4.2. Wave Propagation Perpendicular to the Magnetic Field ($k_\parallel = 0, k_\perp = k$). When the wave vector is perpendicular to the direction of the magnetic field, the motion of the two fluids is constrained and the medium behaves as a concatenation of the two fluids where each fluid has lost its independence. There is a triple degeneracy in phase speeds,

$$\left(V^2 - V_{FS}^2\right)\left(V^2\right)\left(V^2\right) = 0. \tag{37}$$

From this, we can understand that $V^4 = 0$ implies that no propagation and one obtains:

$$V_1 = V_2 = V_3 = 0, \tag{38}$$

and only one mode propagates with a phase speed given by

$$V = V_{FS}^2 = \pm\left\{\frac{\sum_{s=1}^{2}\left(2p_{\perp s}^{0} + B_0^2/4\pi\right)}{\rho_1^0 + \rho_2^0 + 1/c^2\left[\sum_{s=1}^{2}\left(2p_{\perp s}^{0} + p_{\parallel s}^{0}/2 + B_0^2/4\pi\right)\right]}\right\}^{1/2}. \tag{39}$$

This is a stable mode. As noted earlier, the phase speed of this model is also found to be reduced due to relativistic effects as compared with its nonrelativistic counterpart [33]. In this mode, the velocity perturbations are confined to the direction of the wave normal so that the wave propagation is analogous to the usual longitudinal sound wave. This is due to the fact that in this situation, the tension in the magnetic lines of force vanishes; the presence of the magnetic field only adds magnetic pressure to the plasma pressure.

4.3. Oblique Propagation. The parallel and perpendicular propagation of waves only did not contain all information about plasma wave modes. Mahmood et al. [34, 35] studied the arbitrary amplitude solitons propagating obliquely with respect to an external magnetic field in a homogeneous magnetized electron-positron-ion plasma. In many situations in space physics, the velocity distribution function is not Maxwellian but has an enhanced superthermal “tail” and other wave modes resulting from weakly nonlinear turbulent acceleration [36–38]. It is important to be able to model such plasmas and study the effects of the excess superthermal particles on wave behavior. In order to have an insight into the characteristics of these various modes of propagation, it is necessary to discuss the wave modes when the wave vector is inclined at an arbitrary angle with respect to the direction of an ambient magnetic field. As pointed out earlier, the present problem finds a possible application in the propagation of interstellar cosmic rays following a supernova explosion. To investigate such a situation, we use the numerical values for the relevant astrophysical parameters [1]. To do this, we multiply Equation (24) by Equation (28) to obtain an eighth order polynomial equation in U. Then, it may be written as follows:

$$\sum_{n=0}^{8} a_n U^n = 0. \tag{40}$$

From the eighth-order polynomial, we observe the existence of four incoming and four outgoing modes which are the outcome of the two-component plasma. However, this eighth-order polynomial equation is difficult to solve analytically; so, we shall try to analyze it numerically.

4.4. Numerical Analysis for the Wave Modes. In this section, the study of a relativistic anisotropic plasma is presented in the interest of both populations. We also consider the number of other situations in space for further investigations. In order to observe the effect of the relativistic anisotropic for two population plasma, we plot phase speeds of the four modes. We set here the numerical values of $M_A^2 = 0.9, b_{\perp 1}^2 = b_{\parallel 1}^2 = 0.5, b_{\perp 2}^2 = b_{\parallel 2}^2 = 0.0125$ and $d = 10$ as presented [2]. These values are used to solve the already factor out $Alfve'n$ mode solution, Equations (26), and the dispersion relation given by (28) numerically. For the case of nonrelativistic two-component MHD and single-component relativistic limit, we take the values $M_A^2 = 0$ and $d = 0$, respectively, in a polar diagrams using the parameters. First, we plot for the modified $Alfve'n$ mode with its nonrelativistic two population and relativistic single-component counterpart in Figure 1.

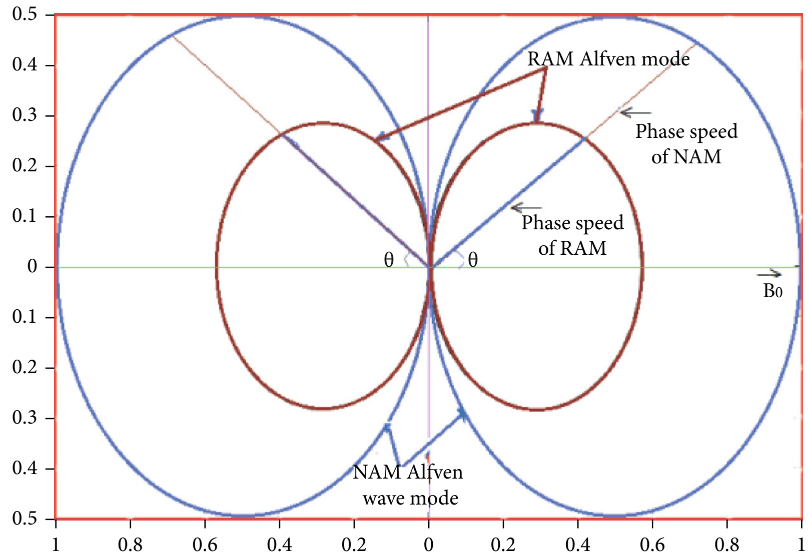

FIGURE 1: Polar plots of the phase speeds of $Alfve'n$ mode for RAM and nonrelativistic anisotropic magnetohydrodynamics (NAM) two component plasma using $M_A^2 = 0.9$, $b_{\perp 1}^2 = b_{\parallel 1}^2 = 0.5$, $b_{\perp 2}^2 = b_{\parallel 2}^2 = 0.0125$, and $d = 10$.

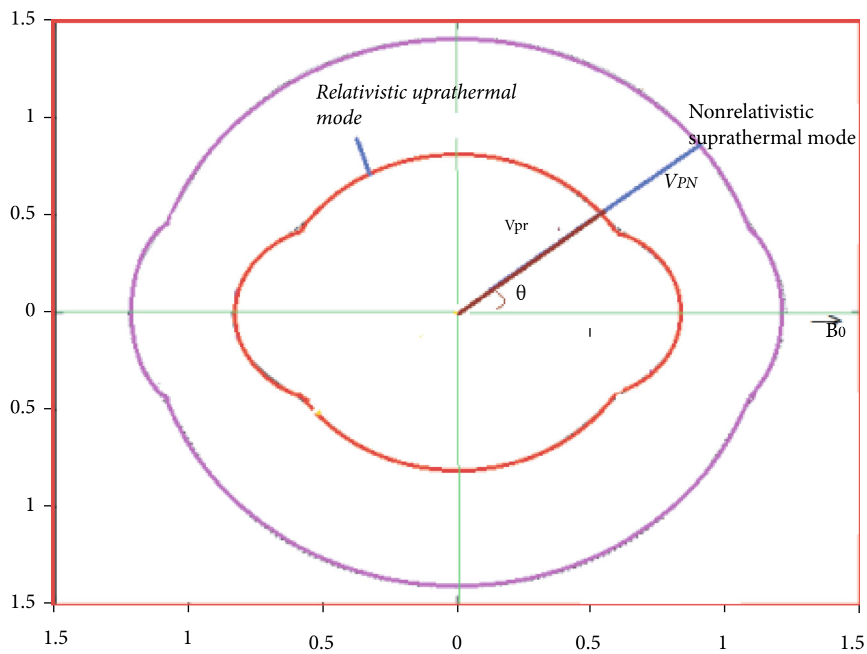

FIGURE 2: Polar plots of the phase speeds of suprathermal mode for RAM and NAM two-component plasma using $M_A^2 = 0.9$, $b_{\perp 1}^2 = b_{\parallel 1}^2 = 0.5$, $b_{\perp 2}^2 = b_{\parallel 2}^2 = 0.0125$, and $d = 10$.

In Figure 1, we represent the modified $Alfve'n$ wave which propagates along the magnetic field. We clearly observe that its speed is reduced due to the relativistic effect and have a difference in phase speeds of relativistic and non-relativistic two components MHD plasma. In all cases, for relativistic and nonrelativistic, there is no propagation orthogonal with the magnetic field.

The above figure, Figure 2, represents the suprathermal mode for relativistic and nonrelativistic two population anisotropic MHD plasma. These wave modes both incoming and outgoing propagate in all directions in the case of the relativistic and nonrelativistic two-component plasmas. The only difference is in the relativistic case, the phase speed is reduced. The suprathermal wave mode is the only mode that persists in a perpendicular direction to the magnetic field. It is the sum of the two waves and is the fast wave mode compared to the others.

Figure 3 shows the phase speeds of the RAM and NAM in two-component plasm fast wave modes in both incoming and outgoing modes. Each mode have the same phase speed in magnitude, and both the modes do not propagate along the perpendicular direction to the magnetic field. One moves

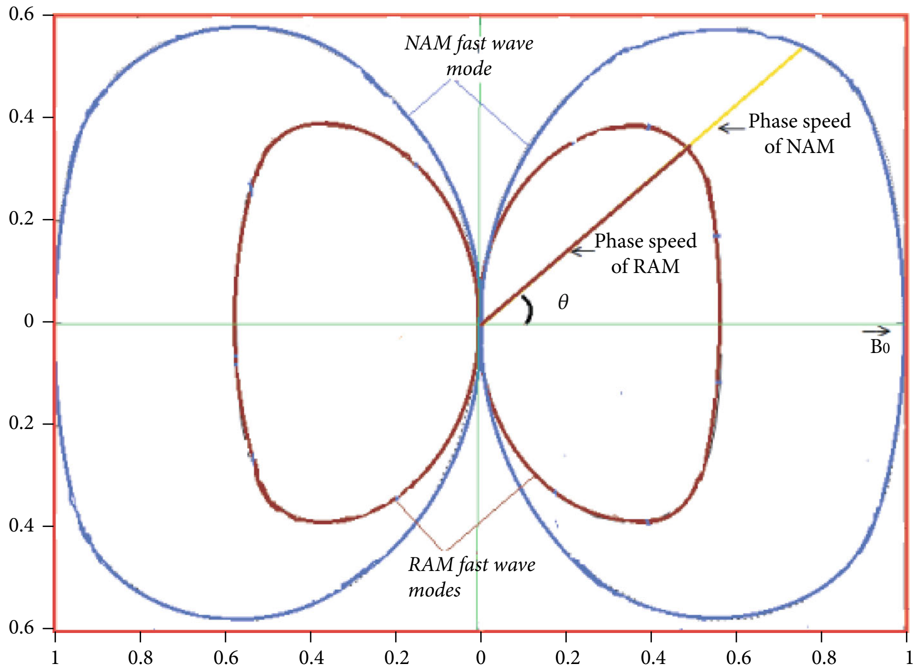

Figure 3: Polar plots of the phase speeds of fast mode for RAM and NAM two-component plasma.

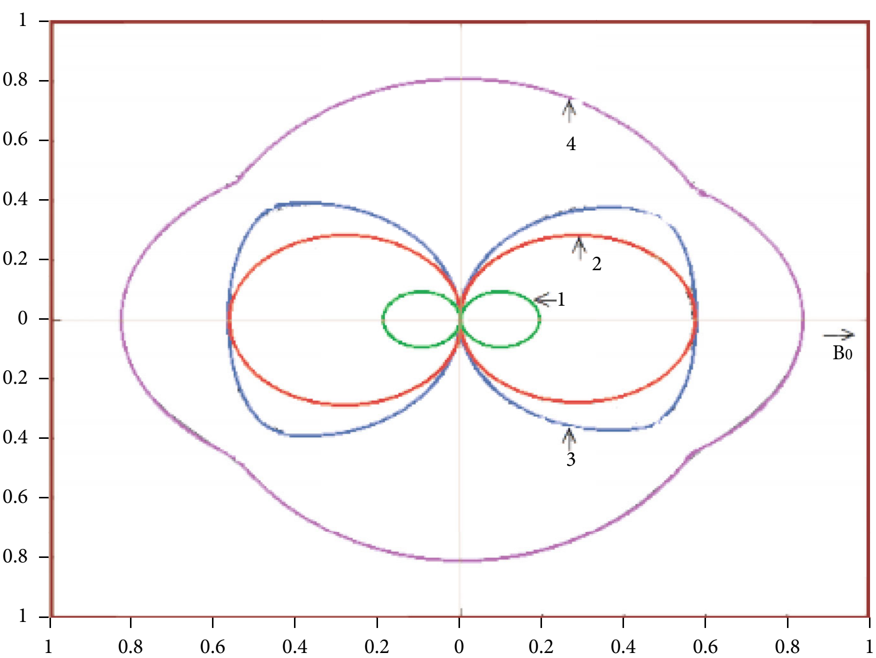

Figure 4: Polar plots of the phase speeds of all the four modes for relativistic anisotropic magnetohydrodynamics two-component plasma using $M_A^2 = 0.9$, $b_{\perp 1}^2 = b_{\parallel 1}^2 = 0.5$, $b_{\perp 2}^2 = b_{\parallel 2}^2 = 0.0125$, and $d = 10$.

to the right and the other to the left. The outer curve represents the nonrelativistic, and the inner one is the relativistic two-component plasma fast wave modes.

Figures 4 depicts that the polar plot of phase speeds normalized to the $Alfve'n$ speed in the relativistic two components of the four modes of propagation. Three of these modes are the analogs of the usual single-population relativistic MHD modes. These are slow, $Alfve'n$ and fast as labeled as 1, 2, and 3, respectively, while the fourth mode called the suprathermal mode, which is labeled as 4, is characteristic of the two-population plasma system and arises due to the interaction of two MHD plasma components. When one of the two components of plasma is made to vanish, the suprathermal mode disappears in such a way that a part of it fuses with the single-component fast mode propagating in a direction normal to the magnetic field. As clearly

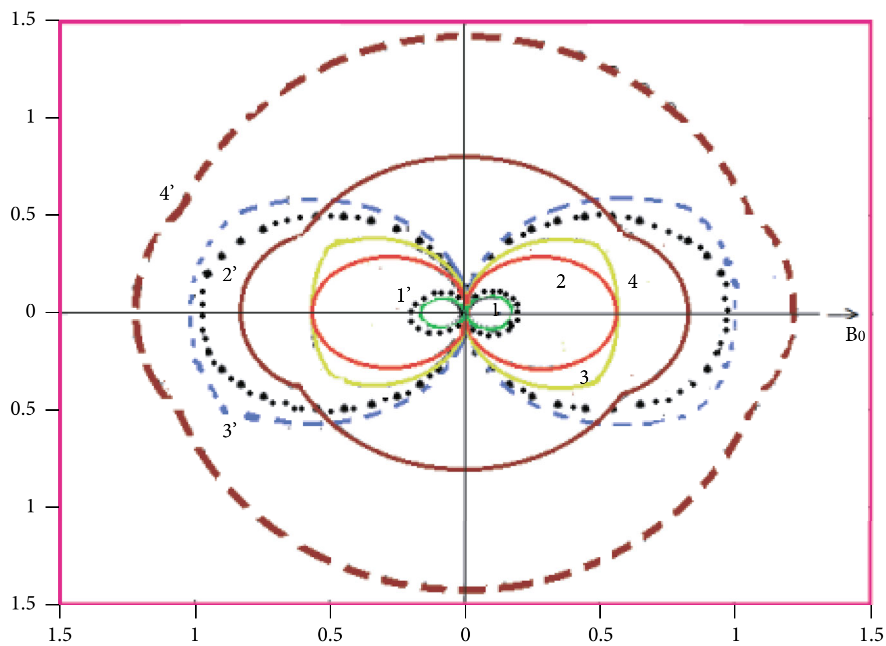

FIGURE 5: Polar plots of the phase speeds of all the four modes for RAM and NAM two-component plasma using $M_A^2 = 0.9$, $b_{\perp 1}^2 = b_{\parallel 1}^2 = 0.5$, $b_{\perp 2}^2 = b_{\parallel 2}^2 = 0.0125$, and $d = 10$.

shown in Figure 4 that the suprathermal mode is the mode that only propagates perpendicular to the direction magnetic field. This is a hybrid of the fast (in a single-component plasma) and the suprathermal modes. The relative magnitude of the phase speeds of these four modes, slow, $Alfve'n$ fast, and suprathermal are $U_S < U_A < U_F < U_{ST}$, respectively. This shows that for the chosen numerical values of the parameters, the modes are in the pseudo-MHD [32] regime.

Finally, we plot both nonrelativistic and relativistic two-component anisotropic MHD plasma dispersion solutions together as shown in Figure 5, the broken lines ($1'$, $2'$, $3'$, and $4'$) indicate the wave modes for NAM and the solid lines (represented by numbers 1, 2, 3, and 4) for the relativistic component.

From this figure, Figure 5, we deduce the wave modes of the relativistic plasma lags than their corresponding nonrelativistic counterpart. As a result, we can conclude from the above plot that we do have all eight modes which become four incoming and four outgoing modes. The wave modes of the relativistic two-component plasma are found to be slower than their nonrelativistic component counterpart. This is because the density of the two mediums is not the same.

5. Conclusion

An in-viscid magneto-hydrodynamic (MHD) model, which describes the mutual interaction of a relativistic anisotropic fluid interstellar plasma and cosmic rays, is used to investigate the stability and propagation of waves. To do so, starting from the relativistic collision-less Boltzmann transport (relativistic Vlasov) equations, the basic equations, and linear dispersion relations are derived. Using parameters relevant to cosmic and interstellar plasmas, the generalized dispersion relation discussed for different cases analytically and numerically, and using polar plots of the phase speed of both relativistic and nonrelativistic cases are drawn and discussed. From the analysis of the dispersion relations, it is found that all existing kinds of MHD waves for the case of two-component anisotropic relativistic plasma definitely have different phase speeds as compared to the standard theory of linear MHD waves in cold collisionless plasma.

On the other hand, from the numerical analysis, polar plots of the phase speeds of different wave modes for different cases are also drawn. Furthermore, for perpendicular propagation, we can definitely notice that the phase speed of magneto-sonic waves in the collision-less plasma with relativistic temperature is much smaller than the same speed in the nonrelativistic case. In the case of parallel propagation, our analysis showed that all the four MHD wave modes propagate, among which the first two are identical and propagate with modified $Alfve'n$ speed, while the third mode propagates with the relativistic sound speed, and the fourth mode is the suprathermal mode which results due to the existence of two-component plasma. However, for normal propagation, it is found that three of the modes vanish and the remaining one specifies a result analogous to ordinary nonrelativistic MHD waves for propagation transverse to the magnetic field, which is the suprathermal mode; the slow mode disappears, and the fast mode has a phase speed whose square is the sum of the squares of the sound and $Alfve'n$ speeds. In addition to the parallel and normal propagation, we have also found from the oblique one that the obtained numerical solutions are plotted to give the result which is in agreement with the above-mentioned outcomes.

In this paper, it may be concluded that the two population results in four-wave mode where the phase speeds are reduced due to the relativistic corrections. It can also conclude from the analytic and numerical solutions that the phase speeds are not only reduced by the relativistic effects but also reduced by the pressure anisotropies ($p_{\parallel 1}, p_{\parallel 2}, p_{\perp 1}$, and $p_{\perp 2}$) which we claim is a new result.

Data Availability

The data used to support the findings of this study are included in the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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