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Research Article

Ion Acoustic Solitary Wave Solutions to mKdV-ZK Model in Homogeneous Magnetized Plasma

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In this exploration, we reflect on the wave transmission of three-dimensional (3D) nonlinear electron–positron magnetized plasma, counting both hot as well as cold ion. Treated equation acquiesces to nonlinear-modified KdV-Zakharov–Kuznetsov (mKdV-ZK) dynamical 3D form. The model is integrated by the φ^6-model expansion scheme and invented few families of ion acoustic solitonic propagation results in term of Jacobi elliptic functions. Various shock waves, bullet like bright soliton, dark soliton, singular soliton, as well as periodic signal solutions, are formed from the Jacobi elliptic solution for different parametric constraints. Some of the solutions are illustrated graphically and analyzed width and height due to change of exist parameters in the solutions. Figures are provided to explain the wave natures and effects of nonlinear and fractional parameters are presented in the same two-dimensional (2D) plots.

1. Introduction

The exploration on nonlinear evolution models (NEMs) has a verity of significant functioning in science and engineering applications. It is proverbial that NEMs are frequently used to illustrate various imperative happenings and progressions in abundant technical scientific fields, resembling epidemiology, stratified internal waves, meteorology, solid-state physics, plasma physics, ocean engineering, ion acoustic, fiber optics, and more others [1–4]. Therefore, the study of accurate exact solutions to realize internal properties of NEMs demonstrates a crucial task for understanding of the majority nonlinear substantial happenings or acquiring novel occurrences. More scientists expended efforts to release inventive proficient procedures designed for explanation of interior properties of NEMs amid with constant coefficients has been established such as a distinct arithmetic structure [5], IRM-CG technique [6], transformed rational function method [7], fractional residual technique [8], new multistage procedure [9], new analytical method [10], extended tanh scheme [11], Hirota-bilinear scheme [12–14], multi exp-expansion technique [15, 16], the Kudryashov and the extended sine-Gordon schemes [17], variable separation method [18], MSE method [19], the nonlinear capacity [20], Jacobi elliptic function [21], the pitchfork bifurcation [22], ansatz method [23], higher order rogue wave [24], fractional natural decomposition method [25], generalized exponential rational function scheme [26], numerical and three analytic schemes [27], $1/G'$-expansion scheme [28], the φ^6-model method [29, 30], and so on. All of the above techniques provide sinusoidal and hyperbolic results. To acquire the solutions, which cover all sinusoidal and hyperbolic solution even express more complex phenomena, Jacobi elliptic function [21] and the φ^6-model expansion [29, 30] schemes are exceptional. These two methods are highly important as it gives Jacobi elliptic function solutions. The sinusoidal and hyperbolic solutions are exceptional suitcases of the Jacobi elliptic function. Owing to this fact, we sport light on the $(3+1)$-D nonlinear mKdV-ZK model in bellow [31–34].

$$U_t + \rho U^2 U_x + U_{xxx} + U_{xyy} + U_{xzz} = 0, \quad (1)$$

where ρ is the nonlinear coefficient and U is the electric field potential. The model is utilized in controlling properties of dimly nonlinear ion-acoustic patterns within electron–positron magnetized plasma, counting the hot–cold mechanism of each type [30–33]. The model also arises in diverse branches of physical areas like plasma physics, optical system, fluid flows, and quantum mechanics. Due to importance of the model, huge effort has been paid on the mKdV-ZK model: Xu [31] applied an elliptic equation technique and derived only five elliptic solution; Kumer and Verma [32] found dynamics of invariant solutions via extended (G'/G)-expansion method of the model; Lu et al. [33] constructed analytical wave solutions via Lie symmetry analysis scheme; Seadawy [34] analyzed the stability and derived stable solutions of the mKdV-ZK model, etc. Dynamical scientists were considered more plasma models [35–38] to analyze plasma fluid, and represented the systems of equation into KP-Burgers [35], ZK and eZK [36], and ZK and mZK [37, 38], with proper explanations of each plasma parameters. The Jacobi elliptic function solution of mKdV-ZK model in fractional differential form is still unexplored in the literature. Owing to this fact, this research is willing to increases the complex nonlinear dynamics in term of Jacobi elliptic function of the model as well.

2. Sketch of the φ^6-Model Expansion Method

An analytical technique of deriving Jacobi elliptic solutions is a recent φ^6-model expansion scheme, which was first invented via Zayed et al. [29]. Due to its effectiveness and novel finding efficient, we aim to execute it on a nonlinear evolution equation (NLEE) whose key steps are given as follows:

Step 1: Let us reflect on the following general NLEE in the form given as follows:

$$\Re(\Theta, \Theta_x, \Theta_t, \Theta_{xx}, \Theta_{xt}, \Theta_{tt}, \ldots\ldots) = 0, \quad (2)$$

where $\Re$ is an expression of $\Theta(x,\ t)$, as well as various order partial derivatives, involving nonlinearity.

Step 2: Assembling wave adaptation relation is given as follows:

$$\Theta(x,\ t) = \Theta(\tau), \tau = \mu(x - vt), \quad (3)$$

where τ, μ, v signify traveling variable, wave number, and phase velocity, respectively. Equation (3) is reduced to Equation (2) into the nonlinear ODE as shown below:

$$\mathrm{H}(\Theta, \Theta', \Theta'' \ldots\ldots) = 0. \quad (4)$$

in which derivatives due to τ are signified with prime.

Step 3: Assume a trial solution of Equation (4) subsists and can be written in a series as follows:

$$\Theta(\tau) = \sum_{i=0}^{2N} a_i S^i(\tau), \quad (5)$$

where a_i, $(i = 0, 1, 2, \ldots\ldots, N)$ is to be evaluated. Using balance law, N can be achieved [26–28], and $S(\tau)$ suits the assisting nonlinear ordinary differential equation (NLODE):

$$\begin{aligned} S'^2(\tau) &= r_0 + r_2 S^2(\tau) + r_4 S^4(\tau) + r_6 S^6(\tau), \\ S''(\tau) &= r_2 S(\tau) + 2r_4 S^3(\tau) + 3r_6 S^5(\tau), \end{aligned} \quad (6)$$

where r_i $(i = 0, 2, 4, 6)$ is in variables that can be revealed afterward.

Step 4: An answer to Equation (6) is in the following form:

$$S(\tau) = \frac{U(\tau)}{\sqrt{fU^2(\tau) + k}}, \quad (7)$$

under setting $0 < f\, U^2(\tau) + k$ and $U(\tau)$ are elliptic function that comes from:

$$U'^2(\tau) = \ell_0 + \ell_2 U^2(\tau) + \ell_4 U^4(\tau), \quad (8)$$

where ℓ_i $(i = 0, 2, 4)$ is unspecified constants and particular values of these will be evaluated, while f and k come from:

$$\begin{aligned} f &= \frac{r_4(\ell_2 - r_2)}{(\ell_2 - r_2)^2 + 3\ell_0\ell_4 - 2\ell_2(\ell_2 - r_2)}, \\ k &= \frac{3\ell_0 r_4}{(\ell_2 - r_2)^2 + 3\ell_0\ell_4 - 2\ell_2(\ell_2 - r_2)}, \end{aligned} \quad (9)$$

with the constraint condition:

$$\begin{aligned} &r_4^2(\ell_2 - r_2)[9\ell_0\ell_4 - (\ell_2 - r_2)(2\ell_2 + r_2)] \\ &\quad + 3r_6[-\ell_2^2 + r_2^2 + 3\ell_0\ell_4]^2 = 0. \end{aligned} \quad (10)$$

Function	$m \to 1$	$m \to 0$	*Function*	$m \to 1$	$m \to 0$
$sn(\tau, m)$	$\tanh(\tau)$	$\sin(\tau)$	$ds(\tau, m)$	$\mathrm{csch}(\tau)$	$\csc(\tau)$
$cn(\tau, m)$	$\mathrm{sech}(\tau)$	$\cos(\tau)$	$sc(\tau, m)$	$\sinh(\tau)$	$\tan(\tau)$
$dn(\tau, m)$	$\mathrm{sech}(\tau)$	1	$sd(\tau, m)$	$\sinh(\tau)$	$\sin(\tau)$
$ns(\tau, m)$	$\coth(\tau)$	$\csc(\tau)$	$nc(\tau, m)$	$\cosh(\tau)$	$\sec(\tau)$
$cs(\tau, m)$	$\mathrm{csch}(\tau)$	$\cot(\tau)$	$cd(\tau, m)$	1	$\cos(\tau)$

(11)

Step 5: According to Zayed et al. [29], it is recognized that the Jacobi elliptic function solution of Equation (8) exhibits for $0<m<1$, otherwise it will be sinusoidal or hyperbolic function solutions for $m \to 0<or\ \ m \to 1$. Determination of solutions of Equation (2) will be completed taking advantages of Equations (7) and (8) into Equation (5).

3. Implementation of the φ^6-Model Expansion Scheme

Let us reflect on the well-known nonlinear evolution model mKdV-ZK EEquation (1) in [31–34], and we can convert it to an ODE with the wave mapping relation given as follows:

$$\tau = \delta_1 x + \delta_2 y + \delta_3 z + wt, \tag{12}$$

where w represents wave velocity and $\delta_1, \delta_2, \delta_3$ are wave numbers in the x, y, z directions, respectively. Accomplish Equation (10) with Equation (12) yields:

$$wU' + \rho U^2 \delta_1 U' + \delta_1^3 U''' + \delta_1 \delta_2^2 U''' + \delta_1 \delta_3^2 U''' = 0. \tag{13}$$

Integrating Equation (13) one time with respect to τ leads to:

$$wU + \rho \frac{\delta_1}{3} U^3 + (\delta_1^3 + \delta_1 \delta_2^2 + \delta_1 \delta_3^2) U'' = 0. \tag{14}$$

Making use of balance principle between U'' with U^3 gives $N = 1$. Inserting $N = 1$ into Equation (5)
we acquire:

$$U(\tau) = g_0 + g_1 S(\tau) + g_2 S^2(\tau), \tag{15}$$

where g_0, g_1, and g_2 are constants.

We attain subsequent equations in reserving Equation (15) together with Equations (6)–(14), as well as putting the coefficients of each $S^i(\tau),\ i = 0, 1, \ldots\ldots, 6$ are equal to zero.

$$\begin{aligned}
&S^0(\tau): 2g_2 r_0 \delta_1 \delta_2^2 + 2g_2 r_0 \delta_1 \delta_3^2 + 2g_2 r_0 \delta_1^3 + \frac{1}{3}\rho \delta_1^2 g_0^3 + wg_0 = 0,\\
&S^1(\tau): \rho \delta_1^2 g_0^2 g_1 + \delta_1^3 g_1 r_2 + \delta_1 \delta_2^2 g_1 r_2 + \delta_1 \delta_3^2 g_1 r_2 + wg_1 = 0,\\
&S^2(\tau): \rho \delta_1^2 g_0^2 g_2 + \rho \delta_1^2 g_0 g_1^2 + 4\delta_1^3 g_2 r_2 + 4\delta_1 \delta_2^2 g_2 r_2 + 4\delta_1 \delta_3^2 g_2 r_2\\
&\qquad + wg_2 = 0,\\
&S^3(\tau): \frac{1}{3}\rho \delta_1^2 g_1^3 + 2g_1 r_4 \delta_1^3 + 2\rho \delta_1^2 g_0 g_1 g_2 + 2g_1 r_4 \delta_1 \delta_2^2\\
&\qquad + 2g_1 r_4 \delta_1 \delta_3^2 = 0,\\
&S^4(\tau): \rho \delta_1^2 g_0 g_2^2 + \rho \delta_1^2 g_1^2 g_2 + 6\delta_1^3 g_2 r_4 + 6\delta_1 \delta_2^2 g_2 r_4\\
&\qquad + 6\delta_1 \delta_3^2 g_2 r_4 = 0,\\
&S^5(\tau): \rho \delta_1^2 g_1 g_2^2 + 3\delta_1^3 g_1 r_6 + 3\delta_1 \delta_2^2 g_1 r_6 + 3\delta_1 \delta_3^2 g_1 r_6 = 0,\\
&S^6(\tau): \frac{1}{3}\rho \delta_1^2 g_2^3 + 8\delta_1^3 g_2 r_6 + 8\delta_1 \delta_3^2 g_2 r_6 + 8\delta_1 \delta_2^2 g_2 r_6 = 0.
\end{aligned} \tag{16}$$

Solving the above equations, lead the subsequently results.

Set 1:

$$\begin{aligned}
&w = -\delta_1^3 r_2 - \delta_1 \delta_2^2 r_2 - \delta_1 \delta_3^2 r_2,\ \ g_0 = 0, g_1 = g_1,\\
&g_2 = 0, r_2 = r_2,\ \ r_4 = -\frac{1}{6}\frac{\rho \delta_1 g_1^2}{\delta_1^2 + \delta_2^2 + \delta_3^2},\ \ r_6 = 0.
\end{aligned} \tag{17}$$

Set 2:

$$\begin{aligned}
&w = -\frac{1}{3}\frac{\rho^2 \delta_1^2 g_0^4 - 36\delta_1^4 r_0 r_4 - 72\delta_1^2 \delta_2^2 r_0 r_4 - 72\delta_1^2 \delta_3^2 r_0 r_4 - 36\delta_2^4 r_0 r_4 - 72\delta_2^2 \delta_3^2 r_0 r_4 - 36\delta_3^4 r_0 r_4}{\rho g_0^2},\\
&g_0 = g_0,\ \ g_1 = 0, g_2 = \frac{-6r_4(\delta_1^2 + \delta_2^2 + \delta_3^2)}{\rho g_0 \delta_1},\\
&r_2 = -\frac{1}{6}\frac{\rho^2 \delta_1^2 g_0^4 + 18\delta_1^4 r_0 r_4 + 36\delta_1^2 \delta_2^2 r_0 r_4 + 36\delta_1^2 \delta_3^2 r_0 r + 18\delta_2^4 r_0 r_4 + 36\delta_2^2 \delta_3^2 r_0 r_4 + 18\delta_3^4 r_0 r}{\rho \delta_1 g_0^2(\delta_1^2 + \delta_2^2 + \delta_3^2)},\\
&r_4 = r_4,\ \ r_6 = -\frac{3}{2}\frac{(\delta_1^2 + \delta_2^2 + \delta_3^2) r_4^2}{\rho \delta_1 g_0^2}.
\end{aligned} \tag{18}$$

Using the constraints *Set 1*, as well as combining Equations (7) and (15) along with the Jacobi elliptic functions from the previous table, the following exact solutions are uttered.

1. If $\ell_0 = 1$, $\ell_2 = -(1 + m^2)$, $\ell_4 = m^2, 0<m<1$, then $U(\tau) = sn(\tau, m)$ or $U(\tau) = cd(\tau, m)$, therefore, solution is:

$$U_{1,1} = g_1 \left[\frac{sn(\tau, m)}{\sqrt{f(sn(\tau, m))^2 + k}}\right], \text{or} \tag{19}$$

$$U_{1,2} = g_1 \left[\frac{cd(\tau, m)}{\sqrt{f(cd(\tau, m))^2 + k}}\right], \tag{20}$$

such that $\tau=\delta_1 x+\delta_2 y+\delta_3 z+wt$ and f, k are specified by:

$$f=\frac{(1+m^2+r_2)r_4}{1-m^2+m^4-r_2^2}, k=\frac{-3r_4}{1-m^2+m^4-r_2^2}, \tag{21}$$

under the constraint $-r_4^2(-1-m^2-r_2)(-1+2m^2-r_2)(-2+m^2+r_2)=0$.

The above Equation (19) turns into a shock wave solution for $m\rightarrow 1$, given as follows:

$$U_{1,3}=g_1\left[\frac{\sqrt{-1+r_2^2}\tan\text{h}(\tau)}{\sqrt{r_4(3-(2+r_2)\tan\text{h}^2(\tau))}}\right], \tag{22}$$

such that $-r_4^2(2+r_2)[-1+r_2]^2=0$.

The above Equation (19) turns into a periodic solution for $m\rightarrow 0$, given as follows:

$$U_{1,4}=g_1\left[\frac{\sqrt{-1+r_2^2}\sin(\tau)}{\sqrt{r_4(3-(2+r_2)\sin^2(\tau))}}\right], \tag{23}$$

such that $r_4^2(-1+r_2)[(-2-r_2)(-1+r_2)]=0$.

2. If $\ell_0=1-m^2$, $\ell_2=2m^2-1$, $\ell_4=-m^2$, $m\in(0,1)$, afterward $U(\tau)=cn(\tau,m)$ and solution yields:

$$U_{1,5}=g_1\left[\frac{cn(\tau,m)}{\sqrt{f(cn(\tau,m))^2+k}}\right], \tag{24}$$

in which f, k are come from $f=(-1+2m^2-r_2)r_4/1-m^2+m^4-r_2^2, k=3(-1+m^2)r_4/1-m^2+m^4-r_2^2$, under constraint $r_4^2(-1+2m^2-r_2)[(-2+m^2+r_2)(1+m^2+r_2)]=0$.

The above result Equation (24) turns into a periodic solution for $m\rightarrow 0$, given as follows:

$$U_{1,6}=g_1\left[\frac{\sqrt{-1+r_2^2}\cos(\tau)}{\sqrt{(3-(1+r_2)\cos^2(\tau))r_4}}\right], \tag{25}$$

such that $r_4^2(-1-r_2)[(-2+r_2)(1+r_2)]=0$.

3. If, $\ell_0=m^2-1$, $\ell_2=2-m^2$, $\ell_4=-1, 0<m<1$, then $U(\tau)=dn(\tau,m)$ and solution yields:

$$U_{1,7}=g_1\left[\frac{dn(\tau,m)}{\sqrt{f(dn(\tau,m))^2+k}}\right], \tag{26}$$

in which f, k are specified with $f=(-2+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k=-3(-1+m^2)r_4/1-m^2+m^4-r_2^2$, under the constraint $r_4^2(2-m^2-r_2)[-(-1+2m^2+r_2)(1+m^2+r_2)]=0$.

4. If, $\ell_0=m^2$, $\ell_2=-(1+m^2)$, $\ell_4=1, m\in(0,1)$, then $U(\tau)=ns(\tau,m)$ or $U(\tau)=dc(\tau,m)$ and solution yields:

$$U_{1,8}=g_1\left[\frac{ns(\tau,m)}{\sqrt{f(ns(\tau,m))^2+k}}\right], \tag{27}$$

or

$$U_{1,9}=g_1\left[\frac{dc(\tau,m)}{\sqrt{f(dc(\tau,m))^2+k}}\right], \tag{28}$$

with f, k are specified via $f=(1+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k=-3m^2r_4/1-m^2+m^4-r_2^2$, with condition $r_4^2(-1-m^2-r_2)[-(-1+2m^2-r_2)(-2+m^2+r_2)]=0$.

The above result Equation (27) turns into a singular dark soliton $m\rightarrow 1$, given as follows:

$$U_{1,10}=g_1\left[\frac{\sqrt{-1+r_2^2}\cot\text{h}(\tau)}{\sqrt{-(-1+r_2+(2+r_2)\operatorname{csc}h^2(\tau)r_4)}}\right], \tag{29}$$

such that $r_4^2(-2-r_2)[-1+r_2]^2=0$.

5. If $\ell_0=-m^2$, $\ell_2=2m^2-1$, $\ell_4=1-m^2, 0<m<1$, then $U(\tau)=nc(\tau,m)$; therefore, solution is:

$$U_{1,11}=g_1\left[\frac{nc(\tau,m)}{\sqrt{f(nc(\tau,m))^2+k}}\right], \tag{30}$$

with f, k are specified via $f=-(-1+2m^2-r_2)r_4/1-m2+m^4-r_2^2, k=3m^2r_4/1-m^2+m^4-r_2^2$, with condition $r_4^2(-1+2m^2-r_2)[(-2+m^2+r_2)(1+m^2+r_2)]=0$.

The above result turns into a solitary wave solution for $m\rightarrow 1$, given as follows:

$$U_{1,12}=g_1\left[\frac{\sqrt{-1+r_2^2}\cos\text{h}(\tau)}{\sqrt{(-3+(1-r_2)\cos\text{h}^2(\tau))r_4}}\right], \tag{31}$$

such that $r_4^2(1-r_2)[-2+r_2+r_2^2]=0$.

6. If $\ell_0=-1$, $\ell_2=2-m^2$, $\ell_4=-(1-m^2)$, $m\in(0,1)$, then $U(\tau)=nd(\tau,m)$; therefore, result is:

$$U_{1,13}=g_1\left[\frac{nd(\tau,m)}{\sqrt{f(nd(\tau,m))^2+k}}\right],\tag{32}$$

with f,k are specified via $f=(-2+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k=3r_4/1-m^2+m^4-r_2^2,$ with condition $r_4^2(2-m^2-r_2)[-(-1+2m^2-r_2)(1+m^2+r_2)]=0.$

7. If, $\ell_0=1,\ \ell_2=2-m^2,\ \ell_4=1-m^2,\ m\in(0,1),$ afterward $U(\tau)=sc(\tau,m)$; therefore, result is:

$$U_{1,14}=g_1\left[\frac{sc(\tau,m)}{\sqrt{f(sc(\tau,m))^2+k}}\right],\tag{33}$$

with f and k are specified via $f=(-2+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k=-3r_4/1-m^2+m^4-r_2^2,$ with condition $r_4^2(2-m^2-r_2)[-(-1+2m^2-r_2)(1+m^2+r_2)]=0.$

The above solution turns into a solitary wave solution for $m\to1,$ given as follows:

$$U_{1,15}=g_1\left[\frac{\sqrt{-1+r_2^2}\,\sin\mathrm{h}(\tau)}{\sqrt{(3+(1-r_2)\sin\mathrm{h}^2(\tau))r_4}}\right],\tag{34}$$

such that $r_4^2(1-r_2)[-2+r_2+r_2^2]=0.$

The above solution turns into a periodic solution for $m\to0,$ given as follows:

$$U_{1,16}=g_1\left[\frac{\sqrt{-1+r_2^2}\,\tan(\tau)}{\sqrt{(3-(-2+r_2)\tan^2(\tau))r_4}}\right],\tag{35}$$

such that $r_4^2(2-r_2)[1+r_2]^2=0.$

8. If $\ell_0=1,\ \ell_2=2m^2-1,\ \ell_4=-m^2(1-m^2),\ m\in(0,1),$ afterward $U(\tau)=sd(\tau,m)$; therefore, solution is:

$$U_{1,17}=g_1\left[\frac{sd(\tau,m)}{\sqrt{f(sd(\tau,m))^2+k}}\right],\tag{36}$$

with f,k are specified via $f=(-1+2m^2-r_2)r_4/1-m^2+m^4-r_2^2, k=-3r_4/1-m^2+m^4-r_2^2,$ with condition $r_4^2(-1+2m^2-r_2)[(-2+m^2+r_2)(1+m^2+r_2)]=0.$

9. If $\ell_0=1-m^2,\ \ell_2=2-m^2,\ \ell_4=1, m\in(0,1),$ afterward $U(\tau)=cs(\tau,m)$; therefore, result is:

$$U_{1,18}=g_1\left[\frac{cs(\tau,m)}{\sqrt{f(cs(\tau,m))^2+k}}\right],\tag{37}$$

here f and k are specified by $f=(-2+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k=3(-1+m^2)r_4/1-m^2+m^4-r_2^2,$ with condition $r_4^2(2-m^2-r_2)[-(-1+2m^2-r_2)(1+m^2+r_2)]=0.$

The above Equation (37) turns into a periodic solution for $m\to0,$ given as follows:

$$U_{1,19}=g_1\left[\frac{\sqrt{-1+r_2^2}\cot(\tau)}{\sqrt{(3+(2-r_2)\cot^2(\tau))r_4}}\right],\tag{38}$$

such that $r_4^2(2-r_2)[(1+r_2)^2]=0.$

10. If $\ell_0=-m^2(1-m^2),\ \ell_2=2m^2-1,\ \ell_4=1, m\in(0,1),$ afterward $U(\tau)=ds(\tau,m)$; therefore, solution is:

$$U_{1,20}=g_1\left[\frac{ds(\tau,m)}{\sqrt{f(ds(\tau,m))^2+k}}\right],\tag{39}$$

here f,k are specified via $f=-(-1+2m^2-r_2)r_4/1-m^2+m^4-r_2^2, k=-3m^2(-1+m^2)r_4/1-m^2+m^4-r_2^2,$ such that $r_4^2(-1+2m^2-r_2)[(-2+m^2+r_2)(1+m^2+r_2)]=0.$

11. If $\ell_0=1-m2/4,\ \ell_2=1+m^2/2,\ \ell_4=1-m^2/4,\ m\in(0,1),$ afterward $U(\tau)=nc(\tau,m)\pm sc(\tau,m)$

or $U(\tau)=cn(\tau,m)\{1\pm sn(\tau,\ m)\}^{-1}$; therefore, solution is:

$$U_{1,21}=g_1\left[\frac{nc(\tau,m)\pm sc(\tau,m)}{\sqrt{f(nc(\tau,m)\pm sc(\tau,m))^2+k}}\right],\tag{40}$$

or

$$U_{1,22}=g_1\left[\frac{cn(\tau,m)}{\sqrt{fcn^2(\tau,m)+k\{1\pm sn(\tau,m)\}^2}}\right],\tag{41}$$

with f and k are specified by $f=-8(1+m^2-2r_2)r_4/1+14m^2+m^4-16r_2^2, k=12(-1+m^2)r_4/1+14m^2+m^4-16r_2^2,$ such that $r_4^2(1/2(1+m^2-2r_2)[1/16(1-6m+m^2+4r_2)(1+6m+m^2)+4r_2)]=0.$

The above solution turns into a periodic solution for $m\to0,$ given as follows:

$$U_{1,23}=g_1\left[\frac{(\sec(\tau)+\tan(\tau))\sqrt{(16r_2^2-1)(-1+\sin(\tau))}}{2\sqrt{(-5+4r_2+(1+4r_2)\sin(\tau))r_4}}\right], \quad (42)$$

or

$$U_{1,24}=g_1\left[\frac{\sqrt{(16r_2^2-1)}\cos\tau}{2\sqrt{((2-4r_2)\cos^2\tau+3(1+\sin(\tau))^2)r_4}}\right], \quad (43)$$

such that $r_4^2(1/2-r_2)[1/16(1+4r_2)^2]=0$.

12. If $\ell_0=-(1-m^2)^2/4$, $\ell_2=1+m^2/2$, $\ell_4=-1/4$, $m\in(0,1)$, afterward $U(\tau)=m\,cn(\tau,m)\pm dn(\tau,m)$; therefore, solution is given as follows:

$$U_{1,25}=g_1\left[\frac{mcn(\tau,m)\pm dn(\tau,m)}{\sqrt{f(mcn(\tau,m)\pm dn(\tau,m))^2+k}}\right], \quad (44)$$

with f, k are specified via $f=-8(1+m^2-2r_2)r_4/1+14m^2+m^4-16r_2^2$, $k=12(-1+m^2)^2r_4/1+14m^2+m^4-16r_2^2$, such that $r_4^2(1/2(1+m^2-2r_2)[1/16(1-6m+m^2+4r_2)(1+6m+m^2)+4r_2)]=0$.

13. If $\ell_0=1/4$, $\ell_2=1-2m^2/2$, $\ell_4=1/4, 0<m<1$, then $U(\tau)=sn(\tau,m)/1\pm cn(\tau,m)$; therefore, solution is given as follows:

$$U_{1,26}=g_1\left[\frac{sn(\tau,m)}{\sqrt{fsn^2(\tau,m)+k\{1\pm cn(\tau,m)\}^2}}\right], \quad (45)$$

with f and k are specified by $f=8(-1+2m^2+2r_2)r_4/1-16m^2+16m^4-16r_2^2$, $k=-12r_4/1-16m^2+16m^4-16r_2^2$, such that $r_4^2(1/2-m^2-r_2)[1/16+2m^2-2m^4+(1/2-m^2)r_2+r_2^2]=0$.

The above solution turns into a dark bright solitary wave solution for $m\to1$, given as follows:

$$U_{1,27}=g_1\left[\frac{\tan\mathrm{h}(\tau)\sqrt{-1+16r_2^2}}{2\sqrt{(3(1+\sec h(\tau))^2-2(1+2r_2)(\tan\mathrm{h}(\tau))^2)r_4}}\right], \quad (46)$$

such that $r_4^2(-1/2-r_2)[-1/16(1-4r_2)^2]=0$.

The above solution turns into a periodic for $m\to0$, given as follows:

$$U_{1,28}=g_1\left[\frac{\sin(\tau)\sqrt{-1+16r_2^2}}{2\sqrt{(3(1+\cos(\tau))^2+2(1-2r_2)\sin^2(\tau))r_4}}\right], \quad (47)$$

such that $r_4^2(1/2-r_2)[1/16(1+4r_2)^2]=0$.

14. If $\ell_0=1/4$, $\ell_2=1+m^2/2$, $\ell_4=(1-m^2)^2/4$, $m\in(0,1)$, afterward $U(\tau)=sn(\tau,m)/cn(\tau,m)\pm dn(\tau,m)$;

therefore, result is given as follows:

$$U_{1,29}=g_1\left[\frac{sn(\tau,m)}{\sqrt{fsn^2(\tau,m)+k(cn(\tau,m)\pm dn(\tau,m))^2}}\right], \quad (48)$$

here f and k are specified via $f=-8(1+m^2-2r_2)r_4/1+14m^2+m^4-16r_2^2$, $k=-12r_4/1+14m^2+m^4-16r_2^2$, such that $r_4^2(1/2(1+m^2-2r_2))[1/16(1-6m+m^2+4r_2)(1+6m+m^2)+4r_2])=0$.

The above solution turns into a solitary for $m\to1$, given as follows:

$$U_{1,30}=g_1\left[\frac{\sin\mathrm{h}(\tau)\sqrt{-1+r_2^2}}{\sqrt{(3+(1-r_2)\sin\mathrm{h}^2(\tau))r_4}}\right], \quad (49)$$

such that $r_4^2(1-r_2)[-2+r_2+r_2^2])=0$.

The above solution turns into a periodic solution for $m\to0$, given as follows:

$$U_{1,31}=g_1\left[\frac{\sin(\tau)\sqrt{-1+16r_2^2}}{2\sqrt{(3(1+\cos(\tau))^2+2(1-2r_2)\sin^2(\tau))r_4}}\right], \quad (50)$$

such that $r_4^2(1/2-r_2)[1/16(1+4r_2)^2]=0$.

Combining *Set 2* with Equations (7) and (15) together with elliptic functions from the previous table, one reach exact results of Equation (10) in the following:

1. If $\ell_0=1$, $\ell_2=-(1+m^2)$, $\ell_4=m^2$, $m\in(0,1)$, afterward $U(\tau)=sn(\tau,\ m)$ or $U(\tau)=cd(\tau,\ m)$ leads solution as:

$$U_{2,1}=g_0\left[1+\frac{4r_6}{r_4}\left(\frac{(sn(\tau,m))^2}{f(sn(\tau,m))^2+k}\right)\right], \quad (51)$$

or

$$U_{2,2}=g_0\left[1+\frac{4r_6}{r_4}\left(\frac{(cd(\tau,m))^2}{f(cd(\tau,m))^2+k}\right)\right], \quad (52)$$

with $\tau=\delta_1x+\delta_2y+\delta_3z+wt$ and f, k are given by $f=(1+m^2+r_2)r_4/1-m^2+m^4-r_2^2$, $k=-3r_4/1-m^2+m^4-r_2^2$, under restriction $-r_4^2(-1-m^2-r_2)(-1+2m^2-r_2)(-2+m^2+r_2)=0$. Equation (51) turns into a bright soliton for $m\to1$, given as follows:

$$U_{2,3} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1+r_2^2)\tan h^2(\tau)}{3-(2+r_2)\tan h^2(\tau)}\right)\right], \tag{53}$$

such that $-r_4^2(2+r_2)[-1+r_2]^2 = 0$.

Equation (51) turns into a periodic solution for $m \to 0$, given as follows:

$$U_{2,4} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1+r_2^2)\sin^2(\tau)}{3-(2+r_2)\sin^2(\tau)}\right)\right], \tag{54}$$

such that $r_4^2(-1+r_2)[(-2-r_2)(-1+r_2)] = 0$.

2. If, $\ell_0 = 1-m^2$, $\ell_2 = 2m^2-1$, $\ell_4 = -m^2$, $m \in (0,1)$ after that $U(\tau) = cn(\tau,m)$, thus solution is given as follows:

$$U_{2,5} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(cn(\tau,m))^2}{f(cn(\tau,m))^2+k}\right)\right], \tag{55}$$

here f,k are specified via $f = (-1+2m^2-r_2)r_4/1-m^2+m^4-r_2^2, k = 3(-1+m^2)r_4/1-m^2+m^4-r_2^2$, under constraint $r_4^2(-1+2m^2-r_2)[(-2+m^2+r_2)(1+m^2+r_2)] = 0$.

Equation (55) turns into a periodic solution for $m \to 0$, given as follows:

$$U_{2,6} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1+r_2^2)\cos^2(\tau)}{-(-3+(1+r_2)\cos^2(\tau))}\right)\right], \tag{56}$$

such that $r_4^2(-1-r_2)[(-2+r_2)(1+r_2)] = 0$.

3. If, $\ell_0 = m^2-1$, $\ell_2 = 2-m^2$, $\ell_4 = -1, m \in (0,1)$, after that $U(\tau) = dn(\tau,m)$, leads to solution as follows:

$$U_{2,7} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(dn(\tau,m))^2}{f(dn(\tau,m))^2+k}\right)\right], \tag{57}$$

here f,k are specified via $f = (-2+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k = -3(-1+m^2)r_4/1-m^2+m^4-r_2^2$, under constraint $r_4^2(2-m^2-r_2)[-(-1+2m^2+r_2)(1+m^2+r_2)] = 0$.

4. If, $\ell_0 = m^2$, $\ell_2 = -(1+m^2)$, $\ell_4 = 1, m \in (0,1)$, after that $U(\tau) = ns(\tau,m)$ or $U(\tau) = dc(\tau,m)$ gives:

$$U_{2,8} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(ns(\tau,m))^2}{f(ns(\tau,m))^2+k}\right)\right], \tag{58}$$

or

$$U_{2,9} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(dc(\tau,m))^2}{f(dc(\tau,m))^2+k}\right)\right], \tag{59}$$

here f,k are agreed with $f = (1+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k = -3m^2r_4/1-m^2+m^4-r_2^2$, under restriction $r_4^2(-1-m^2-r_2)[-(-1+2m^2-r_2)(-2+m^2+r_2)] = 0$.

Equation (58) turns into a dark soliton for $m \to 1$, given as follows:

$$U_{2,10} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1+r_2^2)\cot h^2(\tau)}{(1-r_2)-(2+r_2)\csc h^2(\tau)}\right)\right], \tag{60}$$

such that $r_4^2(-2-r_2)[-1+r_2]^2 = 0$.

5. If, $\ell_0 = -m^2$, $\ell_2 = 2m^2-1$, $\ell_4 = 1-m^2$, $m \in (0,1)$, next $U(\tau) = nc(\tau,m)$, leads to solution as follows:

$$U_{2,11} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(nc(\tau,m))^2}{f(nc(\tau,m))^2+k}\right)\right], \tag{61}$$

here f and k are given by $f = -(-1+2m^2-r_2)r_4/1-m^2+m^4-r_2^2, k = 3m^2r_4/1-m^2+m^4-r_2^2$, under restriction $r_4^2(-1+2m^2-r_2)[(-2+m^2+r_2)(1+m^2+r_2)] = 0$.

Equation (61) turns into a solitary wave solution for $m \to 1$, given as follows:

$$U_{2,12} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1+r_2^2)\cos h^2(\tau)}{-3+(1-r_2)\cos h^2(\tau)}\right)\right], \tag{62}$$

such that $r_4^2(1-r_2)[-2+r_2+r_2^2] = 0$.

6. If $\ell_0 = -1$, $\ell_2 = 2-m^2$, $\ell_4 = -(1-m^2)$, $m \in (0,1)$, afterward $U(\tau) = nd(\tau,m)$; therefore, result is given as follows:

$$U_{2,13} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(nd(\tau,m))^2}{f(nd(\tau,m))^2+k}\right)\right], \tag{63}$$

here f,k are agreed with $f = (-2+m^2+r_2)r_4/1-m^2+m^4-r_2^2, k = 3r_4/1-m^2+m^4-r_2^2$, under constraint $r_4^2(2-m^2-r_2)[-(-1+2m^2-r_2)(1+m^2+r_2)] = 0$.

7. If, $\ell_0 = 1$, $\ell_2 = 2-m^2$, $\ell_4 = 1-m^2$, $m \in (0,1)$, in that case $U(\tau) = sc(\tau,m)$ leads to:

$$U_{2,14} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(sc(\tau, m))^2}{f(sc(\tau, m))^2 + k}\right)\right], \tag{64}$$

here f and k are agreed with $f = (-2 + m^2 + r_2)r_4/1 - m^2 + m^4 - r_2^2, k = -3r_4/1 - m^2 + m^4 - r_2^2,$ under restriction $r_4^2(2 - m^2 - r_2)[-(-1 + 2m^2 - r_2)(1 + m^2 + r_2)] = 0.$

Equation (64) turns into a solitary wave solution for $m \to 1$, given as follows:

$$U_{2,15} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1 + r_2^2)\sin\mathrm{h}^2(\tau)}{3 + (1 - r_2)\sin\mathrm{h}^2(\tau)}\right)\right], \tag{65}$$

such that $r_4^2(1 - r_2)[-2 + r_2 + r_2^2] = 0.$

Equation (64) turns into a periodic solution for $m \to 0$, given as follows:

$$U_{2,16} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1 + r_2^2)\tan^2(\tau)}{3 - (-2 + r_2)\tan^2(\tau)}\right)\right], \tag{66}$$

such that $r_4^2(2 - r_2)[1 + r_2]^2 = 0.$

8. If, $\ell_0 = 1,\ \ell_2 = 2m^2 - 1,\ \ell_4 = -m^2(1 - m^2),\ m \in (0, 1)$, next, $U(\tau) = sd(\tau, m)$ leads to:

$$U_{2,17} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(sd(\tau, m))^2}{f(sd(\tau, m))^2 + k}\right)\right], \tag{67}$$

here f, k are given as $f = (-1 + 2m^2 - r_2)r_4/1 - m^2 + m^4 - r_2^2, k = -3r_4/1 - m^2 + m^4 - r_2^2,$ under constraint $r_4^2(-1 + 2m^2 - r_2)[(-2 + m^2 + r_2)(1 + m^2 + r_2)] = 0.$

Equation (51) turns into a solitary wave solution for $m \to 1$, given as follows:

$$U_{2,18} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1 + r_2^2)\sin\mathrm{h}^2(\tau)}{3 + (1 - r_2)\sin\mathrm{h}^2(\tau)}\right)\right], \tag{68}$$

such that $r_4^2(-2 - r_2)[1 - r_2]^2 = 0.$

Equation (67) turns into a periodic solution for $m \to 0$, given as follows:

$$U_{2,19} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1 + r_2^2)\sin^2(\tau)}{3 + (1 + r_2)\sin^2(\tau)}\right)\right], \tag{69}$$

such that $r_4^2(2 - r_2)[1 + r_2]^2 = 0.$

9. If, $\ell_0 = 1 - m^2,\ \ell_2 = 2 - m^2,\ \ell_4 = 1, m \in (0, 1),$ after that $U(\tau) = cs(\tau, m)$; therefore, solution is given as follows:

$$U_{2,20} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(cs(\tau, m))^2}{f(cs(\tau, m))^2 + k}\right)\right], \tag{70}$$

here f, k are specified via $f = (-2 + m^2 + r_2)r_4/1 - m^2 + m^4 - r_2^2, k = 3(-1 + m^2)r_4/1 - m^2 + m^4 - r_2^2,$ under constraint $r_4^2(2 - m^2 - r_2)[-(-1 + 2m^2 - r_2)(1 + m^2 + r_2)] = 0.$

Equation (70) turns into a periodic solution for $m \to 0$, given as follows:

$$U_{2,21} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1 + r_2^2)\cot^2(\tau)}{3 + (2 - r_2)\cot^2(\tau)}\right)\right], \tag{71}$$

such that $r_4^2(2 - r_2)[(1 + r_2)^2] = 0.$

10. If, $\ell_0 = -m^2(1 - m^2),\ \ell_2 = 2m^2 - 1,\ \ell_4 = 1, m \in (0, 1),$ afterward $U(\tau) = ds(\tau, m)$, leads to:

$$U_{2,22} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(ds(\tau, m))^2}{f(ds(\tau, m))^2 + k}\right)\right], \tag{72}$$

here f, k are specified via $f = (1 - 2m^2 + r_2)r_4/1 - m^2 + m^4 - r_2^2, k = -3m^2(-1 + m^2)r_4/1 - m^2 + m^4 - r_2^2,$ under constraint $r_4^2(-1 + 2m^2 - r_2)[(-2 + m^2 + r_2)(1 + m^2 + r_2)] = 0.$

11. If, $\ell_0 = 1 - m^2/4,\ \ell_2 = 1 + m^2/2,\ \ell_4 = 1 - m^2/4,\ m \in (0, 1),$ afterward $U(\tau) = nc(\tau, m) \pm sc(\tau, m)$ or, $U(\tau) = cn(\tau, m)/1 \pm sn(\tau, m),$ thus solution is given as follows:

$$U_{2,23} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(nc(\tau, m) \pm sc(\tau, m))^2}{f(nc(\tau, m) \pm sc(\tau, m))^2 + k}\right)\right], \tag{73}$$

or

$$U_{2,24} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{cn^2(\tau, m)}{fcn^2(\tau, m) + k\{1 \pm sn(\tau, m)\}^2}\right)\right], \tag{74}$$

here f, k are comes from $f = -8(1 + m^2 - 2r_2)r_4/1 + 14m^2 + m^4 - 16r_2^2, k = 12(-1 + m^2)r_4/1 + 14m^2 + m^4 - 16r_2^2,$ under constraint $r_4^2(1/2(1 + m^2 - 2r_2)[1/16(1 - 6m + m^2 + 4r_2)(1 + 6m + m^2) + 4r_2)] = 0.$

Equation (74) reduces to a periodic solution for $m \to 0$, given as follows:

$$U_{2,25} = g_0\left[1 + \frac{r_6}{r_4^2}\left(\frac{(16r_2^2 - 1)(-1 + \sin(\tau))(\sec(\tau) + \tan(\tau))^2}{(-5 + 4r_2 + (1 + 4r_2)\sin(\tau))}\right)\right], \tag{75}$$

or

$$U_{2,26} = g_0\left[1 + \frac{r_6}{r_4^2}\left(\frac{(16r_2^2 - 1)\cos^2(\tau)}{(2 - 4r_2)\cos^2(\tau) + 3(1 + \sin(\tau))^2}\right)\right], \tag{76}$$

such that $r_4^2\left(\frac{1}{2} - r_2\right)\left[\frac{1}{16}(1 + 4r_2)^2\right] = 0$.

12. If, $\ell_0 = -(1 - m^2)^2/4$, $\ell_2 = 1 + m^2/2$, $\ell_4 = -1/4$, $m \in (0, 1)$, afterward $U(\tau) = m\,cn(\tau, m) \pm dn(\tau, m)$, thus solution is given as follows:

$$U_{2,27} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{(mcn(\tau, m) \pm dn(\tau, m))^2}{f(mcn(\tau, m) \pm dn(\tau, m))^2 + k}\right)\right], \tag{77}$$

here f and k are given by $f = -8(1 + m^2 - 2r_2)r_4/1 + 14m^2 + m^4 - 16r_2^2, k = 12(-1 + m^2)^2 r_4/1 + 14m^2 + m^4 - 16r_2^2$, under restriction $r_4^2(1/2(1 + m^2 - 2r_2)[1/16(1 - 6m + m^2 + 4r_2)(1 + 6m + m^2) + 4r_2\,)] = 0$.

13. If, $\ell_0 = 1/4$, $\ell_2 = 1 - 2m^2/2$, $\ell_4 = 1/4, m \in (0, 1)$, afterward $U(\tau) = sn(\tau, m)/1 \pm cn(\tau, m)$, thus solution is given as follows:

$$U_{2,28} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{\left(\frac{sn(\tau,m)}{1 \pm cn(\tau,m)}\right)^2}{f\left(\frac{sn(\tau,m)}{1 \pm cn(\tau,m)}\right)^2 + k}\right)\right], \tag{78}$$

here f and k are agreed with $f = 8(-1 + 2m^2 + 2r_2)r_4/1 - 16m^2 + 16m^4 - 16r_2^2, k = -12r_4/1 - 16m^2 + 16m^4 - 16r_2^2$, under restriction $r_4^2(1/2 - m^2 - r_2)[1/16 + 2m^2 - 2m^4 + (1/2 - m^2)r_2 + r_2^2] = 0$.

Equation (78) reduces to a dark bright solitary wave solution for $m \to 1$, given as follows:

$$U_{2,29} = g_0\left[1 + \frac{r_6}{r_4^2}\left(\frac{(-1 + 16r_2^2)\tan \mathrm{h}^2(\tau)}{3(1 + \sec h(\tau))^2 - 2(1 + 2r_2)\tan \mathrm{h}^2(\tau)}\right)\right], \tag{79}$$

such that $r_4^2(-1/2 - r_2)[1/16(1 - 4r_2)^2] = 0$.

Equation (78) reduces to a periodic solution for $m \to 0$, given as follows:

$$U_{2,30} = g_0\left[1 + \frac{r_6}{r_4^2}\left(\frac{(-1 + 16r_2^2)\sin^2(\tau)}{3(1 + \cos(\tau))^2 + 2(1 - 2r_2)\sin^2(\tau)}\right)\right], \tag{80}$$

such that $r_4^2(1/2 - r_2)[1/16(1 + 4r_2)^2] = 0$.

14. If, $\ell_0 = 1/4$, $\ell_2 = 1 + m^2/2$, $\ell_4 = (1 - m^2)^2/4$, $m \in (0, 1)$, afterward $U(\tau) = sn(\tau, m)/cn(\tau, m) \pm dn(\tau, m)$ then solution takes the form given as follows:

$$U_{2,31} = g_0\left[1 + \frac{4r_6}{r_4}\left(\frac{\left(\frac{sn(\tau,m)}{cn(\tau,m) \pm dn(\tau,m)}\right)^2}{f\left(\frac{sn(\tau,m)}{cn(\tau,m) \pm dn(\tau,m)}\right)^2 + k}\right)\right], \tag{81}$$

here f and k are given by $f = -8(1 + m^2 - 2r_2)r_4/1 + 14m^2 + m^4 - 16r_2^2, k = -12r_4/1 + 14m^2 + m^4 - 16r_2^2$, under constraint $r_4^2(1/2(1 + m^2 - 2r_2))[1/16(1 - 6m + m^2 + 4r_2)(1 + 6m + m^2) + 4r_2]\,) = 0$.

Equation (81) reduces to a solitary solution for $m \to 1$, given as follows:

$$U_{2,32} = g_0\left[1 + \frac{4r_6}{r_4^2}\left(\frac{(-1 + r_2^2)\sin \mathrm{h}^2(\tau)}{3 + (1 - r_2)\sin \mathrm{h}^2(\tau)}\right)\right], \tag{82}$$

such that $r_4^2(1 - r_2)[-2 + r_2 + r_2^2]\,) = 0$.

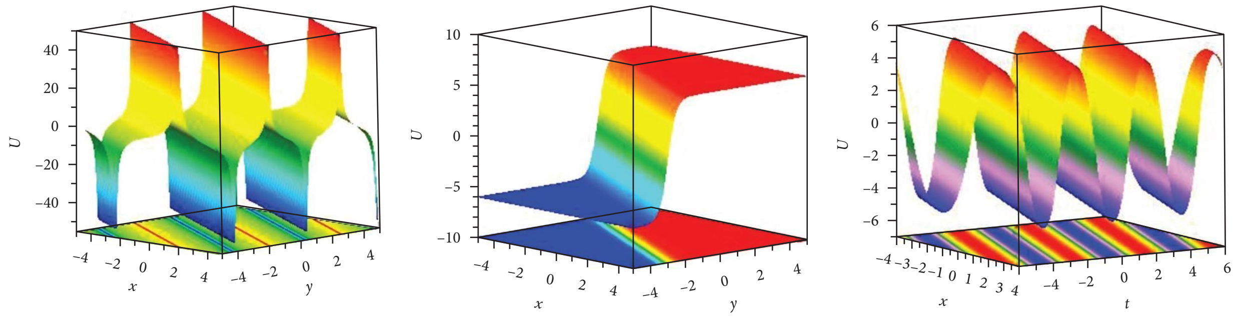
Equation (81) reduces to a periodic wave solution for $m \to 0$, given as follows:

$$U_{2,33} = g_0\left[1 + \frac{r_6}{r_4^2}\left(\frac{(-1 + 16r_2^2)\sin^2(\tau)}{3(1 + \cos(\tau))^2 + 2(1 - 2r_2)\sin^2(\tau)}\right)\right], \tag{83}$$

such that $r_4^2(1/2 - r_2)[1/16(1 + 4r_2)^2] = 0$.

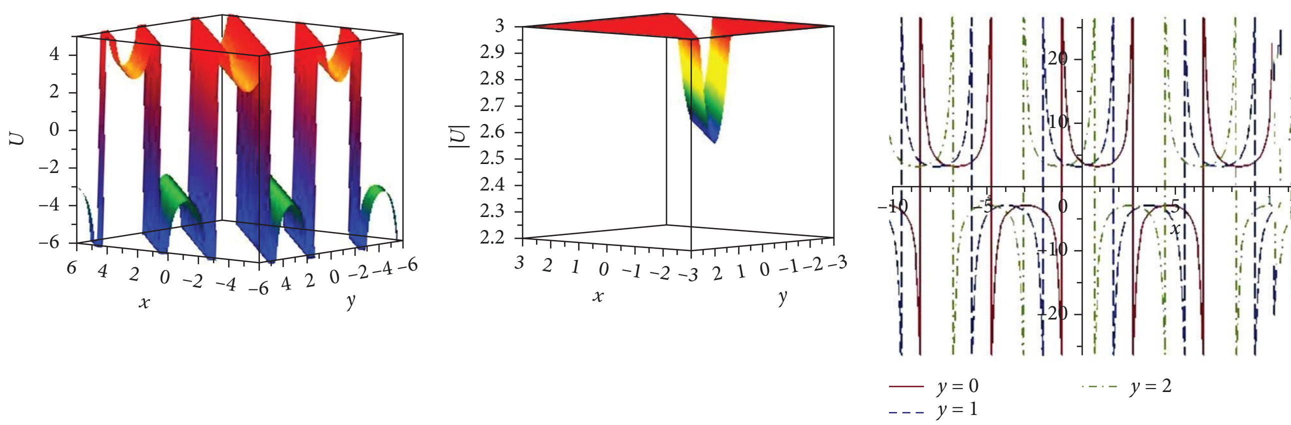
4. Results, Discussions, and Numerical Illustration

We foremost explored the influence of nonlinearity constant on soliton states and changing energy states. Such nonlinearity intensities have a noteworthy influence on the amplitude, width, and stability of solitons, even on dipole and tripole solitons. We investigated the stability of bright and dark solitons by linear stability analysis. Jacobi elliptic wave solution reduces to periodic wave and solitary wave for different parametric states. Few of the solutions are numerically illustrated as superperiodic wave by Equation (19), as shown in Figure 1(a), with $g_1 = 2, \rho = -1, \delta_1 = \delta_3 = 1, \delta_2 = 2, t = 0, z = 1$ as $m = 0.9, r_2 = 1.19$, it exhibits shock wave, as shown in Figure 1(b), due to change of $m = 1, r_2 = -2$ and except unity of m (with $m = 0.8, r_2 = -1.64$) the solution expressed periodic wave, as shown in Figure 1(c). The periodic wave with $m = 0.7, r_2 = 1.51, g_1 = 2, \rho = -2, \delta_1 = \delta_3 = \delta_2 = 1, t = 0, z = 1$ of the solution, as shown in Equation (27), depicted in 3D, as shown in Figure 2(a), and different changing of nonlinear intensity, as shown in Figure 2(c). A single



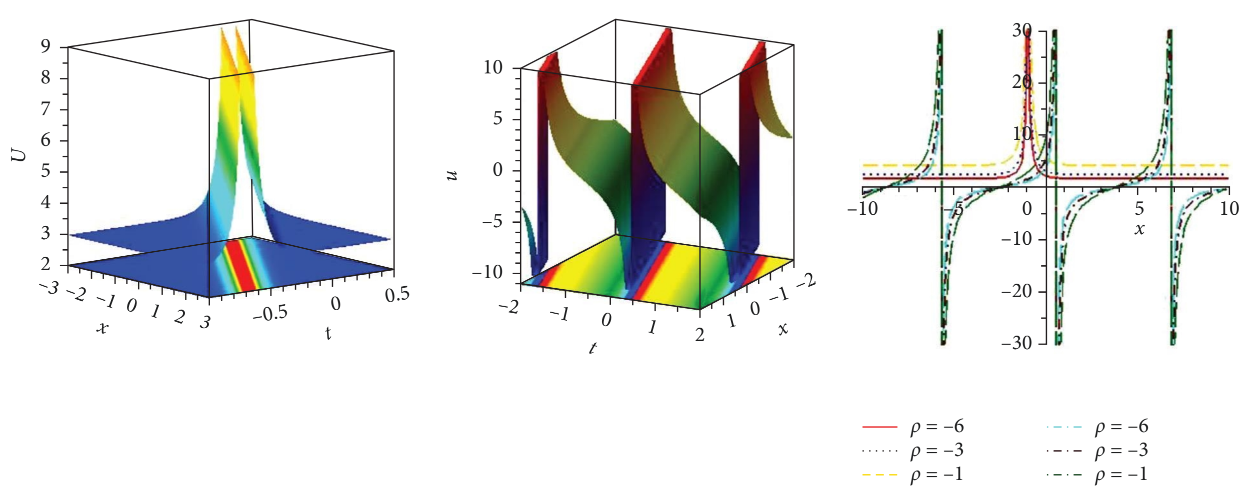
(a) (b) (c)

Figure 1: 3D surface: (a) superperiodic wave of (15) as $m = 0.9$, (b) shock wave of (15) as $m \to 1$ or (17), and (c) periodic wave of (16) as $m = 0.8$.



(a) (b) (c)

Figure 2: 3D surface: (a) periodic wave of (22) as $m = 0.7$, (b) dark soliton of (22) as $m \to 1$ or (24), and (c) 2D shape of (22) as $m = 0.7$.



(a) (b) (c)

Figure 3: 3D surface: (a) singular bright soliton of (26) as $m = 1$, (b) of (38) as $m \to 0$ or (24), and (c) 2D shape of (26) and (38) as nonlinearity varies.

dark soliton is achieved from the same Equation (27) just taking $m = 1$ retaining the rest parametric values are same, as shown in Figure 2(b). Singular bright peak soliton comes via Equation (31) as $m = 1, r_2 = \rho = -2, g_1 = \delta_2 = 2, \delta_1 = \delta_3 = 1, y = 0, z = 1$, as shown in Figure 3(a). Singular periodic wave solutions are visualized via Equation (44) for $r_2 = 0.5$, $\rho = -2, g_1 = \delta_2 = 2, \delta_1 = \delta_3 = 1, y = 0, z = 1$, as shown in Figure 3(b) and its 2D plots for varying nonlinearity are

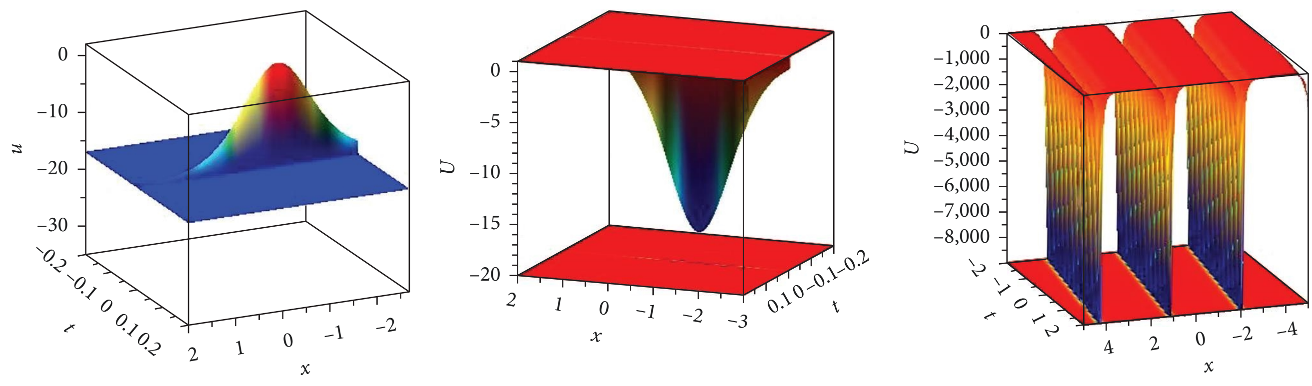


FIGURE 4: 3D surface: (a) bright soliton of (48) as $m=1$, (b) dark solitonvia (50) as $m\rightarrow 1$, and (c) periodic wave of (59) as $m=0$.

presented in Figure 3(c). Bright soliton visualized in Figure 4(a) via Equation (52) with $r_2=-2, \rho=\delta_2=r_0=2, r_4=3g_0=\delta_1=\delta_3=1, y=0, z=1$; dark soliton visualized in Figure 4(b) via Equation (55) with $r_2=-2, \rho=\delta_2=r_0=2, r_4=3g_0=\delta_1=\delta_3=1, y=0, z=1$, and beside this periodic wave visualized in Equation (64) with $r_2=-2, \delta_2=r_0=2, \rho=r_4=g_0=\delta_1=\delta_3=1, y=0, z=1$. Rest of the other solutions are similar to the illustrated solutions, with and without singularities are explained in this research.

5. Conclusions

This paper had achieved ion-acoustic soliton solutions to mKdV-ZK model in homogeneous magnetized plasma media via the φ^6-model expansion scheme. We successfully utilized and invented novel families of ion-acoustic solitons propagation due to Jacobi elliptic forms. Various shock waves, bullet type bright, dark, bright-dark solitons, as well as periodic wave solutions are formed starting the Jacobi elliptic results on different parametric constraints. Some obtained results are numerically illustrated in Figures 1–4 and analyzed the effect of nonlinearity on the change of height and weight of the wave form in 2D, 3D plots. The activities and the transmission of such solutions are discussed in a rating index waveguide for selecting suitable parameters. To best our knowledge, the achieved dynamical natures, in contrast with other's results in literature, are innovative in dissimilar structures. Such critical behaviors can be crucial to recognize attribute of the mKdV-ZK model those are used to explain various phenomena in nonlinear sciences.

Data Availability

Supported data are included inside this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Investigation, methodology, and writing draft, M.R. Pervin; conceptualization, software, and valdation, H.O. Roshid; modifications, corrections, and simulation, P. Dey; supervision and finalization, S. S. Shanta; supervision, idea maker, modification, and checking draft, S. Kumer. All authors have read and agreed to the published version of the manuscript.

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