Hindawi
Advances in Mathematical Physics
Volume 2023, Article ID 1294070, 10 pages
https://doi.org/10.1155/2023/1294070

Research Article

Invariant Solutions and Conservation Laws of the Time-Fractional Telegraph Equation

Ramin Najafi[1], **Ercan Çelik**[2], **and Neslihan Uyanık**[3]

[1]*Department of Mathematics, Maku Branch, Islamic Azad University, Maku, Iran*
[2]*Department of Applied Mathematics and Informatics, Kyrgyz–Turkish Manas University, Bishkek, Kyrgyzstan*
[3]*Department of Mathematics and Science Education, Faculty of Education, Anadolu University, Eskişehir, Turkey*

Correspondence should be addressed to Ramin Najafi; raminnajafi984@gmail.com

Received 7 November 2022; Revised 24 December 2022; Accepted 29 July 2023; Published 29 August 2023

Academic Editor: Igor Freire

In this study, the Lie symmetry analysis is given for the time-fractional telegraph equation with the Riemann–Liouville derivative. This equation is useable to describe the physical processes of models possessing memory. By applying classical and nonclassical Lie symmetry analysis for the telegraph equation with α, β time-fractional derivatives and some technical computations, new infinitesimal generators are obtained. The actual methods give some classical symmetries while the nonclassical approach will bring back other symmetries to these equations. The similarity reduction and conservation laws to the fractional telegraph equation are found.

1. Introduction

For decades, because fractional differential equations have a major role in many different fields of science, the theory of fractional differential equations has attracted broad interest in the different areas of applied sciences [1–5]. The fractional derivatives have also been of basic importance in the mathematical modeling of many systems in control [6], chemistry–biochemistry [7, 8], finance [9], disease transmission dynamics [10–12], and other disciplines. Fractional derivatives supply a perfect and elegant tool to describe diverse materials and processes with memory and hereditary characteristic of a variety of different phenomena. By Xu et al. [13], Zhang et al. [14], Xu and Zhang [15], and Zhang et al. [16], the analytical solutions of the fractional differential equations have been obtained. The telegraph equations describe the current and voltage of wave propagation of electric signals in a cable transmission line to find distance and time, and has many applications such as neutron transport [17], random walk of suspension flows [18], signal analysis for transmission, propagation of electrical signals [19] etc. Since the telegraph equation with classical derivatives can not wells match the abnormal diffusion phenomena during the finite long transmit progress, where the voltage or the current wave possibly exists, in modeling of this equation, from the point of view of memory effects, we obtain the fractional telegraph equations with fractional derivatives [20–22]. The time-fractional telegraph equation can be written as follows:

$$D_t^{\alpha} u + \gamma_1 D_t^{\beta} u + \gamma_2 u = \gamma_3 u_{xx} + f(x, t), 1 < \alpha \leq 2, 0 < \beta \leq 1, \tag{1}$$

where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary constants, $f(x, t)$ is smooth function and $D_t^{\alpha}, D_t^{\beta}$ represent the Riemann–Liouville fractional derivatives. For $\alpha = 2, \beta = 1$, the equation becomes the classical telegraph equation. Recently, the fractional telegraph equation has been studied comprehensively theoretically and numerically [23–26]; however, there are limited works using analytical methods. Lie group theory plays a fundamental role in the theory of differential equations and it is efficiently used to get exact solutions to differential equations and fractional differential equations in some disciplines [27–33]. To obtain new infinitesimal generators of a fractional differential equation and new solutions, we use the class of conditional symmetries or the so-called nonclassical symmetry method, which is proposed by Bluman and Cole [34] for the first time. A number of investigators have applied the method to find analytical solutions for the fractional differential equations [35–40]. Conservation laws have a crucial role in the mathematical physics. They describe the physically conserved quantities such as energy, mass, and other constants of the motion. Noether's [41] theorem produces a link between symmetries and conservation laws of the differential equations.

In this paper, we intend to get the nonclassical Lie group method and conservation laws of the telegraph equation with fractional order. We obtain by using the nonclassical method, infinitesimal generators that are new when compared with obtained symmetries by the classical method. The continuation of the article is as given below: mathematical preliminaries of the Riemann–Liouville fractional derivative and a description of classical and nonclassical Lie symmetry analysis are presented in Section 2. In Section 3 classical, nonclassical Lie symmetries, and similarity reductions of the telegraph equation with fractional order are constructed. Utilizing the obtained symmetries nontrivial conservation laws are given in Section 4.

2. Preliminaries

This section deals with basic concepts, definitions, and lemmas of fractional derivatives and classical, and nonclassical Lie symmetry analysis.

2.1. Basic Definition on Fractional Derivatives

Definition 1. [5] Let $\alpha \in \mathbb{R}_+$. The Riemann–Liouville fractional derivative of order $m-1<\alpha\leq m$ of a function $f\in L_1[0,T]$ is defined by

$$D_t^\alpha f(t)=\frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_0^t (t-s)^{m-\alpha-1}f(s)\mathrm{d}s, \tag{2}$$

for all $t\in(0,T)$.

Proposition 1. *[5] Let f be analytical in $(-h,h)$ for some $h>0$, and let $\alpha\in\mathbb{R}_+$. Then*

$$D_t^\alpha f(t)=\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}f^{(n)}(t), \tag{3}$$

for $0<t<h/2$, where $f^{(n)}(t)$ denotes the n-th derivative of f at t.

Proposition 2. *[5] (Leibniz's formula for Riemann–Liouville fractional derivative) Let $\alpha\in\mathbb{R}_+$ and assume that f and g are analytical on $(-h,h)$ with some $h>0$. Then,*

$$D_t^\alpha[fg](t)=\sum_{n=0}^{\infty}\binom{\alpha}{n}f^{(n)}(t)D_t^{\alpha-n}g(t), \tag{4}$$

for $0<t<h/2$.

Definition 2. [5] Let $\alpha,\beta\in\mathbb{R}_+$. The function $E_{\alpha,\beta}$ defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{5}$$

whenever the series converges is called the two-parameter Mittag–Leffler function with parameters α and β.

2.2. Description of Classical and Nonclassical Lie Symmetry Analysis.

Let us consider a one-parameter Lie group of infinitesimal transformations G on an open subset $M\subset X\times U\simeq R^3$ with coordinate (x,t,u)

$$\bar{x}=x+\varepsilon\xi(x,t,u)+O(\varepsilon^2), \tag{6}$$

$$\bar{t}=t+\varepsilon\tau(x,t,u)+O(\varepsilon^2), \tag{7}$$

$$\bar{u}=u+\varepsilon\varphi(x,t,u)+O(\varepsilon^2), \tag{8}$$

where ε is the group parameter. Then its associated Lie algebra is spanned by the following infinitesimal generators

$$V=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\varphi(x,t,u)\frac{\partial}{\partial u}. \tag{9}$$

We first recall the details of the classical symmetries method for the fractional differential equation. Let us consider the following time-fractional differential equation

$$F(x,t,u,u_x,u_{xx},\cdots,D_t^\alpha u)=0, \tag{10}$$

defined over $M\subset X\times U\simeq R^3$. A one-parameter group of transformations as in Equation (7), is a classical Lie symmetry group of Equation (10) if and only if for every infinitesimal generator V of G we have

$$Pr^{(\alpha,t)}V(F)|_{F=0}=0, \tag{11}$$

where

$$Pr^{(\alpha,t)}V=V+\varphi^x\frac{\partial}{\partial u_x}+\varphi^{xx}\frac{\partial}{\partial u_{xx}}+\cdots+\varphi^{(\alpha,t)}\frac{\partial}{\partial D_t^\alpha u}, \tag{12}$$

with

$$\begin{aligned}\varphi^x&=\mathscr{D}_x(\varphi)-\mathscr{D}_x(\xi)u_x-\mathscr{D}_x(\tau)u_t,\\ \varphi^{xx}&=\mathscr{D}_x(\varphi^x)-\mathscr{D}_x(\xi)u_{xx}-\mathscr{D}_x(\tau)u_{xt},\\ &\cdots\\ \varphi^{(\alpha,t)}&=\mathscr{D}_t^\alpha\varphi+\xi D_t^\alpha u_x-\mathscr{D}_t^\alpha(\xi u_x)-\mathscr{D}_t^{\alpha+1}(\tau u)\\ &\quad+\mathscr{D}_t^\alpha(\mathscr{D}_t(\tau)u)+\tau D_t^{\alpha+1}u.\end{aligned} \tag{13}$$

Using Leibniz's formula for Riemann–Liouville fractional operator and the chain rule, we can write

$$\varphi^{(\alpha,t)} = D_t^\alpha \varphi + (\varphi_u - \alpha \mathscr{D}_t \tau) D_t^\alpha u - u D_t^\alpha \varphi_u + \mu - \sum_{n=1}^{\infty} \binom{\alpha}{n} \mathscr{D}_t^n \xi D_t^{\alpha-n} u_x + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} D_t^n \varphi_u - \binom{\alpha}{n+1} \mathscr{D}_t^{n+1} \tau \right] D_t^{\alpha-n} u, \tag{14}$$

where

$$\mu = \sum_{n=2}^{\infty} \binom{\alpha}{n} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{n}{m} \binom{k}{r} \frac{1}{k!} (-u)^r D_t^m u^{k-r} D_t^{n-m} \left(D_u^k \varphi \right). \tag{15}$$

Here $\varphi_u = \partial\varphi/\partial u$, $D_t^m = \partial^m/\partial t^m$, $\mathscr{D}_t$ denotes the total derivative operator defined by

$$\mathscr{D}_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{16}$$

and $\mathscr{D}_t^\alpha f$, $\mathscr{D}_t f$ denote the total fractional derivative and the total derivative of f, w.r.t. t, respectively.

According to the group of transformations in nonclassical methods Equation (7), we consider the invariant surface condition

$$\Lambda : \xi(x,t,u) u_x + \tau(x,t,u) u_t - \varphi(x,t,u) = 0, \tag{17}$$

and we investigate the invariance of both the original equations together with this equation. Because for every ξ, τ, and φ the invariant surface condition is always invariant under the group of transformations Equation (7), then a one-parameter group Equation (7), is a nonclassical Lie symmetry group of Equation (10) if and only if for every infinitesimal generator V of G, we have

$$Pr^{(\alpha)} V(F) \big|_{F=0, \Lambda=0} = 0. \tag{18}$$

3. Lie Symmetries and Similarity Reductions of the Time-Fractional Telegraph Equation

In this section, we study the invariance properties of the time-fractional telegraph equation. Let us consider the time-fractional telegraph equation

$$D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u = \gamma_3 u_{xx} + f(x,t), 1 < \alpha \le 2, 0 < \beta \le 1, \tag{19}$$

where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary constants and $f(x,t)$ is given smooth function. Now we wish to obtain infinitesimals ξ, τ, and φ so that the one-parameter group Equation (7) be a symmetry group admitted by Equation (19). According to Equation (11) and by applying

$$Pr^{(\alpha,t)} V = V + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{(\beta,t)} \frac{\partial}{\partial D_t^\beta u} + \varphi^{(\alpha,t)} \frac{\partial}{\partial D_t^\alpha u}, \tag{20}$$

to Equation (19), we have the classical Lie's invariance criterion

$$\varphi^{(\alpha,t)} + \gamma_1 \varphi^{(\beta,t)} + \gamma_2 \varphi = \gamma_3 \varphi^{xx} + \xi f_x + \tau f_t, \tag{21}$$

whenever u satisfies Equation (19). Substituting $\varphi^{(\alpha,t)}$, $\varphi^{(\beta,t)}$, φ^{xx} into this equation, replacing

$D_t^\alpha u + \gamma_1 D_t^\beta u$ by $-\gamma_2 u + \gamma_3 u_{xx} + f$ whenever it occurs, and equating the coefficients of the various monomials in the partial derivatives of u, we obtain the classical determining equations for the symmetry group of the time-fractional telegraph equation. Notice that $\varphi^{(\beta,t)}$ is defined similarly to $\varphi^{(\alpha,t)}$. To analyze theses, the coefficients of $u_x u_{xt}$, u_{xt}, $D_t^\beta u$ are

$$\tau_u = 0, \tau_x = 0, \tau_t = 0, \tag{22}$$

respectively, hence τ is constant. On the other hand, due to the lower limit of integral in Definition 2.1 be invariant under the group of transformations Equation (7), we have to suppose that

$$\tau(x,t,u) = 0. \tag{23}$$

Similarly, the coefficients of $u_x u_{xx}$, u_{xx}, $D_t^{\beta-n} u_x$ (or $D_t^{\alpha-n} u_x$) shows that $\xi(x,t,u) = \rho_1$ and the coefficients of u_x, u_x^2 and $D_t^{\beta-n} u$ (or $D_t^{\alpha-n} u$) reveals that

$$\varphi(x,t,u) = \rho_2 u + k(x,t), \tag{24}$$

where ρ_1, ρ_2 are arbitrary constants, k is an arbitrary function and satisfies the following fractional equation

$$D_t^\alpha k + \gamma_1 D_t^\beta k + \gamma_2 k - \gamma_3 k_{xx} + \rho_2 f - \rho_1 f_x = 0. \tag{25}$$

Therefore

$$V = \rho_1 \frac{\partial}{\partial x} + (\rho_2 u + k(x,t)) \frac{\partial}{\partial u}, \tag{26}$$

and the classical symmetry algebra of Equation (19) is spanned by the three vector fields

$$V_1 = \frac{\partial}{\partial x}, V_2 = u \frac{\partial}{\partial u}, V_3 = k(x,t) \frac{\partial}{\partial u}. \tag{27}$$

Example 1. Let $f(x,t) = e^{\lambda x} h(t)$ where λ is arbitrary constant. To obtain a classical invariant solution of Equation (19), we

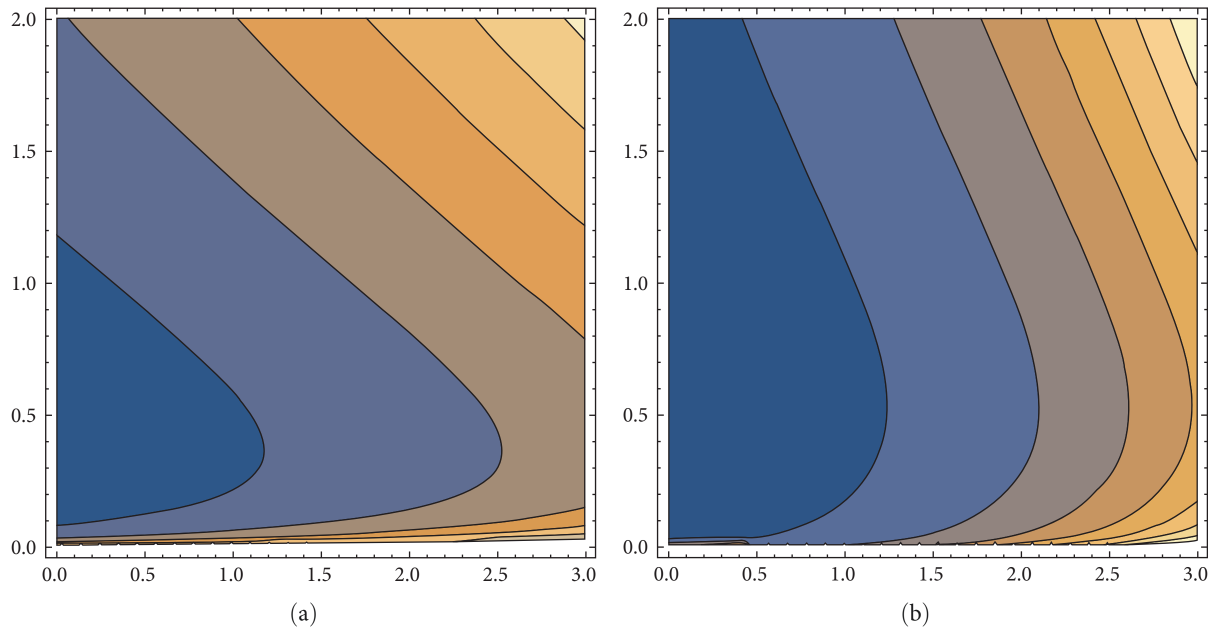


FIGURE 1: The plot of $u(x,t)$ obtained by Example 1 for values of (a) $\alpha=1.5$, $\beta=0.5$, $\lambda=0.3$, $\gamma_1=0.2$, $c_1=0.8$, $c_2=0.9$, and $p=0$ and (b) $\alpha=1.8$, $\beta=0.8$, $\lambda=0.8$, $\gamma_1=1$, $c_1=0.4$, $c_2=0.8$, and $p=0$.

consider the one-parameter group generated by $V=V_1+\lambda V_2$ with invariant solution $u(x,t)=e^{\lambda x}g(t)$. Substituting this solution into Equation (19), the reduced fractional ordinary differential equation (ODE) is

$$D_t^\alpha g(t)+\gamma_1 D_t^\beta g(t)+(\gamma_2-\gamma_3\lambda^2)g(t)=h(t). \tag{28}$$

If, for example, $\gamma_2=\gamma_3\lambda^2$ the solution of this equation can be written in terms of Mittag–Leffler functions

$$g(t)=c_1t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1t^{\alpha-\beta})+c_2t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1t^{\alpha-\beta}) + \sum_{k=0}^{\infty}(-\gamma_1)^kI^{k(\alpha-\beta)+\alpha}h(t), \tag{29}$$

where c_1,c_2 are arbitrary constants and I^α is the Riemann–Liouville fractional integral operator of order α, thus

$$u(x,t)=e^{\lambda x}\Big(c_1t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1t^{\alpha-\beta}) + c_2t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1t^{\alpha-\beta})+\sum_{k=0}^{\infty}(-\gamma_1)^kI^{k(\alpha-\beta)+\alpha}h(t)\Big), \tag{30}$$

is the solution of time-fractional telegraph equation.

If $h(t)=t^p$ where $p\ge 0$ then

$$\begin{aligned}u(x,t)=\ &e^{\lambda x}\big(c_1t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1t^{\alpha-\beta})\\&+c_2t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1t^{\alpha-\beta})\\&+\Gamma(p+1)t^{\alpha+p}E_{\alpha-\beta,\alpha+p+1}(-\gamma_1t^{\alpha-\beta})\big).\end{aligned} \tag{31}$$

Choosing suitable values, the figures for Equation (31) have been displayed in Figures 1 and 2.

Example 2. If there exists a class of solutions $g(x,t,u)=c$ to the Equation (19) admitting group Equation (7), $g(\bar{x},\bar{t},\bar{u})=\bar{c}$ defines a solution too [39], then we have

$$Vg(x,t,u)=1. \tag{32}$$

Now we consider the one-parameter group generated by $V_3=k(x,t)\partial/\partial u$ of the Equation (19), where k satisfies the Equation (25). According to Equation (32), we can write

$$k(x,t)g_u=1. \tag{33}$$

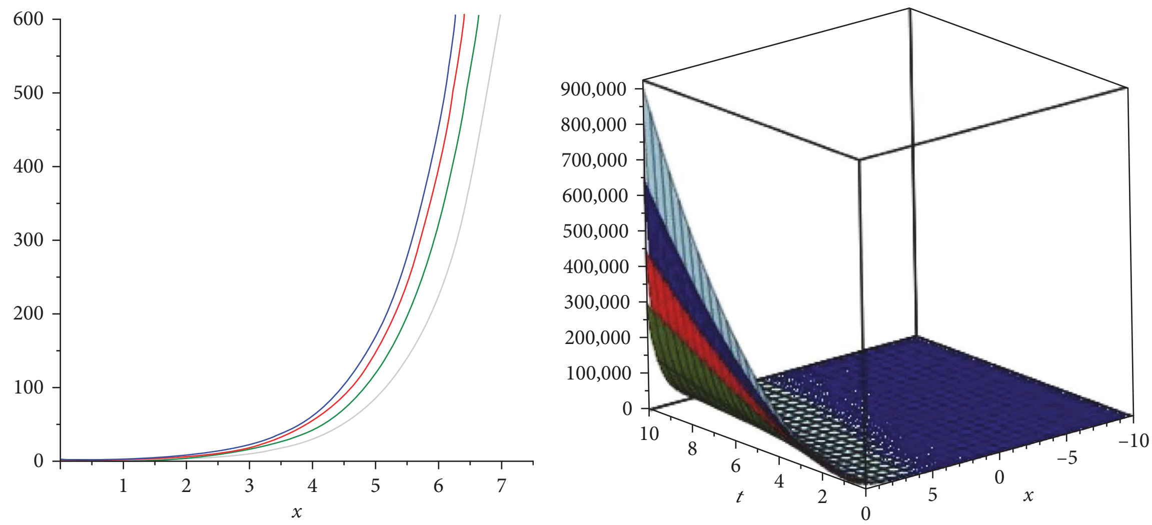
The solution of this equation can be written in the form

$$g(x,t,u)=\frac{u}{k(x,t)}+\frac{w(x,t)}{k(x,t)}=c, \tag{34}$$

so

$$u=ck(x,t)-w(x,t). \tag{35}$$

If we assume, for instance, $f(x,t)=w(x,t)=z(x)h(t)$ therefore



(a) (b)

Figure 2: The plot of $u(x,t)$ obtained by Example 1 for values of $\alpha = 1.25, 1.5, 1.75, 2$ and $\beta = 0.25, 0.5, 0.75, 1$ (a) at the time $t = 1$, (b) for $-10 < x < 10$ and $0 < t < 10$; with parameters $\lambda = \gamma_1 = c_1 = c_2 = p = 1$.

$$u = ck(x,t) - z(x)h(t). \tag{36}$$

Substituting this solution into Equation (19), thus according to Equation (25) and assuming that

$$\gamma_3 z''(x) + c\rho_1 z'(x) - (\gamma_2 + c\rho_2 + 1)z(x) = 0, \tag{37}$$

the reduced fractional ODE is

$$D_t^\alpha h(t) + \gamma_1 D_t^\beta h(t) = 0. \tag{38}$$

The solution of this equation can be written in terms of Mittag–Leffler functions

$$h(t) = c_3 t^{\alpha-1} E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) + c_4 t^{\alpha-2} E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta}), \tag{39}$$

where c_3, c_4 are arbitrary constants. According to Equation (37), if we assume that $\Delta = (c\rho_1)^2 + 4\gamma_3(\gamma_2 + c\rho_2 + 1)$ then we determine the solution of Equation (19) in the following cases:

If $\Delta > 0$, then we have

$$\begin{aligned} u(x,t) = {} & ck(x,t) - \left[c_5 e^{m_1 x} + c_6 e^{m_2 x}\right]\left[c_3 t^{\alpha-1} E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) \right. \\ & \left. + c_4 t^{\alpha-2} E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta})\right], \end{aligned} \tag{40}$$

where m_1, m_2 denote the real roots to the auxiliary equation of Equation (40).

If $\Delta = 0$, then we have

$$\begin{aligned} u(x,t) = {} & ck(x,t) - \left[(c_7 + c_8 x)e^{mx}\right]\left[c_3 t^{\alpha-1} E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) \right. \\ & \left. + c_4 t^{\alpha-2} E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta})\right], \end{aligned} \tag{41}$$

where m denote the repeated real root to the auxiliary equation of Equation (37).

If $\Delta < 0$, then we have

$$\begin{aligned} u(x,t) = {} & ck(x,t) - \left[e^{\theta_1 x}(c_9 \cos\theta_2 x + c_{10} \sin\theta_2 x)\right] \\ & \left[c_3 t^{\alpha-1} E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) + c_4 t^{\alpha-2} E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta})\right], \end{aligned} \tag{42}$$

where $\theta_1 \pm \theta_2 i$ denote the complex roots to the auxiliary equation of Equation (37) and $c_5, \dots, c_{10}$ are arbitrary constants.

Our next goal is now to obtain nonclassical determining equations, employing $Pr^{(\alpha,t)}V$ to Equation (19) and according to Equation (18), the nonclassical infinitesimal invariance criterion is given by

$$\varphi^{(\alpha,t)} + \gamma_1 \varphi^{(\beta,t)} + \gamma_2 \varphi = \gamma_3 \varphi^{xx} + \xi f_x + \tau f_t, \tag{43}$$

which must be satisfied whenever u satisfies Equation (19) and the invariant surface condition Equation (17). According to invariant surface condition, we consider two different cases: $\xi \neq 0$ and $\xi = 0$. In the case $\xi \neq 0$, without loss of generality, we can choose $\xi = 1$, then we have

$$u_x = \varphi - \tau u_t, u_{xt} = \varphi_t + (\varphi_u - \tau_t)u_t - \tau_u u_t^2 - \tau u_{tt}, \tag{44}$$

$$u_{xx} = (\varphi_x + \varphi\varphi_u - \tau\varphi_t) - (2\tau\varphi_u + \tau_x + \tau_u\varphi - \tau\tau_t)u_t + 2\tau\tau_u u_t^2 + \tau^2 u_{tt}, \tag{45}$$

After substituting $\varphi^{(\alpha,t)}, \varphi^{(\beta,t)}, \varphi^{xx}$ into Equation (43) with $\xi = 1$, replacing $D_t^\alpha u + \gamma_1 D_t^\beta u$ by $-\gamma_2 u + \gamma_3 u_{xx} + f$ similar to classical case, and also resubstituting the expressions u_x, u_{xt}, u_{xx} wherever it occurs, we attain the nonclassical determining equations for the symmetry group. Now we analyze the order of the derivatives which appear. The coefficient of $u_t u_{tt}$ is $\tau^2\tau_u = 0$, thus we conclude $\tau = 0$ or $\tau_u = 0$.

If $\tau = 0$ according to coefficient of $D_t^{\beta-n}u$ (or $D_t^{\alpha-n}u$), we have

$$\varphi(x,t,u) = \Theta(x,u) + \Xi(x,t), \tag{46}$$

where $\Theta(x,u), \Xi(x,t)$ are arbitrary functions. After substituting φ into the remaining term gives

$$\varphi(x,t,u) = \theta(x)u + \Xi(x,t), \tag{47}$$

where $\theta(x)$ and $\Xi(x,t)$ satisfy the following system

$$\theta''(x) + 2\theta(x)\theta'(x) = 0, \tag{48}$$

$$D_t^\alpha \Xi(x,t) + \gamma_1 D_t^\beta \Xi(x,t) + \gamma_2 \Xi(x,t) - \gamma_3 \Xi_{xx}(x,t) - 2\gamma_3\theta'(x)\Xi(x,t) + \theta(x)f - f_x = 0. \tag{49}$$

To find the function $\Xi(x,t)$, we solve equation Equation (48) thus

$$\theta(x) = \rho_3, \text{ or } \theta(x) = \frac{1}{x+\rho_4}, \text{ or } \theta(x) = \rho_5 \frac{e^{2\rho_5 x} + \rho_6}{e^{2\rho_5 x} - \rho_6}, \tag{50}$$

where $\rho_3, \ldots, \rho_6$ are arbitrary constants. A comparison with Equation (24) shows that the nonclassical method is more general than the classical method for the symmetry group.

If $\tau \neq 0$ and $\tau_u = 0$ the set of determining equations for the nonclassical symmetry group of Equation (19) are incompatible.

In the case $\xi = 0$ and $\tau \neq 0$, we have

$$u_t = \frac{\varphi}{\tau}, u_{xt} = \frac{1}{\tau^2}[(\tau\varphi_x - \tau_x\varphi) + (\tau\varphi_u - \tau_u\varphi)u_x], \tag{51}$$

and nonclassical determining equations unfortunately is impossible.

Example 3. According to Equation (50), let $\theta(x) = \frac{1}{x+\rho_4}$. *Assume that* $\Xi(x,t) = 0$ *and* $f(x,t) = (x+\rho_4)h(t)$. Then by considering invariat surface condition Equation (20), we have

$$u_x = \frac{u}{x+\rho_4}, \tag{52}$$

whose general solution is given by

$$u(x,t) = (x+\rho_4)g(t). \tag{53}$$

Substituting of this solution into Equation (19) reduces it to following fractional ODE

$$D_t^\alpha g(t) + \gamma_1 D_t^\beta g(t) + \gamma_2 g(t) = h(t). \tag{54}$$

After solving this fractional ODE, taking into account Equation (53) we find the exact solutions of the Equation (19). For example, whenever $\alpha = 2\beta$ and let r_1, r_2 denote the roots of the equation $x^2 + \gamma_1 x + \gamma_2 = 0$, assuming $r_1 \neq r_2$ we have

$$\begin{aligned} g(t) = \frac{1}{r_1 - r_2}&\left\{c_1 t^{\beta-1}\left[E_{\beta,\beta}(r_1 t^\beta) - E_{\beta,\beta}(r_2 t^\beta)\right]\right. \\ &+ c_2 t^{\beta-2}\left[E_{\beta,\beta-1}(r_1 t^\beta) - E_{\beta,\beta-1}(r_2 t^\beta)\right] \\ &+ \int_0^t (t-\tau)^{\beta-1}\left\{E_{\beta,\beta}[r_1(t-\tau)^\beta]\right. \\ &\left.\left.- E_{\beta,\beta}[r_2(t-\tau)^\beta]\right\}h(\tau)d\tau\right\}, \end{aligned} \tag{55}$$

so

$$\begin{aligned} u(x,t) = \frac{x+\rho_4}{r_1 - r_2}&\left\{c_1 t^{\beta-1}\left[E_{\beta,\beta}(r_1 t^\beta) - E_{\beta,\beta}(r_2 t^\beta)\right]\right. \\ &+ c_2 t^{\beta-2}\left[E_{\beta,\beta-1}(r_1 t^\beta) - E_{\beta,\beta-1}(r_2 t^\beta)\right] \\ &+ \int_0^t (t-\tau)^{\beta-1}\left\{E_{\beta,\beta}[r_1(t-\tau)^\beta]\right. \\ &\left.\left.- E_{\beta,\beta}[r_2(t-\tau)^\beta]\right\}h(\tau)d\tau\right\}. \end{aligned} \tag{56}$$

If $r_1 = r_2$, then

$$\begin{aligned} g(t) &= c_1 t^{2\beta-1}\frac{d}{dt}E_{\beta,\beta}(r_1 t^\beta) + c_2 t^{2\beta-2}\frac{d}{dt}E_{\beta,\beta-1}(r_1 t^\beta) - \int_0^t (t-\tau)^{2\beta-1}\frac{d}{d\tau}E_{\beta,\beta}[r_1(t-\tau)^\beta]h(\tau)d\tau, \\ u(x,t) &= (x+\rho_4)\left[c_1 t^{2\beta-1}\frac{d}{dt}E_{\beta,\beta}(r_1 t^\beta) + c_2 t^{2\beta-2}\frac{d}{dt}E_{\beta,\beta-1}(r_1 t^\beta) - \int_0^t (t-\tau)^{2\beta-1}\frac{d}{d\tau}E_{\beta,\beta}[r_1(t-\tau)^\beta]h(\tau)d\tau\right], \end{aligned} \tag{57}$$

where c_1, c_2 are arbitrary constants.

Example 4. Suppose that $\theta(x)=\rho_5 \frac{e^{2\rho_5 x}+\rho_6}{e^{2\rho_5 x}-\rho_6}$, $\Xi(x,t)=0$ *and*

$$f(x,t) = \exp\left(\int \theta(x)dx\right)h(t) = (e^{\rho_5 x} - \rho_6 e^{-\rho_5 x})h(t). \tag{58}$$

Then in view of the invariant surface condition, we have $u_x = \theta(x)u$. Differentiating this equation with respect to x leads to

$$u_{xx} = (\theta'(x) + \theta^2(x))u = \rho_5^2 u. \tag{59}$$

Thus, the general solution of the equation is

$$u(x,t) = e^{-\rho_5 x}F(t) + e^{\rho_5 x}G(t), \tag{60}$$

where F and G are arbitrary functions. Substituting into Equation (19), after some simplifications, we find the reduced fractional ODEs

$$D_t^\alpha F(t) + \gamma_1 D_t^\beta F(t) + (\gamma_2 - \gamma_3\rho_5^2)F(t) = -\rho_6 h(t), \tag{61}$$

$$D_t^\alpha G(t) + \gamma_1 D_t^\beta G(t) + (\gamma_2 - \gamma_3\rho_5^2)G(t) = h(t). \tag{62}$$

We can solve these equations similar to previous examples and obtain the exact solutions to Equation (19).

4. Conservation Laws

In the current section, we will focus on the conservation laws of Equation (19). A vector $C=(C^x, C^t)$ is called a conserved vector for Equation (19), if it fulfills the conservation equation

$$\mathscr{D}_x(C^x) + \mathscr{D}_t(C^t)\big|_{(3.1)} = 0, \tag{63}$$

where the conserved density C^t and the spatial flux C^x are functions of x, t, u, integer-order derivatives, fractional integrals and fractional derivatives of u. Equation (63) is called a conservation law for Equation (19). The new conservation theorem proposed by Ibragimov [42] provides a method to construct conservation laws for differential equations. Based on this theorem, the Lagrangian formal for Equation (19) can be introduced as follows:

$$\mathscr{L} = v(x,t)\left[D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u - \gamma_3 u_{xx} - f(x,t)\right], \tag{64}$$

where $v(x,t)$ is a new dependent variable. Thus the functional of Equation (19) through the formal Lagrangian Equation (64) is presented as follows:

$$\begin{aligned}\mathfrak{L}[u] = \int_{\mathbb{R}}\int_0^T v(x,t)\Big[&D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u - \gamma_3 u_{xx} \\ &- f(x,t)\Big]dtdx.\end{aligned} \tag{65}$$

A necessary condition for functional $\mathfrak{L}[u]$ to have an extremum is

$$\frac{\delta\mathscr{L}}{\delta u} = 0. \tag{66}$$

Here the Euler–Lagrange operator can be represented in the form

$$\begin{aligned}\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + {}_t^C D_T^\alpha \frac{\partial}{\partial D_t^\alpha u} + {}_t^C D_T^\beta \frac{\partial}{\partial D_t^\beta u} - \mathscr{D}_x \frac{\partial}{\partial u_x} \\ + \sum_{k=2}^{\infty}(-1)^k \mathscr{D}_{i_1}\mathscr{D}_{i_2}\ldots\mathscr{D}_{i_k}\frac{\partial}{\partial u_{i_1,i_2,\ldots,i_k}},\end{aligned} \tag{67}$$

where ${}_t^C D_T^\alpha$ and ${}_t^C D_T^\beta$ are the adjoint operators of $D_t^\alpha u, D_t^\beta u$ and are defined by

$$\begin{aligned}{}_t^C D_T^\gamma f(t) = \frac{(-1)^m}{\Gamma(m-\gamma)}\int_t^T (s-t)^{m-\gamma-1}D_s^m f(s)ds, \\ m-1<\gamma<m.\end{aligned} \tag{68}$$

Thus according to Equation (66), the adjoint equation to Equation (19) is given by

$${}_t^C D_T^\alpha v + \gamma_1\, {}_t^C D_T^\beta v + \gamma_2 v - \gamma_3 v_{xx} = 0. \tag{69}$$

We also use fundamental identity [42–44]

$$Pr^{(\alpha,t)}V + \mathscr{D}_t(\tau)I + \mathscr{D}_x(\xi)I = W\frac{\delta}{\delta u} + \mathscr{D}_t(N^t) + \mathscr{D}_x(N^x), \tag{70}$$

where N^t and N^x represent the Noether's operators, I is the identity operator, $Pr^{(\alpha,t)}V$ is given by Equation (20) and we can rewrite in the following form

$$\begin{aligned}Pr^{(\alpha,t)}V = \tau\mathscr{D}_t + \xi\mathscr{D}_x + W\frac{\partial}{\partial u} + \mathscr{D}_x^2(W)\frac{\partial}{\partial u_{xx}} \\ + \mathscr{D}_t^\alpha(W)\frac{\partial}{\partial(D_t^\alpha u)} + \mathscr{D}_t^\beta(W)\frac{\partial}{\partial\left(D_t^\beta u\right)},\end{aligned} \tag{71}$$

where $W = \varphi - \tau u_t - \xi u_x$ and the same as Equation (16), $\mathscr{D}_t$, $\mathscr{D}_x$ are defined by

$$\mathscr{D}_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xxt}\frac{\partial}{\partial u_{xx}} + D_t^{\alpha+1}u\frac{\partial}{\partial (D_t^\alpha u)} + D_t^{\beta+1}u\frac{\partial}{\partial \left(D_t^\beta u\right)}, \tag{72}$$

$$\mathscr{D}_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xxx}\frac{\partial}{\partial u_{xx}} + D_t^{\alpha}u_x\frac{\partial}{\partial (D_t^\alpha u)} + D_t^{\beta}u_x\frac{\partial}{\partial \left(D_t^\beta u\right)}. \tag{73}$$

Therefore according to Equation (70), we have

$$\begin{aligned} &Pr^{(\alpha,t)}V + \mathscr{D}_t(\tau)I + \mathscr{D}_x(\xi)I - W\frac{\delta}{\delta u} \\ &= \mathscr{D}_t\left[\tau I + \mathscr{D}_t^{\alpha-1}(W)\frac{\partial}{\partial(D_t^\alpha u)} - \mathscr{D}_t^{\alpha-2}(W)\mathscr{D}_t\frac{\partial}{\partial(D_t^\alpha u)} + \mathscr{D}_t^{\beta-1}(W)\frac{\partial}{\partial\left(D_t^\beta u\right)} - J\left(W, \mathscr{D}_t^2\frac{\partial}{\partial(D_t^\alpha u)}\right) + J\left(W, \mathscr{D}_t\frac{\partial}{\partial\left(D_t^\beta u\right)}\right)\right] \\ &\quad + \mathscr{D}_x\left[\xi I - W\mathscr{D}_x\frac{\partial}{\partial u_{xx}} + \mathscr{D}_x(W)\frac{\partial}{\partial u_{xx}}\right], \end{aligned} \tag{74}$$

where

$$\mathscr{D}_t(\tau)I + \tau\mathscr{D}_t = \mathscr{D}_t(\tau I) \tag{75}$$

$$\begin{aligned} &\mathscr{D}_t^\alpha(W)\frac{\partial}{\partial(D_t^\alpha u)} - W\ {}_t^C D_T^\alpha\frac{\partial}{\partial(D_t^\alpha u)} \\ &= \mathscr{D}_t\left[\mathscr{D}_t^{\alpha-1}(W)\frac{\partial}{\partial(D_t^\alpha u)} - \mathscr{D}_t^{\alpha-2}(W)\mathscr{D}_t\frac{\partial}{\partial(D_t^\alpha u)} - J\left(W, \mathscr{D}_t^2\frac{\partial}{\partial(D_t^\alpha u)}\right)\right] \end{aligned} \tag{76}$$

$$\begin{aligned} &\mathscr{D}_t^\beta(W)\frac{\partial}{\partial\left(D_t^\beta u\right)} - W\ {}_t^C D_T^\beta\frac{\partial}{\partial\left(D_t^\beta u\right)} \\ &= \mathscr{D}_t\left[\mathscr{D}_t^{\beta-1}(W)\frac{\partial}{\partial\left(D_t^\beta u\right)} + J\left(W, \mathscr{D}_t\frac{\partial}{\partial\left(D_t^\beta u\right)}\right)\right] \end{aligned} \tag{77}$$

$$\mathscr{D}_x^2(W)\frac{\partial}{\partial u_{xx}} - W\mathscr{D}_x^2\frac{\partial}{\partial u_{xx}} = \mathscr{D}_x\left[-W\mathscr{D}_x\frac{\partial}{\partial u_{xx}} + \mathscr{D}_x(W)\frac{\partial}{\partial u_{xx}}\right], \tag{78}$$

and the integral operator J is defined by

$$J(f(t), g(t)) = \frac{1}{\Gamma(m-\alpha)}\int_0^t\int_t^T (s-\nu)^{m-\alpha-1}f(\nu)g(s)dsd\nu. \tag{79}$$

This integral satisfies the property

$$J(f(t), g(t)) = f(t)_{(t)}I_T^{m-\gamma} - g(t)_0 I_t^{m-\gamma}, m-1<\gamma<m. \tag{80}$$

So for Equation (19), the operator N^t and N^x can be written by the following formula

$$\begin{aligned} N^t = {} & \tau I + \mathscr{D}_t^{\alpha-1}(W)\frac{\partial}{\partial(D_t^\alpha u)} - \mathscr{D}_t^{\alpha-2}(W)\mathscr{D}_t\frac{\partial}{\partial(D_t^\alpha u)} \\ & + \mathscr{D}_t^{\beta-1}(W)\frac{\partial}{\partial\left(D_t^\beta u\right)} - J\left(W, \mathscr{D}_t^2\frac{\partial}{\partial(D_t^\alpha u)}\right) \\ & + J\left(W, \mathscr{D}_t\frac{\partial}{\partial\left(D_t^\beta u\right)}\right), \end{aligned} \tag{81}$$

$$N^x = \xi I - W\mathscr{D}_x\frac{\partial}{\partial u_{xx}} + \mathscr{D}_x(W)\frac{\partial}{\partial u_{xx}}. \tag{82}$$

Since for all infinitesimal generator V admitted by Equation (19) and its any solution

$$Pr^{(\alpha,t)}V(\mathscr{L}) + \mathscr{D}_t(\tau)\mathscr{L} + \mathscr{D}_x(\xi)\mathscr{L} - W\frac{\delta\mathscr{L}}{\delta u}\Big|_{(3.1)} = 0, \tag{83}$$

therefore, in view of Equation (70), the conservation law of Equation (19) can be written as follows:

$$\mathscr{D}_t(N^t\mathscr{L}) + \mathscr{D}_x(N^x\mathscr{L})\Big|_{(3.1)} = 0. \tag{84}$$

Now, we present the components of conserved vectors for Equation (19). Using Equations (81) and (82), we have

$$
\begin{aligned}
C^t &= N^t \mathscr{L} = \tau \mathscr{L} + \mathscr{D}_t^{\alpha-1}(W)v \\
&\quad - \mathscr{D}_t^{\alpha-2}(W)v_t + \gamma_1 \mathscr{D}_t^{\beta-1}(W)v - J(W, v_{tt}) + \gamma_1 J(W, v_t), \\
C^x &= N^x \mathscr{L} = \xi \mathscr{L} + \gamma_3 W v_x - \gamma_3 \mathscr{D}_x(W)v,
\end{aligned}
\tag{85}
$$

where $v(x, t)$ is an arbitrary nontrivial solution of Equation (69) and for different infinitesimal generator $V_i, i = 1, \ldots, 3$, of Equation (19), W_i are defined as follows:

$$
W_1 = -u_x, W_2 = u, W_3 = k(x, t). \tag{86}
$$

5. Conclusion

In this paper, the methods of classical and nonclassical Lie symmetry analysis are applied to the time-fractional telegraph equation. The Lie symmetry method has had an enormous success when applied to a diverse range of differential equations. The idea of looking for the exact solutions and conservation law for the more derivatives of fractional and integer-order will motivate the nonclassical Lie symmetry method to study such types of fractional partial differential equations (PDEs). The results shown in this work demonstrate how the nonclassical approaches grab the new exact solutions to differential equations with fractional order. Moreover, we constructed conservation laws for the time-fractional telegraph equation by Ibragimovâ€s conservation theorem adapted to the time-fractional PDEs.

Data Availability

Data supporting this research article are available on request.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, 2006.

[3] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*, John Wiley & Sons, Inc, 2014.

[4] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, 2010.

[5] K. Diethelm, *The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo Type*, Springer, 2010.

[6] Z. Gong, C. Liu, K. L. Teo, S. Wang, and Y. Wu, "Numerical solution of free final time fractional optimal control problems," *Applied Mathematics and Computation*, vol. 405, Article ID 126270, 2021.

[7] G. Karamali, M. Dehghan, and M. Abbaszadeh, "Numerical solution of a time-fractional PDE in the electroanalytical chemistry by a local meshless method," *Engineering with Computers*, vol. 35, pp. 87–100, 2019.

[8] M. A. Dokuyucu, E. Çelik, H. Bulut, and H. M. Baskonus, "Cancer treatment model with the Caputo–Fabrizio fractional derivative," *The European Physical Journal Plus*, vol. 133, Article ID 92, 2018.

[9] O. I. Tacha, J. M. Munoz-Pacheco, E. Zambrano-Serrano, I. N. Stouboulos, and V.-T. Pham, "Determining the chaotic behavior in a fractional-order finance system with negative parameters," *Nonlinear Dynamics*, vol. 94, pp. 1303–1317, 2018.

[10] M. H. DarAssi, M. A. Safi, M. A. Khan, A. Beigi, A. A. Aly, and M. Y. Alshahrani, "A mathematical model for SARS-CoV-2 in variable-order fractional derivative," *The European Physical Journal Special Topics*, vol. 231, pp. 1905–1914, 2022.

[11] O. T. Kolebaje, O. R. Vincent, U. E. Vincent, and P. V. E. McClintock, "Nonlinear growth and mathematical modelling of COVID-19 in some African countries with the Atangana–Baleanu fractional derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 105, Article ID 106076, 2022.

[12] D. Baleanu, F. A. Ghassabzade, J. J. Nieto, and A. Jajarmi, "On a new and generalized fractional model for a real cholera outbreak," *Alexandria Engineering Journal*, vol. 61, no. 11, pp. 9175–9186, 2022.

[13] B. Xu, Y. Zhang, and S. Zhang, "Line soliton interactions for shallow ocean waves and novel solutions with peakon, ring, conical, columnar, and lump structures based on fractional KP equation," *Advances in Mathematical Physics*, vol. 2021, Article ID 6664039, 15 pages, 2021.

[14] S. Zhang, Y. Wei, and B. Xu, "Fractional soliton dynamics and spectral transform of time-fractional nonlinear systems: a concrete example," *Complexity*, vol. 2019, Article ID 7952871, 9 pages, 2019.

[15] B. Xu and S. Zhang, "Riemann–Hilbert approach for constructing analytical solutions and conservation laws of a local time-fractional nonlinear Schrödinger type equation," *Symmetry*, vol. 13, no. 9, Article ID 1593, 2021.

[16] S. Zhang, S. Zhao, and Y. Zhang, "Extending operator method to local fractional evolution equations in fluids," *Thermal Science*, vol. 23, Part B, no. 6, pp. 3759–3766, 2019.

[17] M. R. Altahhan, M. S. Nagy, H. H. Abou-Gabal, and A. E. Aboanber, "Formulation of a point reactor kinetics model based on the neutron telegraph equation," *Annals of Nuclear Energy*, vol. 91, pp. 176–188, 2016.

[18] A. C. Metaxas and R. J. Meredith, *Industrial Microwave, Heating*, Peter Peregrinus, 1983.

[19] S. T. R. Rizvi, K. Ali, A. Bekir, B. Nawaz, and M. Younis, "Investigation on the single and multiple dromions for nonlinear telegraph equation in electrical transmission line," *Qualitative Theory of Dynamical Systems*, vol. 21, Article ID 12, 2022.

[20] O. Nikan, Z. Avazzadeh, and J. A. Tenreiro Machado, "Numerical approximation of the nonlinear time-fractional telegraph equation arising in neutron transport," *Communications in Nonlinear Science and Numerical Simulation*, vol. 99, Article ID 105755, 2021.

[21] V. A. Vyawahare, P. S. V. Nataraj, G. Espinosa-Paredes, and R.-I. Cázares-Ramírez, "Nuclear reactor with subdiffusive neutron transport: development of linear fractional-order models," *International Journal of Dynamics and Control*, vol. 5, pp. 1182–1200, 2017.

[22] Z. Zhao and C. Li, "Fractional difference/finite element approximations for the time–space fractional telegraph equation," *Applied Mathematics and Computation*, vol. 219, no. 6, pp. 2975–2988, 2012.

[23] M. M. A. Khater, K. S. Nisar, and M. S. Mohamed, "Numerical investigation for the fractional nonlinear space–time telegraph equation via the trigonometric Quintic B-spline scheme," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 6, pp. 4598–4606, 2021.

[24] N. Vieira, M. M. Rodrigues, and M. Ferreira, "Time-fractional telegraph equation of distributed order in higher dimensions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 102, Article ID 105925, 2021.

[25] A. Kumar, A. Bhardwaj, and S. Dubey, "A local meshless method to approximate the time-fractional telegraph equation," *Engineering with Computers*, vol. 37, pp. 3473–3488, 2021.

[26] X. Yang, W. Qiu, H. Zhang, and L. Tang, "An efficient alternating direction implicit finite difference scheme for the three-dimensional time-fractional telegraph equation," *Computers & Mathematics with Applications*, vol. 102, pp. 233–247, 2021.

[27] M. I. El Bahi and K. Hilal, "Symmetry analysis, invariant solutions, and conservation laws of fractional KdV–Like equation," *Advances in Mathematical Physics*, vol. 2022, Article ID 5825938, 11 pages, 2022.

[28] Y. Yang and L. Wang, "Lie symmetry analysis, conservation laws and separation variable type solutions of the time-fractional porous medium equation," *Waves in Random and Complex Media*, vol. 32, no. 2, pp. 980–999, 2022.

[29] Y. Chatibi, E. H. El Kinani, and A. Ouhadan, "On the discrete symmetry analysis of some classical and fractional differential equations," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 4, pp. 2868–2878, 2021.

[30] T. Bakkyaraj, "Lie symmetry analysis of system of nonlinear fractional partial differential equations with Caputo fractional derivative," *The European Physical Journal Plus*, vol. 135, Article ID 126, 2020.

[31] A. Akbulut, M. Mirzazadeh, M. S. Hashemi, K. Hosseini, S. Salahshour, and C. Park, "Triki–Biswas model: its symmetry reduction, Nucci's reduction and conservation laws," *International Journal of Modern Physics B*, vol. 37, no. 7, 2023.

[32] Y.-M. Chu, M. Inc, M. S. Hashemi, and S. Eshaghi, "Analytical treatment of regularized Prabhakar fractional differential equations by invariant subspaces," *Computational and Applied Mathematics*, vol. 41, Article ID 271, 2022.

[33] M. S. Hashemi, F. Bahrami, and R. Najafi, "Lie symmetry analysis of steady-state fractional reaction–convection–diffusion equation," *Optik*, vol. 138, pp. 240–249, 2017.

[34] G. W. Bluman and J. D. Cole, "The general similarity solution of the heat equation," *Journal of Mathematics and Mechanics*, vol. 18, no. 11, pp. 1025–1042, 1969.

[35] R. Najafi, F. Bahrami, and S. Shahmorad, "Fractional differential equations, compatibility, and exact solutions," *Computational and Applied Mathematics*, vol. 41, Article ID 23, 2022.

[36] M. S. Hashemi, A. Haji-Badali, F. Alizadeh, and M. Inc, "Nonclassical Lie symmetry and conservation laws of the nonlinear time-fractional Korteweg-de Vries equation," *Communications in Theoretical Physics*, vol. 73, no. 9, Article ID 095006, 2021.

[37] R. Najafi, "Approximate nonclassical symmetries for the time-fractional KdV equations with the small parameter," *Computational Methods for Differential Equations*, vol. 8, no. 1, pp. 111–118, 2020.

[38] R. Najafi, "Group-invariant solutions for time-fractional Fornberg–Whitham equation by Lie symmetry analysis," *Computational Methods for Differential Equations*, vol. 8, no. 2, pp. 251–258, 2020.

[39] R. Najafi, F. Bahrami, and M. S. Hashemi, "Classical and nonclassical Lie symmetry analysis to a class of nonlinear time-fractional differential equations," *Nonlinear Dynamics*, vol. 87, pp. 1785–1796, 2017.

[40] F. Bahrami, R. Najafi, and M. S. Hashemi, "On the invariant solutions of space/time-fractional diffusion equations," *Indian Journal of Physics*, vol. 91, pp. 1571–1579, 2017.

[41] E. Noether, "Invariant variation problems," *Transport Theory and Statistical Physics*, vol. 1, no. 3, pp. 186–207, 1971.

[42] N. H. Ibragimov, "A new conservation theorem," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 311–328, 2007.

[43] S. Y. Lukashchuk, "Conservation laws for time-fractional subdiffusion and diffusion-wave equations," *Nonlinear Dynamics*, vol. 80, pp. 791–802, 2015.

[44] R. K. Gazizov, N. H. Ibragimov, and S. Y. Lukashchuk, "Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 23, no. 1–3, pp. 153–163, 2015.