Research Article

Invariant Solutions and Conservation Laws of the Time-Fractional Telegraph Equation

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In this study, the Lie symmetry analysis is given for the time-fractional telegraph equation with the Riemann–Liouville derivative. This equation is useful to describe the physical processes of models possessing memory. By applying classical and nonclassical Lie symmetry analysis for the telegraph equation with \( \alpha, \beta \) time-fractional derivatives and some technical computations, new infinitesimal generators are obtained. The actual methods give some classical symmetries while the nonclassical approach will bring back other symmetries to these equations. The similarity reduction and conservation laws to the fractional telegraph equation are found.

1. Introduction

For decades, because fractional differential equations have a major role in many different fields of science, the theory of fractional differential equations has attracted broad interest in the different areas of applied sciences [1–5]. The fractional derivatives have also been of basic importance in the mathematical modeling of many systems in control [6], chemistry–biochemistry [7, 8], finance [9], disease transmission dynamics [10–12], and other disciplines. Fractional derivatives supply a perfect and elegant tool to describe diverse materials and processes with memory and hereditary characteristic of a variety of different phenomena. By Xu et al. [13], Zhang et al. [14], Xu and Zhang [15], and Zhang et al. [16], the analytical solutions of the fractional differential equations have been obtained. The telegraph equations describe the current and voltage of wave propagation of electric signals in a cable transmission line to find distance and time, and has many applications such as neutron transport [17], random walk of suspension flows [18], signal analysis for transmission, propagation of electrical signals [19] etc. Since the telegraph equation with classical derivatives can not well match the abnormal diffusion phenomena during the finite long transmit progress, where the voltage or the current wave possibly exists, in modeling of this equation, from the point of view of memory effects, we obtain the fractional telegraph equations with fractional derivatives [20–22]. The time-fractional telegraph equation can be written as follows:

\[
D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u = \gamma_3 u_{xx} + f(x, t), \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1.
\]  

(1)

where \( \gamma_1, \gamma_2, \gamma_3 \) are arbitrary constants, \( f(x, t) \) is smooth function and \( D_t^\alpha, D_t^\beta \) represent the Riemann–Liouville fractional derivatives. For \( \alpha = 2, \beta = 1 \), the equation becomes the classical telegraph equation. Recently, the fractional telegraph equation has been studied comprehensively theoretically and numerically [23–26]; however, there are limited works using analytical methods. Lie group theory plays a fundamental role in the theory of differential equations and it is efficiently used to get exact solutions to differential equations and fractional differential equations in some disciplines [27–33]. To obtain new infinitesimal generators of a fractional differential equation and new solutions, we use the class of conditional symmetries or the so-called nonclassical symmetry method, which is proposed by Bluman and Cole [34] for the first time. A number of investigators have applied the method to find analytical solutions for the fractional differential equations [35–40]. Conservation laws have a
crucial role in the mathematical physics. They describe the physically conserved quantities such as energy, mass, and other constants of the motion. Noether’s [41] theorem produces a link between symmetries and conservation laws of the differential equations.

In this paper, we intend to get the nonclassical Lie group method and conservation laws of the telegraph equation with fractional order. We obtain by using the nonclassical method, infinitesimal generators that are new when compared with obtained symmetries by the classical method. The continuation of the article is as given below: mathematical preliminaries of obtained symmetries by the classical method. The continuation of the article is as given below: mathematical preliminaries of the Riemann–Liouville fractional derivative and a description of classical and nonclassical Lie symmetry analysis are presented in Section 2. In Section 3 classical, nonclassical Lie symmetries, and similarity reductions of the telegraph equation are presented in Section 2. In Section 3 classical, nonclassical Lie symmetries, and similarity reductions of the telegraph equation with fractional order are constructed. Utilizing the obtained symmetries nontrivial conservation laws are given in Section 4.

2. Preliminaries

This section deals with basic concepts, definitions, and lemmas of fractional derivatives and classical, and nonclassical Lie symmetry analysis.

2.1. Basic Definition on Fractional Derivatives

Definition 1. [5] Let $\alpha \in \mathbb{R}_+$. The Riemann–Liouville fractional derivative of order $m-1<\alpha \leq m$ of a function $f \in L^1[0, T]$ is defined by

$$D^\alpha_t f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{m-\alpha-1} f(s)ds,$$

for all $t \in (0, T)$.

Proposition 1. [5] Let $f$ be analytical in $(-h, h)$ for some $h>0$, and let $\alpha \in \mathbb{R}_+$. Then

$$D^\alpha_t f(t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{t^{\alpha-n}}{\Gamma(n+1)} f^{(n)}(t),$$

for $0 < t < h/2$, where $f^{(n)}(t)$ denotes the $n$-th derivative of $f$ at $t$.

Proposition 2. [5] (Leibniz’s formula for Riemann–Liouville fractional derivative) Let $\alpha \in \mathbb{R}_+$ and assume that $f$ and $g$ are analytical on $(-h, h)$ with some $h>0$. Then,

$$D^\alpha_t [fg](t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} f^{(n)}(t) D^{\alpha-n}_t g(t),$$

for $0 < t < h/2$.

Definition 2. [5] Let $\alpha, \beta \in \mathbb{R}_+$. The function $E_{\alpha, \beta}$ defined by

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+\beta)},$$

whenever the series converges is called the two-parameter Mittag–Leffler function with parameters $\alpha$ and $\beta$.

2.2. Description of Classical and Nonclassical Lie Symmetry Analysis

Let us consider a one-parameter Lie group of infinitesimal transformations $G$ on an open subset $M \subset X \times U \cong R^3$ with coordinate $(x, t, u)$

$$ \bar{x} = x + \epsilon \xi(x, t, u) + O(\epsilon^2), $$

$$ \bar{t} = t + \epsilon \tau(x, t, u) + O(\epsilon^2), $$

$$ \bar{u} = u + \epsilon \varphi(x, t, u) + O(\epsilon^2), $$

where $\epsilon$ is the group parameter. Then its associated Lie algebra is spanned by the following infinitesimal generators

$$ V = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \varphi(x, t, u) \frac{\partial}{\partial u}. $$

We first recall the details of the classical symmetries method for the fractional differential equation. Let us consider the following time-fractional differential equation

$$ F(x, t, u, u_x, \ldots, D^\alpha_t u) = 0, $$

defined over $M \subset X \times U \cong R^3$. A one-parameter group of transformations as in Equation (7), is a classical Lie symmetry group of Equation (10) if and only if for every infinitesimal generator $V$ of $G$ we have

$$ P^\epsilon V(F)|_{F=0} = 0, $$

where

$$ P^\epsilon V = V + \varphi^x \frac{\partial}{\partial u_x} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \cdots + \varphi^{(\alpha)} \frac{\partial}{\partial D^\alpha_t u}, $$

with

$$ \varphi^x = \partial_x \varphi - \partial_x (\xi) u_t - \partial_x (\tau) u_x, $$

$$ \varphi^{xx} = \partial_x (\varphi^x) - \partial_x (\xi) u_{xx} - \partial_x (\tau) u_{xx}, $$

$$ \ldots $$

$$ \varphi^{(\alpha)} = \partial_x \varphi + \xi \partial_x D^\alpha_t u_x - \partial_x (\xi u_x) - \partial_x (\tau) u_{xx} + \partial_x (\partial_t (\tau) u) + \partial_t (\partial_t T u) + t D^{\alpha+1}_t u. $$

Using Leibniz’s formula for Riemann–Liouville fractional operator and the chain rule, we can write
\[ q^{(\alpha, t)} = D_t^\alpha \phi + (\phi_u - \alpha D_t \phi) D_t^\alpha u - u D_t^\alpha \phi_u + \mu - \sum_{n=1}^{\infty} \left( \frac{\alpha}{n} \right) D_t^n \phi_u + \sum_{n=1}^{\infty} \left[ \left( \frac{\alpha}{n} \right) D_t^n \phi_u - \left( \frac{\alpha}{n+1} \right) D_t^{n+1} \phi_u \right] D_t^{n-u}, \]

where

\[ \mu = \sum_{n=0}^{\infty} \left( \frac{n}{n+1} \right) \sum_{m=2}^{n} \sum_{k=1}^{m} \frac{(-1)^m}{k!} D_t^n u^{m-k} D_t^{m-k} (D_\phi \phi). \]

Here \( \phi_u = \partial \phi / \partial u, \) \( D_t^n = \partial^n / \partial t^n, \) \( D_t \) denotes the total derivative operator defined by

\[ D_t = \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + \frac{\partial}{\partial u_1} + \ldots. \]

and \( D_t f, D_t f \) denote the total fractional derivative and the total derivative of \( f, \) w.r.t. \( t, \) respectively.

According to the group of transformations in nonclassical methods Equation (7), we consider the invariant surface condition

\[ A : \xi(x, t, u) u_x + \tau(x, t, u) u_t - \phi(x, t, u) = 0, \]

and we investigate the invariance of both the original equations together with this equation. Because for every \( \xi, \tau, \) and \( \phi \) the invariant surface condition is always invariant under the group of transformations Equation (7), then a one-parameter group Equation (7), is a nonclassical Lie symmetry group of Equation (10) if and only if for every infinitesimal generator \( V \) of \( G, \) we have

\[ P_{\xi(t)} V(F) \bigg|_{F=0, A=0} = 0. \]

### 3. Lie Symmetries and Similarity Reductions of the Time-Fractional Telegraph Equation

In this section, we study the invariance properties of the time-fractional telegraph equation. Let us consider the time-fractional telegraph equation

\[ D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u = \gamma_3 u_x + f(x, t), 1 < \alpha \leq 2, 0 < \beta \leq 1, \]

where \( \gamma_1, \gamma_2, \gamma_3 \) are arbitrary constants and \( f(x, t) \) is given smooth function. Now we wish to obtain infinitesimals \( \xi, \tau, \) and \( \phi \) so that the one-parameter group Equation (7) be a symmetry group admitted by Equation (19). According to Equation (11) and by applying

\[ P_{\xi(t)} V = V + \phi_u x \frac{\partial V}{\partial u_x} + \phi^{(\beta,t)} x \frac{\partial V}{\partial u_x} + \phi^{(\alpha,t)} x \frac{\partial V}{\partial u_x} \]

to Equation (19), we have the classical Lie's invariance criterion

\[ \phi^{(\alpha,t)} x + \gamma_1 \phi^{(\beta,t)} x + \gamma_2 \phi = \gamma_3 \phi x x + \xi x + \tau f, \]

whenever \( u \) satisfies Equation (19). Substituting \( \phi^{(\alpha,t)} x, \phi^{(\beta,t)} x, \phi^{(\alpha,t)} x \) into this equation, replacing \( D_t^\alpha u + \gamma_1 D_t^\beta u \) by \( -\gamma_2 u + \gamma_3 u_x + f \) whenever it occurs, and equating the coefficients of the various monomials in the partial derivatives of \( u, \) we obtain the classical determining equations for the symmetry group of the time-fractional telegraph equation. Notice that \( \phi^{(\beta,t)} x \) is defined similarly to \( \phi^{(\alpha,t)} x \).

To analyze these, the coefficients of \( u_x u_x, u_x, D_t^\beta u \) are

\[ \tau_u = 0, \tau_x = 0, \tau_t = 0, \]

respectively, hence \( \tau \) is constant. On the other hand, due to the lower limit of integral in Definition 2.1 be invariant under the group of transformations Equation (7), we have to suppose that

\[ \tau(x, t, u) = 0. \]

Similarly, the coefficients of \( u_x u_x, u_x, D_t^\beta u \) (or \( D_t^\alpha u_x \)) shows that \( \xi(x, t, u) = \rho_1 \) and the coefficients of \( u_x, D_t^\beta u \) and \( D_t^\alpha u \) (or \( D_t^\alpha u_x \)) reveals that

\[ \phi(x, t, u) = \rho_2 u + k(x, t), \]

where \( \rho_1, \rho_2 \) are arbitrary constants, \( k \) is an arbitrary function and satisfies the following fractional equation

\[ D_t^\beta k + \gamma_1 D_t^\beta k + \gamma_2 u - \gamma_3 k_x + \rho_2 f - \rho_1 f_x = 0. \]

Therefore

\[ V = \rho_1 \frac{\partial}{\partial x} + (\rho_2 u + k(x, t)) \frac{\partial}{\partial u}, \]

and the classical symmetry algebra of Equation (19) is spanned by the three vector fields

\[ V_1 = \frac{\partial}{\partial x}, V_2 = u \frac{\partial}{\partial u}, V_3 = k(x, t) \frac{\partial}{\partial u}. \]

**Example 1.** Let \( f(x, t) = e^{i\lambda h(t)} \) where \( \lambda \) is arbitrary constant. To obtain a classical invariant solution of Equation (19), we
consider the one-parameter group generated by \( V = V_1 + \lambda V_2 \) with invariant solution \( u(x, t) = e^{i\omega} g(t) \). Substituting this solution into Equation (19), the reduced fractional ordinary differential equation (ODE) is

\[
D^\alpha_t g(t) + \gamma_1 D^\beta_t g(t) + (\gamma_2 - \gamma_3 \lambda^2) g(t) = h(t). \tag{28}
\]

If, for example, \( \gamma_2 = \gamma_3 \lambda^2 \) the solution of this equation can be written in terms of Mittag–Leffler functions

\[
g(t) = c_1 t^{\alpha-1} E_{\alpha-\beta, \alpha}( -\gamma_1 t^{\alpha-\beta} ) + c_2 t^{\alpha-2} E_{\alpha-\beta, \alpha-1}( -\gamma_1 t^{\alpha-\beta} ) \\
+ \sum_{k=0}^{\infty} ( -\gamma_1 )^k k^{(\alpha-\beta)+\alpha} h(t), \tag{29}
\]

where \( c_1, c_2 \) are arbitrary constants and \( t^\alpha \) is the Riemann–Liouville fractional integral operator of order \( \alpha \), thus

\[
u(x, t) = e^{i\omega} \left( c_1 t^{\alpha-1} E_{\alpha-\beta, \alpha}( -\gamma_1 t^{\alpha-\beta} ) \\
+ c_2 t^{\alpha-2} E_{\alpha-\beta, \alpha-1}( -\gamma_1 t^{\alpha-\beta} ) \\
+ \sum_{k=0}^{\infty} ( -\gamma_1 )^k k^{(\alpha-\beta)+\alpha} h(t) \right), \tag{30}
\]

is the solution of time-fractional telegraph equation.

If \( h(t) = t^p \) where \( p \geq 0 \) then

\[
u(x, t) = e^{i\omega} \left( c_1 t^{\alpha-1} E_{\alpha-\beta, \alpha}( -\gamma_1 t^{\alpha-\beta} ) \\
+ c_2 t^{\alpha-2} E_{\alpha-\beta, \alpha-1}( -\gamma_1 t^{\alpha-\beta} ) \\
+ \Gamma(p+1) t^{\alpha+p} E_{\alpha-\beta, \alpha+p+1}( -\gamma_1 t^{\alpha-\beta} ) \right). \tag{31}
\]

Choosing suitable values, the figures for Equation (31) have been displayed in Figures 1 and 2.

Example 2. If there exists a class of solutions \( g(x, t, u) = c \) to the Equation (19) admitting group Equation (7), \( g(x, \tilde{t}, \tilde{u}) = \tilde{c} \) defines a solution too \([39]\), then we have

\[
V g(x, t, u) = 1, \tag{32}
\]

Now we consider the one-parameter group generated by \( V_2 = k(x, t) \partial / \partial u \) of the Equation (19), where \( k \) satisfies the Equation (25). According to Equation (32), we can write

\[
k(x, t) g_{\tilde{u}} = 1. \tag{33}
\]

The solution of this equation can be written in the form

\[
g(x, t, u) = \frac{u}{k(x, t)} + \frac{w(x, t)}{k(x, t)} = c, \tag{34}
\]

so

\[
u = c k(x, t) - w(x, t). \tag{35}
\]

If we assume, for instance, \( f(x, t) = w(x, t) = z(x) h(t) \) therefore
the reduced fractional ODE is

\[ u = ck(x,t) - z(x)h(t). \]  

Substituting this solution into Equation (19), thus according to Equation (25) and assuming that

\[ \gamma_3 z''(x) + c\gamma_1 z'(x) - (\gamma_2 + c\gamma_2 + 1)z(x) = 0, \]  

the reduced fractional ODE is

\[ D_{\alpha}^{\gamma_1}h(t) + \gamma_1 D_{\beta}^{\gamma_1}h(t) = 0. \]  

The solution of this equation can be written in terms of Mittag-Leffler functions

\[ h(t) = c_1 t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) + c_4 t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta}), \]  

where \( c_1, c_4 \) are arbitrary constants. According to Equation (37), if we assume that \( \Delta = (c\rho_1)^2 + 4\gamma_3(c\rho_2 + 1) \) then we determine the solution of Equation (19) in the following cases:

If \( \Delta > 0 \), then we have

\[ u(x,t) = ck(x,t) - [c_5 e^{m_1 x} + c_6 e^{m_2 x}]\left[ c_1 t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) + c_4 t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta}) \right], \]  

where \( m_1, m_2 \) denote the real roots to the auxiliary equation of Equation (40).

If \( \Delta = 0 \), then we have

\[ u(x,t) = ck(x,t) - \left[ (c_7 + c_8 x)e^{m_1 t} \right] \left[ c_1 t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) + c_4 t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta}) \right], \]  

where \( m \) denote the repeated real root to the auxiliary equation of Equation (37).

If \( \Delta < 0 \), then we have

\[ u(x,t) = ck(x,t) - \left[ \phi^{(\alpha-\beta)}(c\cos \theta x + c\theta \sin \theta x) \right] \left[ c_1 t^{\alpha-1}E_{\alpha-\beta,\alpha}(-\gamma_1 t^{\alpha-\beta}) + c_4 t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-\gamma_1 t^{\alpha-\beta}) \right], \]  

where \( \theta_1 \pm \theta_2 i \) denote the complex roots to the auxiliary equation of Equation (37) and \( c_5, \ldots, c_{10} \) are arbitrary constants.

Our next goal is now to obtain nonclassical determining equations, employing \( D_{\alpha}^{\gamma_1}V \) to Equation (19) and according to Equation (18), the nonclassical infinitesimal invariance criterion is given by

\[ \phi^{(\alpha,t)} + \gamma_1 \phi^{(\beta,t)} + \gamma_2 \phi = \gamma_3 \phi_{xx} + \xi f_x + \tau f_t, \]  

which must be satisfied whenever \( \phi \) satisfies Equation (19) and the invariant surface condition Equation (17). According to invariant surface condition, we consider two different cases: \( \xi \neq 0 \) and \( \xi = 0 \). In the case \( \xi \neq 0 \), without loss of generality, we can choose \( \xi = 1 \), then we have

\[ u_x = \phi - \tau u_t, u_{xt} = \phi_x + (\phi_u - \tau)u_t - \tau u_{tt} + \tau u_{uu}, \]
where \( u_{xx} = (\varphi_x + \varphi u_x - \varphi t) - (2\varphi u_x + \tau_x + \tau u\varphi - \tau t)u_t + 2\tau u_{tt} + \tau^2 u_{tt}, \)

\[
(45)
\]

After substituting \( \varphi^{(n, t)}, \varphi^{(n, t)} \), \( \varphi^{xx} \) into Equation (43) with \( \xi = 1 \), replacing \( D^\gamma_t u + \gamma_1 D^\beta_t u \) by \( -\gamma_2 u + \gamma_3 u_{xx} + f \) similar to classical case, and also substituting the expressions \( u_x, u_{xt}, u_{xx} \) wherever it occurs, we attain the nonclassical determining equations for the symmetry group. Now we analyze the order of the derivatives which appear. The coefficient of \( u_t, u_{xt}, u_{xx} \) is \( \tau^3 \), thus we conclude \( \tau = 0 \) or \( \tau_u = 0 \).

If \( \tau = 0 \) according to coefficient of \( D^\beta_t u \) (or \( D^\gamma_t u \)), we have

\[
\varphi(x, t, u) = \Theta(x, u) + \Xi(x, t),
\]

where \( \Theta(x, u), \Xi(x, t) \) are arbitrary functions. After substituting \( \varphi \) into the remaining term gives

\[
\varphi(x, t, u) = \theta(x)u + \Xi(x, t),
\]

where \( \theta(x) \) and \( \Xi(x, t) \) satisfy the following system

\[
\theta''(x) + 2\theta(x)\theta'(x) = 0,
\]

\[
D^\gamma_t \Xi(x, t) + \gamma_1 D^\beta_t \Xi(x, t) + \gamma_2 \Xi(x, t) - \gamma_3 \Xi_{xx}(x, t) - 2\gamma_3 \theta'(x)\Xi(x, t) + \theta(x)f - f_x = 0.
\]

(49)

To find the function \( \Xi(x, t) \), we solve equation Equation (48) thus

\[
\theta(x) = \rho_3, \text{ or } \theta(x) = \frac{1}{x + \rho_4}, \text{ or } \theta(x) = \rho_5 \frac{e^{2\varphi x} + \rho_6}{e^{2\varphi x} - \rho_6},
\]

where \( \rho_3, ..., \rho_6 \) are arbitrary constants. A comparison with Equation (24) shows that the nonclassical method is more general than the classical method for the symmetry group.

If \( \tau \neq 0 \) and \( \tau_u = 0 \) the set of determining equations for the nonclassical symmetry group of Equation (19) are incompatible.

In the case \( \xi = 0 \) and \( \tau \neq 0 \), we have

\[
u_t = \frac{\varphi}{\tau}, u_{xt} = \frac{1}{\tau^2} [(\tau u_x - \tau_u \varphi) + (\tau u_x - \tau_u \varphi)u_x],
\]

and nonclassical determining equations unfortunately is impossible.

Example 3. According to Equation (50), let \( \Theta(x) = \frac{1}{x + \rho_4} \). Assume that \( \Xi(x, t) = 0 \) and \( f(x, t) = (x + \rho_4)h(t) \). Then by considering invariant surface condition Equation (20), we have

\[
u_x = \frac{u}{x + \rho_4},
\]

whose general solution is given by

\[
u(x, t) = (x + \rho_4)g(t).
\]

(53)

Substituting of this solution into Equation (19) reduces it to following fractional ODE

\[
D^\gamma_t g(t) + \gamma_1 D^\beta_t g(t) + \gamma_2 g(t) = h(t).
\]

(54)

After solving this fractional ODE, taking into account Equation (53) we find the exact solutions of the Equation (19). For example, whenever \( \alpha = \beta \) and let \( r_1, r_2 \) denote the roots of the equation \( x^2 + \gamma_1 x + \gamma_2 = 0 \), assuming \( r_1 \neq r_2 \) we have

\[
g(t) = \frac{1}{r_1 - r_2} \left\{ c_1 t^{\beta-1} [E_{\beta, \beta}(r_1 t^\beta) - E_{\beta, \beta}(r_2 t^\beta)] + c_2 t^{\beta-2} [E_{\beta, \beta-1}(r_1 t^\beta) - E_{\beta, \beta-1}(r_2 t^\beta)] + \int_0^t (t - \tau)^{\beta-1} \left[ E_{\beta, \beta}[r_1 (t - \tau)^\beta] - E_{\beta, \beta}[r_2 (t - \tau)^\beta] \right] h(\tau) d\tau \right\}.
\]

(55)

so

\[
u(x, t) = \frac{x + \rho_4 \left\{ c_1 t^{\beta-1} [E_{\beta, \beta}(r_1 t^\beta) - E_{\beta, \beta}(r_2 t^\beta)] + c_2 t^{\beta-2} [E_{\beta, \beta-1}(r_1 t^\beta) - E_{\beta, \beta-1}(r_2 t^\beta)] + \int_0^t (t - \tau)^{\beta-1} \left[ E_{\beta, \beta}[r_1 (t - \tau)^\beta] - E_{\beta, \beta}[r_2 (t - \tau)^\beta] \right] h(\tau) d\tau \right\}}{r_1 - r_2}.
\]

(56)

If \( r_1 = r_2 \), then

\[
g(t) = \frac{1}{r_1} \left\{ c_1 t^{\beta-1} [E_{\beta, \beta}(r_1 t^\beta) - E_{\beta, \beta}(r_2 t^\beta)] + c_2 t^{\beta-2} [E_{\beta, \beta-1}(r_1 t^\beta) - E_{\beta, \beta-1}(r_2 t^\beta)] + \int_0^t (t - \tau)^{\beta-1} \left[ E_{\beta, \beta}[r_1 (t - \tau)^\beta] - E_{\beta, \beta}[r_2 (t - \tau)^\beta] \right] h(\tau) d\tau \right\}.
\]

(57)
where $c_1, c_2$ are arbitrary constants.

Example 4. Suppose that $\theta(x) = \rho_5 \frac{\partial^3}{\partial y^3} + \rho_6 \Xi(x, t) = 0$ and

$$f(x, t) = \exp \left( \int \theta(x) dx \right) h(t) = (e^{\rho_5 x} - \rho_6 e^{-\rho_5 x}) h(t).$$

(58)

Then in view of the invariant surface condition, we have $u_x = \theta(x) u$. Differentiating this equation with respect to $x$ leads to

$$u_{xx} = (\theta'(x) + \Theta^2(x)) u = \rho_5^2 u.$$  

(59)

Thus, the general solution of the equation is

$$u(x, t) = e^{-\rho_5 x} F(t) + e^{\rho_5 x} G(t),$$

(60)

where $F$ and $G$ are arbitrary functions. Substituting into Equation (19), after some simplifications, we find the reduced fractional ODEs

$$D_t^\alpha F(t) + \gamma_1 D_t^\beta F(t) + (\gamma_2 - \gamma_3 \rho_5^2) F(t) = -\rho_6 h(t),$$

(61)

$$D_t^\alpha G(t) + \gamma_1 D_t^\beta G(t) + (\gamma_2 - \gamma_3 \rho_5^2) G(t) = h(t).$$

(62)

We can solve these equations similar to previous examples and obtain the exact solutions to Equation (19).

4. Conservation Laws

In the current section, we will focus on the conservation laws of Equation (19). A vector $C = (C^x, C^\tau)$ is called a conserved vector for Equation (19), if it fulfills the conservation equation

$$\mathcal{D}_x (C^x) + \mathcal{D}_t (C^\tau) |_{(3.1)} = 0,$$

(63)

where the conserved density $C^x$ and the spatial flux $C^\tau$ are functions of $x, t, u$, integer-order derivatives, fractional integrals and fractional derivatives of $u$. Equation (63) is called a conservation law for Equation (19). The new conservation theorem proposed by Ibragimov [42] provides a method to construct conservation laws for differential equations. Based on this theorem, the Lagrangian formal for Equation (19) can be introduced as follows:

$$\mathcal{L} = v(x, t) \left[ D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u - \gamma_3 u_{xx} - f(x, t) \right],$$

(64)

where $v(x, t)$ is a new dependent variable. Thus the functional of Equation (19) through the formal Lagrangian Equation (64) is presented as follows:

$$\mathcal{L}[u] = \int_0^T \int_0^1 v(x, t) \left[ D_t^\alpha u + \gamma_1 D_t^\beta u + \gamma_2 u - \gamma_3 u_{xx} - f(x, t) \right] dx dt.$$  

(65)

A necessary condition for functional $\mathcal{L}[u]$ to have an extremum is

$$\frac{\delta \mathcal{L}}{\delta u} = 0.$$  

(66)

Here the Euler–Lagrange operator can be represented in the form

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \frac{C}{t} \frac{\partial}{\partial D_t^\alpha u} + \frac{C}{t} \frac{\partial}{\partial D_t^\beta u} - \mathcal{D}_x \frac{\partial}{\partial u_x},$$

(67)

where $C_t D_t^\alpha$ and $C_t D_t^\beta$ are the adjoint operators of $D_t^\alpha u$, $D_t^\beta u$ and are defined by

$$C_t D_t^\alpha f(t) = \frac{(-1)^m}{\Gamma(m-\gamma)} \int_0^T (s-t)^{m-\gamma-1} D_t^\alpha f(s) ds,$$

(68)

$$m - 1 < \gamma < m.$$

Thus according to Equation (66), the adjoint equation to Equation (19) is given by

$$C_t D_t^\alpha v + \gamma_1 C_t D_t^\beta v + \gamma_3 v - \gamma_3 v_{xx} = 0.$$  

(69)

We also use fundamental identity [42–44]

$$\mathcal{P}_r^{(\alpha, \tau)} V + \mathcal{D}_x (\xi) I + \mathcal{D}_t (\xi) I = W \frac{\delta}{\delta u} + \mathcal{D}_x (N^1) + \mathcal{D}_t (N^2),$$

(70)

where $N^1$ and $N^2$ represent the Noether’s operators, $I$ is the identity operator, $\mathcal{P}_r^{(\alpha, \tau)} V$ is given by Equation (20) and we can rewrite in the following form

$$\mathcal{P}_r^{(\alpha, \tau)} V = \tau \mathcal{D}_x + \xi \mathcal{D}_x + W \frac{\delta}{\delta u} + \mathcal{D}_x (W) \frac{\partial}{\partial u_x}$$

$$+ \mathcal{D}_t (W) \frac{\partial}{\partial u_x} + \mathcal{D}_t (W) \frac{\partial}{\partial (D_t^\alpha u)},$$

(71)

where $W = \varphi - \tau u - \xi u_x$ and the same as Equation (16), $\mathcal{D}_x$, $\mathcal{D}_t$ are defined by
\[ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_{xx}} + D_t^{p-1} u \frac{\partial}{\partial (D_t^p u)} + D_t^{p+1} u \frac{\partial}{\partial (D_t^p u)}. \]  

(72)

\[ D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_{xx}} + D_t^p u_x \frac{\partial}{\partial (D_t^p u)} + D_t^p u \frac{\partial}{\partial (D_t^p u)}. \]  

(73)

Therefore according to Equation (70), we have

\[ p_{t(x)} V + D_t(\tau) I + D_x(\xi) I - W \frac{\delta}{\delta u} \]

\[ = D_t \left[ \tau I + D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} - D_t^{p-2}(W) D_t \frac{\partial}{\partial (D_t^p u)} + D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} - f \left( W, \frac{\partial}{\partial (D_t^p u)} \right) \right] \]

\[ + D_x \left[ \delta I - W \frac{\partial}{\partial u_{xx}} + D_x(W) \frac{\partial}{\partial u_{xx}} \right]. \]  

(74)

where

\[ D_t(\tau) I + \tau D_t = D_t(\tau I) \]  

(75)

\[ D_t^p(W) = \frac{\partial}{\partial (D_t^p u)} - W \frac{\partial}{\partial u} \]

\[ = D_t \left[ D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} - D_t^{p-2}(W) D_t \frac{\partial}{\partial (D_t^p u)} \right] \]

\[ + D_t \left[ D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} - f \left( W, \frac{\partial}{\partial (D_t^p u)} \right) \right] \]  

(76)

\[ D_t^p(W) = \frac{\partial}{\partial (D_t^p u)} - W \frac{\partial}{\partial u} \]

\[ = D_t \left[ D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} + f \left( W, \frac{\partial}{\partial (D_t^p u)} \right) \right] \]

\[ + D_t \left[ D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} \right] \]  

(77)

\[ D_t^2(W) \frac{\partial}{\partial u_{xx}} - W D_t^2 \frac{\partial}{\partial u_{xx}} = D_x \left[ -W D_x \frac{\partial}{\partial u_{xx}} + D_x(W) \frac{\partial}{\partial u_{xx}} \right]. \]  

(78)

and the integral operator \( f \) is defined by

\[ f(t, g(t)) = \frac{1}{\Gamma(m - \alpha)} \int_0^T \int_t^T (s - \nu)^{m-\alpha-1} f(\nu) g(s) ds d\nu. \]  

(79)

This integral satisfies the property

\[ f(t, g(t)) = f(t) \left[ t^{m-\gamma} - g(t) \right] t^{m-\gamma}, m - 1 < \gamma < m. \]  

(80)

So for Equation (19), the operator \( N^t \) and \( N^x \) can be written by the following formula

\[ N^t = \tau I + D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} - D_t^{p-2}(W) D_t \frac{\partial}{\partial (D_t^p u)} \]

\[ + D_t^{p-1}(W) \frac{\partial}{\partial (D_t^p u)} - f \left( W, \frac{\partial}{\partial (D_t^p u)} \right) \]

\[ + f \left( W, \frac{\partial}{\partial (D_t^p u)} \right) \]  

(81)

\[ N^x = \xi I - W D_x \frac{\partial}{\partial u_{xx}} + D_x(W) \frac{\partial}{\partial u_{xx}}. \]  

(82)

Since for all infinitesimal generator \( V \) admitted by Equation (19) and its any solution

\[ p_{t(x)} \mathcal{L} + D_t(\tau) \mathcal{L} + D_x(\xi) \mathcal{L} - W \frac{\delta \mathcal{L}}{\delta u} \]  

(3.1) = 0,

(83)

therefore, in view of Equation (70), the conservation law of Equation (19) can be written as follows:

\[ D_t(N^t \mathcal{L}) + D_x(N^x \mathcal{L}) \]  

(3.1) = 0.  

(84)

Now, we present the components of conserved vectors for Equation (19). Using Equations (81) and (82), we have
\[ C' = N_i \mathcal{L} = \tau \mathcal{L} + \partial_{t}^{q-1} (W) \nu - \partial_{x}^{q-2} (W) \nu + \gamma_1 \partial_{t}^{q-1} (W) \nu - J(W, V_{\nu}) + \gamma_1 J(W, V_{\nu}), \]

\[ C^* = N_i \mathcal{L} = \xi \mathcal{L} + \gamma_3 W_{V_{x}} - \gamma_3 \partial_{x} (W) \nu, \]  \tag{85}

where \( \nu(x, t) \) is an arbitrary nontrivial solution of Equation (69) and for different infinitesimal generator \( V_i, i = 1, \ldots, 3, \) of Equation (19), \( W_i \) are defined as follows:

\[ W_1 = -u_x, W_2 = u, W_3 = k(x, t). \]  \tag{86}

5. Conclusion

In this paper, the methods of classical and nonclassical Lie symmetry analysis are applied to the time-fractional telegraph equation. The Lie symmetry method has had an enormous success when applied to a diverse range of differential equations. The idea of looking for the exact solutions and conservation law for the time-fractional telegraph equation by Ibragimov's conservation theorem adapted to the time-fractional PDEs. The results shown in this work demonstrate how the nonclassical approaches grab the new exact solutions to fractional partial differential equations. Moreover, we constructed conservation laws for the time-fractional telegraph equation by Ibragimov's conservation theorem adapted to the time-fractional PDEs.

Data Availability

Data supporting this research article are available on request.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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