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Research Article

Sharp Threshold of Global Existence and Mass Concentration for the Schrödinger–Hartree Equation with Anisotropic Harmonic Confinement

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This article is concerned with the initial-value problem of a Schrödinger–Hartree equation in the presence of anisotropic partial/whole harmonic confinement. First, we get a sharp threshold for global existence and finite time blow-up on the ground state mass in the L^2-critical case. Then, some new cross-invariant manifolds and variational problems are constructed to study blow-up versus global well-posedness criterion in the L^2-critical and L^2-supercritical cases. Finally, we research the mass concentration phenomenon of blow-up solutions and the dynamics of the L^2-minimal blow-up solutions in the L^2-critical case. The main ingredients of the proofs are the variational characterisation of the ground state, a suitably refined compactness lemma, and scaling techniques. Our conclusions extend and compensate for some previous results.

1. Introduction

In this paper, we consider the initial-value problem of the following Schrödinger–Hartree equation in the presence of anisotropic partial/whole harmonic confinement:

$$\begin{cases} i\varphi_t + \Delta\varphi - \sum_{j=1}^{k} \nu_j^2 x_j^2 \varphi + \lambda (I_\alpha * |\varphi|^p)|\varphi|^{p-2}\varphi = 0, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $\varphi : [0,T) \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $0 < T \leq \infty$ and φ_0 is a given function in $\mathbb{R}^N$, $1 \leq k \leq N$, $\nu_j \neq 0$ and $\nu_j \in \mathbb{R}$ $(1 \leq j \leq k)$, $\lambda > 0$, $2 \leq p < (N+\alpha)/(N-2)$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by the following:

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}, \tag{2}$$

with $0 < \alpha < N$ and Γ is the Gamma function.

Nonlinear Schrödinger equations of Hartree-type have a broad physical background. They often appear as models of quantum semiconductor devices [1]. When $k = N$, Equation (1), known as Schrödinger–Hartree equation with complete harmonic confinement, can be used to characterise Bose–Einstein condensation (BEC) in a gas with very weak two-body interactions, which was found in ^{23}Na or ^{87}Rb atomic systems [2]. When $1 \leq k < N$, Equation (1) is called a nonlinear Hartree equation with partial confinement, arising also as a typical model to describe the BEC [3]. When removing the harmonic confinement in Equation (1), for $N = 3$, $p = 2$, and $\alpha = 2$, Equation (1) is used to describe electrons trapped in their own holes, which is similar to the Hartree–Fock theory of single component plasma to some extent [4].

When $k = N$, Equation (1) with complete harmonic confinement has been well-studied. In the special case $\nu_1 = \nu_2 = \cdots = \nu_N$ and $p = 2$, Huang et al. [5] applied the Hamiltonian invariants and the Gagliardo–Nirenberg inequality of convolution type and scaling technique to investigate the sharp threshold of global existence and showed the stability of standing waves in the mass-critical case $\alpha = N - 2$. Wang [6] proved the existence of blow-up solutions and studied the strong instability of standing waves by variational methods in the mass-supercritical case $2<N-\alpha<\min\{4,N\}$. It's worth mentioning that, Feng [7] derived the sharp threshold for global existence and finite time blow-up on mass for $\nu_1=\nu_2=\cdots=\nu_N$ and $p=1+(2+\alpha)/N\geq 2$ in Equation (1), by using the variational characterisation of the ground state solution to a nonlinear Schrödinger–Hartree equation without potential (see Equation (14)). Moreover, in the general L^2-supercritical case $1+(2+\alpha)/N\leq p<(N+\alpha)/(N-2)$ with $0<\alpha<N$, Feng [7] obtained blow-up versus global well-posedness criteria in both the L^2-critical and L^2-supercritical cases by constructing some cross-invariant manifolds and variational problems and studied the stability and instability of standing waves. If the nonlinearity $(I_\alpha*|\varphi|^p)|\varphi|^{p-2}\varphi$ is replaced by $|\varphi|^{p-1}\varphi$, there exist extensive literatures on the Cauchy problem of nonlinear Schrödinger equation with complete harmonic potential, see, e.g., [8–10]. In particular, Shu and Zhang [9] and Zhang [10] derived the sharp criterion of global existence to Equation (1) in the L^2-critical and L^2-supercritical cases by variational methods and constructing different cross-constrained variational problems and so-called invariant sets.

When $1\leq k<N$, the main difference between nonlinear Schrödinger-type equation with partial harmonic confinement and complete confinement is that the embedding from natural energy space Σ (see Section 2) to $L^p(\mathbb{R}^N)(p\in[2,2N/(N-2)))$ is lack of compactness, resulting the main difficulty on the study of blow-up dynamics and stability of standing waves to the corresponding Cauchy problem. Due to the fact, the existence of stable standing waves, global and blow-up dynamics, and sharp criterion of global existence to the nonlinear Schrödinger-type equations with partial confinement have attracted considerable interest. A lot of studies have been made in these directions to Equation (1) with power type nonlinearity $|x|^{-b}|\varphi|^{p-1}\varphi\,(b\geq 0)$, see [11–16] for example and the references therein. More precisely, in the L^2-supercritical case with $b=0$ and $k=2<N=3$, Bellazzini et al. [11] applied the concentration compactness principle to overcome the lack of compactness and obtained the existence and stability of normalised standing waves. Ardila and Carles [12] studied the criteria of blow-up and scattering below the ground state in the focusing L^2-supercritical case. Zhang [13] and Pan and Zhang [14] studied the sharp threshold for finite time blow-up and global existence in the mass-critical case by making full use of the ground state to a classical nonlinear elliptic equation without harmonic confinement and Hamilton conservation, as well as scaling arguments. It is worth noting that by exploiting the refined compactness lemma proposed by Hmidi and Keraani [17] and the variational characterisation of the ground state and scaling techniques, Pan and Zhang [14] investigated the mass concentration properties and limiting profile of the blow-up solutions possessing small super-critical mass for the 2D L^2-critical Schrödinger equation with $k=1$. More recently, when $k=2$ and $N=3$, Wang and Zhang [15] derived the sharp condition for global existence and blow-up to the solutions by constructing cross-constrained variational problems and invariant manifolds of the evolution flow. Liu et al. [16] studied the existence and stability of normalised standing waves for Equation (1) with anisotropic partial confinement and inhomogeneous nonlinearity $|x|^{-b}|\varphi|^{p-1}\varphi\,(b>0)$ by making use of profile decomposition theory and concentration compactness principle. Besides, based on the ideas of Bellazzini et al. [11], the existence and orbital stability of standing waves for Equation (1) with $k=N-1$ were obtained by Xiao et al. [18]. As far as we know, the research of the sharp threshold of global existence and mass concentration phenomenon to the blow-up solutions of nonlinear Schrödinger-type equation with Hartree nonlinearity $(I_\alpha*|\varphi|^p)|\varphi|^{p-2}\varphi$ and partial confinement are still open, which is greatly pursued in physics. This is the main motivation for us to study the Cauchy problem (1).

In the absence of harmonic confinement in Equation (1), the corresponding equation is also known as the Choquard equation, which has also been extensively studied, see for instance [8, 19–23]. In particular, by constructing invariant sets and using variational methods, Chen and Guo [19] obtained the existence of blow-up solutions for some suitable initial data and showed strong instability of standing waves in the case $N=3$ and $2<N-\alpha<3$. Miao et al. [20] studied the mass concentration properties of blow-up solutions as well as the dynamics of blow-up solutions with minimal mass for Equation (1) in the L^2-critical case with $\alpha=2$ and $N=4$. When $p=1+4/N\ (N=3,4)$, Genev and Venkov [21] gave a sharp sufficient condition of global existence to Equation (1). Furthermore, they proved the existence of blow-up solutions and considered the blow-up dynamics to the solutions in the L^2-critical setting, i.e., $p=1+(2+\alpha)/N$ with $\alpha=2$. Notice that Feng and Yuan [22] not only considered the local and global well-posedness and finite time blow-up to the corresponding initial-value problem (1) in the general case $2\leq p<(N+\alpha)/(N-2)$ with $\max\{0,N-4\}<\alpha<N$, but also took into account the concentration phenomenon of blow-up solutions and the blow-up dynamics of blow-up solutions possessing minimal mass in the case $p=1+(2+\alpha)/N\geq 2$, by establishing a new refined compactness lemma with respect to the nonlocal nonlinearity $(I_\alpha*|\varphi|^p)|\varphi|^{p-2}\varphi$.

To the best of our knowledge, there are few papers dealing with the global well-posedness and blow-up dynamics to the Cauchy problem (1) in the presence of anisotropic partial/whole harmonic confinement. Inspired by the literatures aforementioned, the purposes of this present article are devoted to investigate the sharp criterion of global existence and mass-concentration phenomenon of blow-up solutions as well as the dynamical properties of minimal mass blow-up solutions of Equation (1) with anisotropic partial/whole confinement. The main difficulties come from the presence of anisotropic harmonic confinement $\sum_{j=1}^{k}\nu_j^2x_j^2$ and the nonlocal nonlinearity $(I_\alpha*|\varphi|^p)|\varphi|^{p-2}\varphi$, resulting in the loss of compactness and pseudo-conformal transformation. Motivated by Feng [7], Zhang [13], Pan and Zhang [14], and Zhang [24], we utilise the ground state to the nonlinear Schrödinger–Hartree Equation (14), which is without any confined potential, to study the blow-up phenomenon and overcome the difficulties. First, we get a sharp threshold for global existence and finite time blow-up on the ground state mass for the Schrödinger–Hartree equation with anisotropic partial/whole harmonic confinement in the L^2-critical case. This result extends and compensates the work of Feng [7], which only considered the case with complete confinement for $k=N$ and $\nu_1=\nu_2=\ldots=\nu_k=1$ in Equation (1). Then, in the L^2-critical and L^2-supercritical cases, by constructing some new cross-invariant manifolds of the evolution flow and some variational problems associated with Equation (1), we derive blow-up versus global well-posedness criterion for Equation (1). In the present case, the constructed cross-invariant sets and variational problems are in light of Shu and Zhang [9], which differ from those of Feng [7], and some new criterion of global existence is derived, generalising and complementing the corresponding result of. Finally, based on the ideas of Pan and Zhang [14], Hmidi and Keraani [17], and Feng and Yuan [22], we research the mass concentration phenomenon of blow-up solutions and the dynamics of the L^2-minimal blow-up solutions, including the precise mass-concentration and blow-up rate of the minimal mass blow-up solutions. The main ingredients of the proofs are the variational characterisation of the ground state to Equation (14), a refined compactness lemma established by Feng and Yuan [22], and scaling techniques. Our conclusions about the mass concentration phenomenon of blow-up solutions and the dynamics of the L^2-minimal blow-up solutions extend the results of Feng and Yuan [22], in which the case without any potential was considered, to the Schrödinger–Hartree equation with anisotropic partial/whole confinement.

The rest of this paper is organised as follows: in Section 2, some notations and preliminaries are given. Section 3 considers the sharp threshold for global existence and finite time blow-up of Equation (1) in both the L^2-critical and L^2-supercritical cases. The last section focuses on the mass concentration phenomenon of blow-up solutions and the dynamics of the L^2-minimal blow-up solutions.

2. Notations and Preliminaries

Throughout this paper, we use $\int \cdot\, dx$ to represent $\int_{\mathbb{R}^N} \cdot\, dx$ and denote $\|\varphi\|_p=\|\varphi\|_{L^p(\mathbb{R}^N)}=\left(\int|\varphi|^p dx\right)^{\frac{1}{p}}$, and use C to stand for positive constants, which may vary from line to line. Without loss of generality, we assume $\lambda=1$ in this and subsequent sections.

For Equation (1), we equip the natural energy space

$$\Sigma=\left\{\varphi\in H^1(\mathbb{R}^N),\ \int\sum_{j=1}^{k}\nu_j^2x_j^2|\varphi|^2dx<\infty,\ \nu_j\in\mathbb{R}\setminus\{0\}\right\}, \tag{3}$$

with the inner product

$$\langle\phi,\varphi\rangle_\Sigma=Re\int\left(\phi\bar{\varphi}+\nabla\phi\cdot\nabla\bar{\varphi}+\sum_{j=1}^{k}\nu_j^2x_j^2\phi\bar{\varphi}\right)dx,\ \forall\phi,\varphi\in\Sigma, \tag{4}$$

and the corresponding norm is given by the following:

$$\|\varphi\|_\Sigma^2=\|\varphi\|_2^2+\|\nabla\varphi\|_2^2+\int\sum_{j=1}^{k}\nu_j^2x_j^2|\varphi|^2dx,\ \forall\varphi\in\Sigma. \tag{5}$$

The energy function associated with Equation (1) is defined as follows:

$$E(\varphi(t))=\frac{1}{2}\int\left(|\nabla\varphi(t)|^2+\sum_{j=1}^{k}\nu_j^2x_j^2|\varphi(t)|^2-\frac{1}{p}(I_\alpha*|\varphi|^p)|\varphi|^p\right)dx,\ \varphi\in\Sigma. \tag{6}$$

Let us now state the local well-posedness of Equation (1) in energy space Σ according to Feng [7] and Feng and Yuan [22].

Proposition 1. *Let $\varphi_0\in\Sigma$ and $2\le p<(N+\alpha)/(N-2)$. Then there exists $T=T(\|\varphi_0\|_\Sigma)$ such that Equation (1) admits a unique solution $\varphi(t,x)\in C([0,T),\Sigma)$. Let $[0,T)$ be the maximal time interval such that the solution $\varphi(t,x)$ is well-defined. If $T<\infty$, then $\lim_{t\to T}\|\varphi(t)\|_\Sigma=\infty$ (blow-up). Furthermore, $\varphi(t,x)$ depends continuously on initial data φ_0 and for any $t\in[0,T)$, the following conservation laws of mass and energy hold,*

$$\int|\varphi(t)|^2dx=\int|\varphi_0|^2dx, \tag{7}$$

$$E(\varphi(t))=E(\varphi_0). \tag{8}$$

Then we introduce some vital lemmas.

Lemma 2 (see [25]). *Let $0<\lambda<N$ and $s, r>1$ be constants such that*

$$\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{N}=2. \tag{9}$$

Assume that $g\in L^r(\mathbb{R}^N)$ and $h\in L^s(\mathbb{R}^N)$. Then

$$\left|\int\int g(x)|x-y|^{-\lambda}h(y)dxdy\right|\le C(N,s,\lambda)\|g\|_r\|h\|_s. \tag{10}$$

By inequality Equation (10), we can obtain the following generalised Gagliardo–Nirenberg inequality

$$\int(I_\alpha*|\varphi|^p)|\varphi|^pdx\le C_{\alpha,p}\left(\int|\nabla\varphi|^2dx\right)^{\frac{Np-N-\alpha}{2}}\left(\int|\varphi|^2dx\right)^{\frac{N+\alpha-Np+2p}{2}}. \tag{11}$$

Following Weinstein [26], Feng and Yuan [22] derived the best constant in the inequality Equation (11) by discussing the existence of the minimiser of the functional

$$J_{\alpha,p}(\varphi)=\frac{\left(\int|\nabla\varphi|^2dx\right)^{\frac{Np-N-\alpha}{2}}\left(\int|\varphi|^2dx\right)^{\frac{N+\alpha-Np+2p}{2}}}{\int(I_\alpha*|\varphi|^p)|\varphi|^pdx}. \tag{12}$$

Lemma 3 (see [22]). *It follows that the best constant in the generalised Gagliardo-Nirenberg inequality Equation (11) is*

$$C_{\alpha,p}=\frac{2p}{2p-Np+N+\alpha}\left(\frac{2p-Np+N+\alpha}{Np-N-\alpha}\right)^{\frac{Np-N-\alpha}{2}}\|Q(x)\|_2^{2-2p}, \tag{13}$$

where $Q(x)$ is the ground state of the elliptic equation

$$-\triangle\varphi+\varphi-(I_\alpha*|\varphi|^p)|\varphi|^{p-2}\varphi=0. \tag{14}$$

In particular, in the L^2-critical case, $C_{\alpha,p}=p\|Q(x)\|_2^{2-2p}$.

It is known that the ground state is of great importance in studying global existence and blow-up dynamics to the initial-value problem of the nonlinear Schrödinger equation; in the following lemma, we recall some existence results and properties of the ground state solution to Equation (14).

Lemma 4 (see [27]). *Let $\alpha\in(0,N)$ and $(N+\alpha)/N<p<(N+\alpha)/(N-2)$. It follows that Equation (14) admits a ground state solution $Q(x)$ in $H^1(\mathbb{R}^N)$. Every ground state $Q(x)$ of Equation (14) is in $L^1\cap C^\infty$, and there exist $x_0\in\mathbb{R}^N$ and a monotone real function $\tau\in C^\infty(0,\infty)$ such that for every $x\in\mathbb{R}^N$, $Q(x)=\tau(|x-x_0|)$. Moreover, the following Pohožaev identity holds,*

$$\begin{aligned}\frac{N-2}{2}\int|\nabla Q(x)|^2dx+\frac{N}{2}\int|Q(x)|^2dx\\=\frac{N+\alpha}{2p}\int(I_\alpha*|Q(x)|^p)|Q(x)|^pdx,\end{aligned} \tag{15}$$

$$\begin{aligned}\int|\nabla Q(x)|^2dx+\int|Q(x)|^2dx\\=\int(I_\alpha*|Q(x)|^p)|Q(x)|^pdx.\end{aligned} \tag{16}$$

From Equations (15) and (16), one has

$$p\int|\nabla Q(x)|^2dx=\int(I_\alpha*|Q(x)|^p)|Q(x)|^pdx. \tag{17}$$

In order to study the blow-up phenomenon of Equation (1), we also need the following lemma obtained in Weinstein [26].

Lemma 5 (see [26]). *Let $\varphi\in H^1(\mathbb{R}^N)$, then we have that*

$$\int|\varphi|^2dx\le\frac{2}{N}\left(\int|\nabla\varphi|^2dx\right)^{\frac{1}{2}}\left(\int|x|^2|\varphi|^2dx\right)^{\frac{1}{2}}. \tag{18}$$

Following the idea of Glassey [28] (see also Feng [7]), we will adopt the convexity method to study the existence of blow-up solutions. More precisely, we need to consider the variance

$$V(t)=\int|x|^2|\varphi(t,x)|^2dx, \tag{19}$$

and show that there exists time $T>0$ such that $V(T)=0$. With some formal computations (which can be rigorously proved by Cazenave [8]), we have the following virial identities:

Proposition 6. *Let $2\le p<(N+\alpha)/(N-2)$ and assume that $\varphi(t,x)$ is a solution of problem (1) in $C([0,T);\Sigma)$ with $\varphi_0\in H^1(\mathbb{R}^N)$ and $|x|\varphi_0\in L^2(\mathbb{R}^N)$. Then the function $t\to|\cdot|\varphi(t,\cdot)$ belongs to $C([0,T),L^2)$. Furthermore, the function $V(t)=\int|x|^2|\varphi(t,x)|^2dx$ belongs to $C^2[0,T)$, then we obtain that*

$$V'(t)=4Im\int x\nabla\varphi\bar{\varphi}dx, \tag{20}$$

and

$$\begin{aligned}V''(t)=&\ 8\int|\nabla\varphi|^2dx-8\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx\\&-\frac{4Np-4N-4\alpha}{p}\int(I_\alpha*|\varphi|^p)|\varphi|^pdx\\=&\ 8(Np-N-\alpha)E(\varphi)\\&+(8+4N+4\alpha-4Np)\int|\nabla\varphi|^2dx\\&+(4N+4\alpha-4Np-8)\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx,\end{aligned} \tag{21}$$

for all $t\in[0,T)$. In particular, when $p=1+(2+\alpha)/N$, we have

$$V''(t)=16E(\varphi_0)-16\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx. \tag{22}$$

Using Lemma 5 and Proposition 6, we can easily get the following sufficient conditions on the existence of blow-up solutions.

Corollary 7. *Assume that* $\max\{0,N-4\}<\alpha<N$ *and* $\max\{1+(2+\alpha)/N,2\}\le p<(N+\alpha)/(N-2)$. *Let $\varphi_0\in H^1(\mathbb{R}^N)$ and $|x|\varphi_0\in L^2(\mathbb{R}^N)$, and satisfy one of the following conditions:*

Case (1): $E(\varphi_0)<0$;

Case (2): $E(\varphi_0)=0$ *and* $Im\int x\nabla\varphi_0\bar{\varphi}_0dx<0$;

Case (3): *$E(\varphi_0)>0$ and $Im\int x\nabla\varphi_0\bar{\varphi}_0dx+(2V(0)E(\varphi_0))^{\frac{1}{2}}\le 0$.*

Then the corresponding solution $\varphi(t,x)$ of Equation (1) blows up in finite time.

3. Sharp Threshold for Global Existence and Blow-up

3.1. The L^2-Critical Case. The aim of this subsection is mainly to consider the global existence and blow-up of the solutions to Equation (1) in the L^2-critical case, i.e., $p=1+(2+\alpha)/N$. The ground state mass $\|Q(x)\|_2$ gives a sufficient condition on the global existence of the solution to Equation (1).

Theorem 8. *Let $p=1+(2+\alpha)/N\ge 2$ and $Q(x)$ be the positive radially symmetric ground state solution of Equation (14). If $\varphi_0\in\Sigma$ and φ_0 satisfies*

$$\|\varphi_0\|_2<\|Q(x)\|_2, \tag{23}$$

then the Cauchy problem (1) has a global solution $\varphi(t,x)$ in $C([0,\infty),\Sigma)$. Furthermore, we have for any $0\le t<\infty$,

$$\begin{aligned}&\int\left(|\nabla\varphi|^2+\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2\right)dx\\&<\frac{2E(\varphi_0)}{1-\|Q(x)\|_2^{2-2p}\left(\int|\varphi_0|^2dx\right)^{p-1}}+2E(\varphi_0).\end{aligned} \tag{24}$$

Proof. Let $\varphi(t,x)$ be the corresponding solution of Equation (1) in $C([0,T),\Sigma)$ with initial value $\varphi_0\in\Sigma$. By Equation (8), Equation (6), Lemma 3, and Equation (7), we obtain

$$\begin{aligned}E(\varphi_0)&=E(\varphi)=\frac{1}{2}\int\left(|\nabla\varphi|^2+\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2-\frac{1}{p}(I_\alpha*|\varphi|^p)|\varphi|^p\right)dx\\&\ge\frac{1}{2}\int\left(|\nabla\varphi|^2+\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2\right)dx-\frac{1}{2p}p\|Q(x)\|_2^{2-2p}\int|\nabla\varphi|^2dx\left(\int|\varphi|^2dx\right)^{p-1}\\&=\frac{1}{2}\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx+\frac{1}{2}\int\left(1-\|Q(x)\|_2^{2-2p}\left(\int|\varphi|^2dx\right)^{p-1}\right)|\nabla\varphi|^2dx\\&=\frac{1}{2}\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx+\frac{1}{2}\int\left(1-\|Q(x)\|_2^{2-2p}\left(\int|\varphi_0|^2dx\right)^{p-1}\right)|\nabla\varphi|^2dx.\end{aligned} \tag{25}$$

From Equations (25) and (23), we have for all $t\in[0,T)$, where T is arbitrary and $T<\infty$, there exists C such that

$$\int|\nabla\varphi|^2dx+\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx\le C. \tag{26}$$

Then, according to Proposition 1, $\varphi(t,x)$ exists globally in time. Moreover, we have

$$\int|\nabla\varphi|^2dx<\frac{2E(\varphi_0)}{1-\|Q(x)\|_2^{2-2p}\left(\int|\varphi_0|^2dx\right)^{p-1}}, \tag{27}$$

and

$$\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx<2E(\varphi_0). \tag{28}$$

It follows from Equations (27) and (28) that Equation (24) holds true. □

Remark 9.

(i) When $k=N$ and $\nu_1=\nu_2=\cdots=\nu_k=1$ in Equation (1), Feng [7] proved that the solution $\varphi(t,x)$ of Equation (1) exists globally (see Theorem 3.2 by Feng [7]). Theorem 8 can be viewed as the complement of the corresponding result of Feng [7] for Equation (1) with whole harmonic confinement.

(ii) We give an explicit bound to the global solution of Equation (1) in Σ (see Equation (24)).

By using the variational characterisation of the ground state solution to Equation (14), some scaling arguments and energy conservation, we can get the existence result of blow-up solutions to Equation (1).

Theorem 10. *Let $Q(x)$ be the positive radially symmetric solution of Equation (14), $p=1+(2+\alpha)/N\ge 2$. Then for any $\varepsilon>0$, there exist $\varphi_0\in\Sigma$ and $\int|x|^2|\varphi_0|^2dx<\infty$ such that*

$$\|\varphi_0\|_2^2=\|Q(x)\|_2^2+\varepsilon, \tag{29}$$

and the solution $\varphi(t,x)$ of the Cauchy problem (1) blows up in finite time.

Proof. For any $a>1,\ b>0$, we take $Q_{a,b}(x)=ab^{\frac{N}{2}}Q(bx)$. Based on some scaling arguments, one has that

$$\int |Q_{a,b}(x)|^2 dx = a^2\int |Q(x)|^2 dx, \tag{30}$$

$$\int |\nabla Q_{a,b}(x)|^2 dx = a^2 b^2\int |\nabla Q(x)|^2 dx, \tag{31}$$

$$\int \sum_{j=1}^{k}\nu_j^2 x_j^2 |Q_{a,b}(x)|^2 dx = a^2 b^{-2}\int \sum_{j=1}^{k}\nu_j^2 x_j^2 |Q(x)|^2 dx, \tag{32}$$

$$\int \left(I_\alpha * |Q_{a,b}(x)|^p\right)|Q_{a,b}(x)|^p dx = a^{2+\frac{2(2+\alpha)}{N}} b^2 \int (I_\alpha * |Q(x)|^p)|Q(x)|^p dx. \tag{33}$$

Now we set

$$a = \frac{\int |Q(x)|^2 dx + \varepsilon}{\int |Q(x)|^2 dx} > 1,$$

$$b > \left[\frac{\int \sum_{j=1}^{k}\nu_j^2 x_j^2 |Q(x)|^2 dx}{\left(a^{\frac{2(2+\alpha)}{N}}-1\right)\int |\nabla Q(x)|^2 dx}\right]^{\frac{1}{4}},\ \text{and}\ \varphi_0(x)=ab^{\frac{N}{2}}Q(bx), \tag{34}$$

then we have $\varphi_0(x)\in\Sigma$ and $\int |x|^2|\varphi_0|^2 dx<\infty$. In fact, since $Q_{a,\,b}(x)=ab^{\frac{N}{2}}Q(bx)\in H^1(\mathbb{R}^N)$, by utilising the exponential decay of ground state solution $Q(x)$ (see [27]):

$$Q(|x|), \nabla Q(|x|) = O\left(|x|^{-\frac{N-1}{2}} e^{-|x|}\right),\ as\ |x|\to\infty, \tag{35}$$

we conclude that $Q_{a,b}(x)\in L^2(\mathbb{R}^N)$ and so $\varphi_0 = ab^{\frac{N}{2}}Q(bx)\in H^1(\mathbb{R}^N)$ and $\int |x|^2|\varphi_0|^2 dx<\infty$. Thus, we also deduce that $\varphi_0\in\Sigma$. Moreover, it follows from Equation (30) that

$$\int |\varphi_0|^2 dx = \int |Q(x)|^2 dx + \varepsilon. \tag{36}$$

From Equations (6), (8), (17), and (31)–(33), we get

$$\begin{aligned} E(\varphi) &= E(\varphi_0) = \frac{1}{2}\int \left(|\nabla\varphi_0|^2 + \sum_{j=1}^{k}\nu_j^2 x_j^2|\varphi_0|^2 - \frac{1}{p}(I_\alpha * |\varphi_0|^p)|\varphi_0|^p\right)dx \\ &= \frac{1}{2}\left(1-a^{\frac{2(2+\alpha)}{N}}\right)a^2b^2\int |\nabla Q(x)|^2 dx + \frac{a^2}{2b^2}\int \sum_{j=1}^{k}\nu_j^2 x_j^2 |Q(x)|^2 dx \\ &= \frac{1}{2}a^2b^2\left(\left(1-a^{\frac{2(2+\alpha)}{N}}\right)\int |\nabla Q(x)|^2 dx + \frac{1}{b^4}\int \sum_{j=1}^{k}\nu_j^2 x_j^2|Q(x)|^2 dx\right) \\ &< 0. \end{aligned} \tag{37}$$

Thus, it follows from Corollary 7 that the solution $\varphi(t,x)$ of Equation (1) blows up in finite time. □

Remark 11.

(i) When $k=N$ and $\nu_1=\nu_2=\cdots=\nu_k=1$ in Equation (1), Feng [7] proved the existence of blow-up solutions (see Theorem 3.2 by Feng [7]). When considering Equation (1) in the presence of anisotropic partial/complete harmonic confinement, we derive the corresponding blow-up result by scaling approach, which differs from the method of Feng [7].

(ii) Theorems 8 and 10 declare that $\|Q(x)\|_2$ provides a sharp threshold for global existence and blow-up to Equation (1) in terms of the initial data, which is called minimal mass for the blow-up solutions.

3.2. The L^2-Supercritical Case. For $\varphi\in\Sigma$ and $(N+\alpha)/N< p<(N+\alpha)/(N-2)$, define the following functionals:

$$I(\varphi) = \frac{1}{2}\int \left(|\nabla\varphi|^2 + |\varphi|^2 + \sum_{j=1}^{k}\nu_j^2 x_j^2|\varphi|^2\right)dx - \frac{1}{2p}\int (I_\alpha * |\varphi|^p)|\varphi|^p dx; \tag{38}$$

$$J(\varphi) = \int (|\nabla\varphi|^2 + |\varphi|^2 - (I_\alpha * |\varphi|^p)|\varphi|^p)dx; \tag{39}$$

$$S(\varphi) = \int \left(|\nabla\varphi|^2 - \frac{Np-N-\alpha}{2p}(I_\alpha * |\varphi|^p)|\varphi|^p\right)dx. \tag{40}$$

Then, we define the set

$$\mathbb{M} = \{\varphi \in \Sigma \setminus \{0\}, J(\varphi) < 0, S(\varphi) = 0\}, \quad (41)$$

and consider the following two constrained minimisation problems:

$$d_1 = \inf_{\{\varphi \in \Sigma \setminus \{0\}, J(\varphi)=0\}} I(\varphi), \quad (42)$$

$$d_2 = \inf_{\varphi \in \mathbb{M}} I(\varphi). \quad (43)$$

Proposition 12. *If* $1 + (2+\alpha)/N \le p < (N+\alpha)/(N-2)$, *then* $d_2 > 0$.

Proof. First, we prove $\mathbb{M} \neq \emptyset$. According to Lemma 3, there exists $\varphi \in \Sigma \setminus \{0\}$ such that φ is a solution of Equation (14). By multiplying both sides of Equation (14) by φ and integrating over $\mathbb{R}^N$, we get

$$\int |\nabla \varphi|^2 dx + \int |\varphi|^2 dx = \int (I_\alpha * |\varphi|^p)|\varphi|^p dx. \quad (44)$$

It follows from Equation (44) that $J(\varphi) = 0$. Moreover, by taking the inner product of Equation (14) with $x \cdot \nabla \varphi$, we have the following Pohǒzaev identity

$$-\frac{N-2}{2}\int |\nabla \varphi|^2 dx - \frac{N}{2}\int |\varphi|^2 + \frac{N+\alpha}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx = 0. \quad (45)$$

Then multiplying both sides of Equation (44) by $N/2$, we have

$$\frac{N}{2}\int (|\nabla \varphi|^2 + |\varphi|^2 - (I_\alpha * |\varphi|^p)|\varphi|^p) dx = 0. \quad (46)$$

From Equations (45) and (46), one has that

$$\int |\nabla \varphi|^2 dx + \left(\frac{N+\alpha-Np}{2p}\right)\int (I_\alpha * |\varphi|^p)|\varphi|^p dx = 0, \quad (47)$$

which implies $S(\varphi) = 0$. Thus, there exists $\varphi \in \Sigma \setminus \{0\}$ such that $S(\varphi) = 0$ and $J(\varphi) = 0$.

Set

$$u(x) = \mu^{\frac{2+\alpha}{2p-2}} \varphi(\mu x), \quad \mu > 0. \quad (48)$$

By some simple computations, we obtain

$$J(u(x)) = \mu^{\frac{2p+\alpha-Np+N}{p-1}}\left(\int |\nabla \varphi|^2 dx - \int (I_\alpha * |\varphi|^p)|\varphi|^p dx\right) + \mu^{\frac{2+\alpha}{p-1}-N}\int |\varphi|^2 dx, \quad (49)$$

and

$$S(u(x)) = \mu^{\frac{2p+\alpha-Np+N}{p-1}}\left(\int |\nabla \varphi|^2 dx - \frac{Np-N-\alpha}{2p}\int (I_\alpha * |\varphi|^p)|\varphi|^p dx\right). \quad (50)$$

Note that $S(\varphi) = 0$. Thus $S(u(x)) = 0$ for every $\mu > 0$. Moreover,

$$J(u(x)) = \left(\mu^{\frac{2+\alpha}{p-1}-N} - \mu^{\frac{2p+\alpha-Np+N}{p-1}}\right)\int |\varphi|^2 dx = \mu^{\frac{2+\alpha}{p-1}-N}(1-\mu^2)\int |\varphi|^2 dx. \quad (51)$$

Thus, there exists $\mu > 1$ such that $J(u(x)) < 0$. Therefore, when $\mu > 1$, we have $S(u(x)) = 0$ and $J(u(x)) < 0$ which implies $\mathbb{M} \neq \emptyset$.

Next, we prove $d_2 > 0$. Let $\varphi \in \mathbb{M}$, from $J(\varphi) < 0$, we get $\varphi \neq 0$. Since $S(\varphi) = 0$, we have

$$I(\varphi) = \left(\frac{1}{2} - \frac{1}{Np-N-\alpha}\right)\int |\nabla \varphi|^2 dx + \frac{1}{2}\int \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 dx + \frac{1}{2}\int |\varphi|^2 dx. \quad (52)$$

It follows from $1 + (2+\alpha)/N \le p < (N+\alpha)/(N-2)$, Equation (52) and $\varphi \neq 0$ that $I(\varphi) > 0$ for all $\varphi \in \mathbb{M}$. Thus, by Equation (43), we obtain $d_2 \ge 0$. In the following, we will divide the proof into two cases: the L^2-supercritical case and the L^2-critical case.

We first consider the L^2-supercritical case $2 \le 1 + (2+\alpha)/N < p < (N+\alpha)/(N-2)$. In this case, it follows from Equation (10) that

$$\int (I_\alpha * |\varphi|^p)|\varphi|^p dx \le C\left(\int |\varphi|^{\frac{2Np}{N+\alpha}} dx\right)^{\frac{N+\alpha}{N}} \le C\left(\int (|\nabla \varphi|^2 + |\varphi|^2) dx\right)^p, \quad (53)$$

which together with $J(\varphi)<0$ implies

$$\begin{aligned}\int(|\nabla\varphi|^2+|\varphi|^2)dx &< \int(I_\alpha*|\varphi|^p)|\varphi|^p dx\\ &\le C\left(\int(|\nabla\varphi|^2+|\varphi|^2)dx\right)^p.\end{aligned} \tag{54}$$

Thus, one has that

$$\int(|\nabla\varphi|^2+|\varphi|^2)dx \ge C>0. \tag{55}$$

Since $p>1+(2+\alpha)/N$, we deduce from Equation (52) and (55) that

$$I(\varphi)\ge C>0,\ \text{for all}\ \varphi\in\mathbb{M}, \tag{56}$$

which implies $d_2>0$ for $1+(2+\alpha)/N<p<(N+\alpha)/(N-2)$.

Now we deal with the L^2-critical case $p=1+(2+\alpha)/N$. Suppose by contradiction that $d_2=0$, then we derive from Equation (43) that there exists a sequence $\{\varphi_n\}\subset\mathbb{M}$ such that $S(\varphi_n)=0$, $J(\varphi_n)<0$ and $I(\varphi_n)\to 0$ as $n\to\infty$. Since $p=1+(2+\alpha)/N$, one can derive from Equation (52) that

$$\int|\varphi_n|^2dx\to 0,\ \int\sum_{j=1}^k \nu_j^2x_j^2|\varphi_n|^2dx\to 0,\ \text{as}\ n\to\infty. \tag{57}$$

On the other hand, it follows from $J(\varphi_n)<0$ and Equation (11) that

$$\begin{aligned}&\int(|\nabla\varphi_n|^2+|\varphi_n|^2)dx<\int(I_\alpha*|\varphi_n|^p)|\varphi_n|^p dx\\ &\le C\int|\nabla\varphi_n|^2dx\left(\int|\varphi_n|^2dx\right)^{p-1}.\end{aligned} \tag{58}$$

However, when n is sufficiently large, from Equation (57), one has that

$$\int(|\nabla\varphi_n|^2+|\varphi_n|^2)dx>C\int|\nabla\varphi_n|^2dx\left(\int|\varphi_n|^2dx\right)^{p-1}. \tag{59}$$

It is obvious that Equation (59) contradicts Equation (58). Thus, $d_2>0$ for $p=1+(2+\alpha)/N$. Therefore, we have $d_2>0$ for $1+(2+\alpha)/N\le p<(N+\alpha)/(N-2)$. □

Now we define

$$d=\min\{d_1,d_2\}. \tag{60}$$

Then, we have the following conclusion.

Proposition 13. *Let* $1+(2+\alpha)/N\le p<(N+\alpha)/(N-2)$, *then* $d>0$.

Proof. From Equations (38) and (39), we obtain

$$\begin{aligned}d_1=&\inf_{\{\varphi\in\Sigma\setminus\{0\},J(\varphi)=0\}}\left(\frac{1}{2}-\frac{1}{2p}\right)\int(|\nabla\varphi|^2+|\varphi|^2)dx\\ &+\frac{1}{2}\int\sum_{j=1}^k\nu_j^2x_j^2|\varphi|^2dx.\end{aligned} \tag{61}$$

Therefore, $d_1>0$. This, together with Proposition 12, implies that the proposition holds true. □

To study the sharp threshold of global existence for Equation (1) in the L^2-supercritical case, we introduce some new cross-constrained invariant sets as follows:

Proposition 14. *Define*

$$\begin{aligned}\mathbb{K}&=\{\varphi\in\Sigma, I(\varphi)<d, S(\varphi)<0, J(\varphi)<0\},\\ \mathbb{K}_+&=\{\varphi\in\Sigma, I(\varphi)<d, S(\varphi)>0, J(\varphi)<0\},\\ \mathbb{R}_-&=\{\varphi\in\Sigma, I(\varphi)<d, J(\varphi)<0\},\\ \mathbb{R}_+&=\{\varphi\in\Sigma, I(\varphi)<d, J(\varphi)>0\}.\end{aligned} \tag{62}$$

Then $\mathbb{K}$, $\mathbb{K}_+$, $\mathbb{R}_-$, $\mathbb{R}_+$ *are invariant sets of Equation (1), that is, if* $\varphi_0\in\mathbb{K}$, $\mathbb{K}_+$, $\mathbb{R}_-$ *or* $\mathbb{R}_+$ *then the solution* $\varphi(t,x)$ *of the Equation (1) also satisfies* $\varphi(t,x)\in\mathbb{K}$, $\mathbb{K}_+$, $\mathbb{R}_-$ *or* $\mathbb{R}_+$ *for any* $t\in[0,T)$.

Proof. We first prove that $\mathbb{K}$ is an invariant set of Equation (1). Let $\varphi_0\in\Sigma$ and $\varphi(t,x)$ be the corresponding solution of Equation (1). From Equations (7) and (8), one has that

$$I(\varphi)=I(\varphi_0),\ \text{for}\ t\in[0,T). \tag{63}$$

Thus $I(\varphi_0)<d$ implies that $I(\varphi)<d$ for any $t\in[0,T)$.

Now we show $J(\varphi)<0$ for $t\in[0,T)$. If otherwise, by the continuity of $J(\varphi)$ on t, there exists $t_0\in[0,T)$ such that $J(\varphi(t_0,\cdot))=0$. By Equation (63), we have $\varphi(t_0,\cdot)\neq 0$. It is clear that Equations (42) and (60) imply $I(\varphi(t_0,\cdot))\ge d$. This is contradictory to $I(\varphi(t,\cdot))<d$ for $t\in[0,T)$. Thus $J(\varphi(t,\cdot))<0$ for all $t\in[0,T)$.

Then we show $S(\varphi(t,\cdot))<0$ for all $t\in[0,T)$. On the contrary, from the continuity, there exists $t'\in[0,T)$ such that $S(\varphi(t',\cdot))=0$. Because we have shown $S(\varphi(t',\cdot))=0$ and $J(\varphi(t',\cdot))<0$, it follows that $\varphi(t',\cdot)\in\mathbb{M}$. Thus, Equations (43) and (60) imply $I(\varphi(t',\cdot))\ge d_2\ge d$. This contradicts to $I(\varphi(t,\cdot))<d$ for all $t\in[0,T)$. Therefore $S(\varphi(t,\cdot))<0$ for all $t\in[0,T)$. From the above we have proved $\varphi(t,x)\in\mathbb{K}$ for any $t\in[0,T)$.

Similar to the proof above, we can also prove that $\mathbb{K}_+$, $\mathbb{R}_-$, $\mathbb{R}_+$ are invariant manifolds. □

In the below, we will use the cross-constrained variational approach to investigate the sharp condition of global existence for Equation (1).

Theorem 15. *If $\varphi_0 \in \mathbb{K}_+ \cup \mathbb{R}_+$, then the solution $\varphi(t,x)$ of the Cauchy problem (1) globally exists.*

Proof. For $\varphi_0 \in \mathbb{K}_+$, we have $\varphi(t,x) \in \mathbb{K}_+$ for $t \in [0,T)$ by Proposition 14. For $t \in [0,T)$, one has $I(\varphi) < d$ and $S(\varphi) > 0$. It follows from Equations (38) and (40) that

$$\int \left(\left[\frac{1}{2} - \frac{1}{Np - N - \alpha} \right] |\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 + \frac{1}{2}\sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 \right) dx < d. \tag{64}$$

First, we deal with the L^2-critical case $p = 1 + (2+\alpha)/N$. In this case, we infer from Equation (64) that

$$\frac{1}{2}\int \left(|\varphi|^2 + \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 \right) dx < d. \tag{65}$$

Denote $\varphi^\omega(x) = \omega^{\frac{\alpha+N}{2p}} \varphi(\omega x)$, then one has

$$S(\varphi^\omega(x)) = \omega^{\frac{2\alpha+4}{N+2+\alpha}} \int |\nabla\varphi(x)|^2 dx + \frac{N+\alpha-Np}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx. \tag{66}$$

It follows from $S(\varphi) > 0$ that there exists $0 < \omega_1 < 1$ such that $S(\varphi^{\omega_1}(x)) = 0$. Combining Equation (38) with Equation (40), we deduce that

$$\begin{aligned} I(\varphi^{\omega_1}(x)) &= \frac{1}{2}\int \left(|\varphi^{\omega_1}(x)|^2 + \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi^{\omega_1}(x)|^2 \right) dx \\ &= \frac{1}{2}\int \left(\omega_1^{-\frac{2N}{N+2+\alpha}} |\varphi(x)|^2 + \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi(x)|^2 \right) dx, \end{aligned} \tag{67}$$

which, together with Equation (64), yields

$$I(\varphi^{\omega_1}(x)) < \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} d. \tag{68}$$

Now we see $J(\varphi^{\omega_1})$, which only has two possibilities. One is $J(\varphi^{\omega_1}) < 0$. In this case, noting that $S(\varphi^{\omega_1}) = 0$, we infer from Equations (43) and (60) that

$$I(\varphi^{\omega_1}) \geq d_2 \geq d > I(\varphi). \tag{69}$$

Thus,

$$I(\varphi) - I(\varphi^{\omega_1}) < 0. \tag{70}$$

That is,

$$\begin{aligned} &\frac{1}{2}\left(1 - \omega_1^{\frac{4+2\alpha}{N+2+\alpha}}\right) \int |\nabla\varphi|^2 dx + \frac{1}{2}\left(1 - \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}}\right) \int \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 dx \\ &\quad + \frac{1}{2}\left(1 - \omega_1^{-\frac{2N}{N+2+\alpha}}\right) \int |\varphi|^2 dx < 0. \end{aligned} \tag{71}$$

It follows that

$$\int |\nabla\varphi|^2 dx < C \int \left(\sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 + |\varphi|^2 \right) dx. \tag{72}$$

By Equation (65), we obtain

$$\int |\nabla\varphi|^2 dx < C. \tag{73}$$

For $J(\varphi^{\omega_1})$, the other possible case is $J(\varphi^{\omega_1}) \geq 0$. In the present case, we deduce from the inequality Equation (68) that

$$I(\varphi^{\omega_1}(x)) - \frac{1}{2p} J(\varphi^{\omega_1}(x)) < \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} d. \tag{74}$$

Since $S(\varphi^{\omega_1}) = 0$ and Equation (74), one has

$$\begin{aligned} &\omega_1^{\frac{4+2\alpha}{N+2+\alpha}} \int |\nabla\varphi|^2 dx + \frac{p}{p-1} \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} \int \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 dx \\ &\quad + \omega_1^{-\frac{2N}{N+2+\alpha}} \int |\varphi|^2 dx < \frac{2p}{p-1} \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} d. \end{aligned} \tag{75}$$

It follows from Equation (75) that

$$\int |\nabla\varphi|^2 dx < C. \tag{76}$$

Thus, according to Proposition 1, we obtain that the solution $\varphi(t,x)$ is global in time.

When $1 + (2+\alpha)/N < p < (N+\alpha)/(N-2)$, by Equation (64), we also have

$$\int |\nabla\varphi|^2 + \sum_{j=1}^{k} \nu_j^2 x_j^2 |\varphi|^2 dx < C. \tag{77}$$

By Proposition 1, the solution $\varphi(t,x)$ of Equation (1) exists globally. Thus, the solution $\varphi(t,x)$ of Equation (1) with initial data $\varphi_0 \in \mathbb{K}_+$ exists globally on $t \in [0, +\infty)$.

Now we consider $\varphi_0 \in \mathbb{R}_+$. In view of Proposition 14, this gives immediately that the solution $\varphi(t,x)$ of Equation (1) satisfies that $\varphi(t,x) \in \mathbb{R}_+$ for $t \in [0,T)$. That is, $I(\varphi) < d$ and $J(\varphi) > 0$ for $t \in [0,T)$. By Equations (38) and (39), we get

$$\left(\frac{1}{2}-\frac{1}{2p}\right)\int\left(|\nabla\varphi|^2+|\varphi|^2+\sum_{j=1}^{k}\nu_j^2x_j^2|\varphi|^2\right)dx<d. \quad (78)$$

Thus, the solution of $\varphi(t,x)$ of Equation (1) exists globally. This completes the proof. □

Theorem 16. *Let* $\max\{2, 1+(2+\alpha)/N\}\le p<(N+\alpha)/(N-2)$. *If* $\int|x|^2|\varphi_0|^2dx<\infty$ *and* $\varphi_0\in\mathbb{K}$, *then the solution* $\varphi(t,x)$ *of Cauchy problem (1) blows up in finite time.*

Proof. For $\varphi_0\in\mathbb{K}$, we know from Proposition 14 that the solution $\varphi(t,x)$ of Equation (1) satisfies: $\varphi(t,x)\in\mathbb{K}$ for $t\in[0,T)$. For $V(t)=\int|x|^2|\varphi|^2dx$, it follows from Equations (22) and (40) that

$$V''(t)<8S(\varphi(t,\cdot)),\quad \text{for } t\in[0,T). \quad (79)$$

Thus for $t\in[0,T)$, φ satisfies that $S(\varphi)<0$, $J(\varphi)<0$. For $\mu>0$, we take $\varphi_\mu=\mu^{\frac{N+\alpha}{2p}}\varphi(\mu x)$. Thus

$$\begin{aligned} J(\varphi_\mu) &= \mu^{\frac{N+\alpha+2p-Np}{p}}\int|\nabla\varphi|^2dx+\mu^{\frac{N+\alpha-Np}{p}}\int|\varphi|^2dx \\ &\quad -\int(I_\alpha*|\varphi|^p)|\varphi|^pdx, \\ S(\varphi_\mu) &= \mu^{\frac{N+\alpha+2p-Np}{p}}\int|\nabla\varphi|^2dx-\frac{Np-N-\alpha}{2p} \\ &\quad \int(I_\alpha*|\varphi|^p)|\varphi|^pdx. \end{aligned} \quad (80)$$

Since $1+(2+\alpha)/N\le p<(N+\alpha)/(N-2)$, $S(\varphi)<0$, then there exists $\mu_1>1$ such that $S(\varphi_{\mu_1})=0$, and when $\mu\in[1,\mu_1)$, $S(\varphi_\mu)<0$. For $\mu\in[1,\mu_1]$, since $J(\varphi)<0$, $J(\varphi_\mu)$ has the following two cases:

(i) $J(\varphi_\mu)<0$ for $\mu\in[1,\mu_1]$;
(ii) There exists $1<\mu_2\le\mu_1$ such that $J(\varphi_{\mu_2})=0$.

For the case (i), we have $S(\varphi_{\mu_1})=0$ and $J(\varphi_{\mu_1})<0$. It follows from Equations (43) and (60) that

$$I(\varphi_{\mu_1})\ge d_2\ge d. \quad (81)$$

Furthermore, one has

$$\begin{aligned} I(\varphi)-I(\varphi_{\mu_1}) &= \frac{1}{2}\left(1-\mu_1^{\frac{N+\alpha+2p-Np}{p}}\right)\int|\nabla\varphi|^2dx \\ &\quad +\frac{1}{2}\left(1-\mu_1^{\frac{N+\alpha-Np}{p}}\right)\int|\varphi|^2dx \\ &\quad +\frac{1}{2}\left(1-\mu_1^{\frac{N+\alpha-2p-Np}{p}}\right)\int\sum_{j=1}^{k}\nu_j^2x_j^2|\varphi|^2dx, \end{aligned} \quad (82)$$

$$S(\varphi)-S(\varphi_{\mu_1})=\frac{1}{2}\left(1-\mu_1^{\frac{N+\alpha+2p-Np}{p}}\right)\int|\nabla\varphi|^2dx. \quad (83)$$

Taking into account that $\mu_1>1$ and $1+(2+\alpha)/N\le p<(N+\alpha)/(N-2)$, we infer from Equations (82) and (83) that

$$I(\varphi)-I(\varphi_{\mu_1})\ge S(\varphi)-S(\varphi_{\mu_1})=\frac{1}{2}S(\varphi). \quad (84)$$

For the case (ii), we have $J(\varphi_{\mu_2})=0$ and $S(\varphi_{\mu_2})\le 0$. Thus, Equations (42) and (60) yield that

$$I(\varphi_{\mu_2})\ge d_1\ge d. \quad (85)$$

It follows from Equations (82) and (83) that

$$I(\varphi)-I(\varphi_{\mu_2})\ge S(\varphi)-S(\varphi_{\mu_2})=\frac{1}{2}S(\varphi). \quad (86)$$

Since $I(\varphi_{\mu_1})\ge d$, $I(\varphi_{\mu_2})\ge d$, from Equations (84) and (86), we obtain

$$S(\varphi)<2[I(\varphi)-d]. \quad (87)$$

From $I(\varphi)=I(\varphi_0)$, $\varphi_0\in\mathbb{K}$ and Equation (87), one can estimate as follows:

$$V''(t)<8S(\varphi)<16[I(\varphi_0)-d]<0. \quad (88)$$

Then, by the convexity method introduced by Glassey [28], there must exist time $0<T<\infty$ such that $V(T)=0$. Then from Proposition 1 or Lemma 5, we have

$$\lim_{t\to T}\|\varphi\|_\Sigma=\infty. \quad (89)$$

Thus, the proof is completed. □

Remark 17. When $k=N$ and $\nu_1=\nu_2=\cdots=\nu_k=1$ in Equation (1), Feng [7] derived the sharp threshold for global existence and blow-up to the solutions of Equation (1) (see Theorem 3.10 and Theorem 3.11 by Feng [7]). Our results in Theorems 15 and 16 extend and compensate for the ones of Feng [7] for Equation (1) with anisotropic partial/whole harmonic confinement by constructing some new cross-invariant sets and minimisation problems.

Remark 18. It is obvious that

$$\{\varphi\in\Sigma\setminus\{0\}, I(\varphi)<d\}=\mathbb{R}_+\cup\mathbb{K}_+\cup\mathbb{K}. \quad (90)$$

In this sense, Theorem 16 implies that Theorem 15 is sharp when $\int|x|^2|\varphi_0|^2dx<\infty$.

By the above corollary, we immediately have

Corollary 19. *Let* $\max\{2, 1+(2+\alpha)/N\} \le p < (N+\alpha)/(N-2)$ *and* φ_0 *satisfy* $\int |x|^2|\varphi_0|^2 dx < \infty$ *and* $I(\varphi) < d$. *Then the solution* $\varphi(t,x)$ *of Equation (1) blows up in finite time if and only if* $\varphi_0 \in \mathbb{K}$.

By Theorem 15, we can get another sufficient condition of the global existence of Equation (1).

Corollary 20. *If* $\varphi_0 \in \Sigma$ *and* $\|\varphi_0\|_\Sigma^2 < 2d$, *then the corresponding solution* $\varphi(t,x)$ *of Equation (1) exists globally.*

Proof. Since $\|\varphi_0\|_\Sigma^2 < 2d$, we have $I(\varphi_0) < d$. Thus, we only need to prove $J(\varphi_0) > 0$. If otherwise, there exists γ with $0 < \gamma \le 1$, such that $J(\gamma\varphi_0) = 0$. From Equations (40), (59), and $J(\gamma\varphi_0) = 0$, we have

$$I(\gamma\varphi_0) \ge d. \tag{91}$$

On the other hand,

$$\|\gamma\varphi_0\|_\Sigma^2 = \gamma^2\|\varphi_0\|_\Sigma^2 < 2\gamma^2 d \le 2d. \tag{92}$$

Therefore, we have $I(\gamma\varphi_0) < d$, which gives a contradiction. Thus one has $\varphi_0 \in \mathbb{R}_+$. It follows from Theorem 15 that the corollary holds true. □

4. Mass Concentration and Dynamics of the L^2-Minimal Blow-up Solutions

In this section, we are devoted to the dynamical properties of blow-up solutions to Equation (1) with partial/whole harmonic confinement. We first study the mass concentration phenomenon and then the dynamics of the L^2-minimal blow-up solutions, including the precise mass-concentration and blow-up rate to the blow-up solutions with minimal mass.

In order to study the dynamical properties of the blow-up solutions of Equation (1), we recall the refined compactness lemma established by Feng and Yuan [22].

Lemma 21. *Let* $p = 1+(2+\alpha)/N$, $\{v_n\}_{n=1}^\infty$ *be a bounded sequence in* $H^1(\mathbb{R}^N)$ *and satisfy*

$$\limsup_{n\to\infty} \|\nabla v_n\|_2^2 \le M, \quad \limsup_{n\to\infty} \int (I_\alpha * |v_n|^p)|v_n|^p dx \ge m. \tag{93}$$

Then, there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ *such that, up to a subsequence,*

$$v_n(x+x_n) \rightharpoonup U \text{ weakly in } H^1(\mathbb{R}^N), \tag{94}$$

with $\|U\|_2 \ge (m/pM)^{\frac{1}{2p-2}}\|Q(x)\|_2$.

Using the refined compactness lemma, we can establish the following concentration property to the blow-up solutions of Equation (1).

Theorem 22. *(L^2-concentration) Assume* $N-2 \le \alpha < N$ *and* $p = 1+(2+\alpha)/N$. *Let* $\varphi(t,x)$ *be a solution of Equation (1) that blows up in finite time* T, *and* $s(t)$ *be a real-valued nonnegative function on* $[0,T)$ *such that* $s(t)\|\nabla\varphi\|_2 \to \infty$ *as* $t \to T$. *Then there exists a function* $x(t) \in \mathbb{R}^N$ *for* $t < T$ *such that*

$$\liminf_{t\to T} \int_{|x-x(t)|\le s(t)} |\varphi(t,x)|^2 dx \ge \int Q(x)^2 dx, \tag{95}$$

where $Q(x)$ *is the ground state solution of Equation (14).*

Proof. Set

$$\rho(t) = \frac{\|\nabla Q(x)\|_2}{\|\nabla\varphi\|_2}, \quad v(t,x) = \rho(t)^{\frac{N}{2}}\varphi(t,\rho(t)x). \tag{96}$$

Let $\{t_n\}_{n=1}^\infty$ be an arbitrary time sequence such that $t_n \to T$ as $n \to \infty$, and denote $\rho_n = \rho(t_n)$ and $v_n(x) = v(t_n,x)$. By Equations (7), (8), and (96), we obtain

$$\begin{aligned}&\|v_n\|_2 = \|\varphi(t_n)\|_2 = \|\varphi_0\|_2, \quad \|\nabla v_n\|_2 = \rho_n\|\nabla\varphi(t_n)\|_2\\ &\quad = \|\nabla Q(x)\|_2.\end{aligned} \tag{97}$$

For $f(x) \in H^1(\mathbb{R}^N)$, we define the functional

$$H(f(x)) = \frac{1}{2}\int \left(|\nabla f(x)|^2 - \frac{1}{p}(I_\alpha * |f(x)|^p)|f(x)|^p\right)dx. \tag{98}$$

From Equations (97), (6), and (96), one has that

$$\begin{aligned} H(v_n) &= \frac{1}{2}\int \left(|\nabla v_n|^2 - \frac{1}{p}(I_\alpha * |v_n|^p)|v_n|^p\right)dx\\ &= \rho_n^2 \int \frac{1}{2}\left(|\nabla\varphi(t_n)|^2 - \frac{1}{p}(I_\alpha * |\varphi(t_n)|^p)|\varphi(t_n)|^p\right)dx\\ &= \rho_n^2\left(E(\varphi(t_n)) - \frac{1}{2}\int \sum_{j=1}^k \nu_j^2 x_j^2 |\varphi(t_n)|^2 dx\right)\\ &\le \rho_n^2 E(\varphi(t_n)) = \rho_n^2 E(\varphi_0) \to 0 \text{ since } \rho_n \to 0 \text{ as } n \to \infty, \end{aligned} \tag{99}$$

which yields, in particular,

$$\int (I_\alpha * |\varphi(t_n)|^p)|\varphi(t_n)|^p dx \to p\|\nabla Q(x)\|_2^2 \text{ as } n \to \infty. \tag{100}$$

Take $M = \|\nabla Q(x)\|_2^2$ and $m = p\|\nabla Q(x)\|_2^2$. Then by Lemma 21, there exist $U(x) \in H^1(\mathbb{R}^N)$ and $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(\cdot+x_n)=\rho_n^{\frac{N}{2}}\varphi(t_n,\rho_n\cdot+x_n)\rightharpoonup U \text{ weakly in } H^1(\mathbb{R}^N), \tag{101}$$

with $\|U\|_2\geq\|Q(x)\|_2$. From Equation (101), it follows that

$$v_n(\cdot+x_n)\rightharpoonup U \text{ weakly in } L^2(\mathbb{R}^N). \tag{102}$$

Then, from Equation (102) and the weakly lower semicontinuous of the L^2-norm, it ensures that for any $A>0$,

$$\liminf_{n\to\infty}\int_{|x|\leq A}\rho_n^N|\varphi(t_n,\rho_n x+x_n)|^2dx\geq\int_{|x|\leq A}|U|^2dx. \tag{103}$$

Since

$$\lim_{n\to\infty}\frac{s(t_n)}{\rho_n}=\infty, \tag{104}$$

then there exists $n_0>0$ such that for any $n>n_0$, we obtain that $A\rho_n<s(t_n)$. It follows from $A\rho_n<s(t_n)$ and Equation (103) that

$$\begin{aligned}\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leq s(t_n)}|\varphi(t_n,x)|^2dx &\geq \liminf_{n\to\infty}\int_{|x-x_n|\leq A\rho_n}|\varphi(t_n,x)|^2dx\\ &=\liminf_{n\to\infty}\int_{|x|\leq A}\rho_n^N|\varphi(t_n,\rho_n x+x_n)|^2dx\\ &\geq\int_{|x|\leq A}|U|^2dx, \text{ for any } A>0,\end{aligned} \tag{105}$$

which implies that

$$\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leq s(t_n)}|\varphi(t_n,x)|^2dx\geq\int|U|^2dx=\|U\|_2^2. \tag{106}$$

Due to the arbitrariness of the sequence $\{t_n\}_{n=1}^{\infty}$, from $\|U\|_2\geq\|Q(x)\|_2$, we get that

$$\liminf_{t\to T}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leq s(t)}|\varphi(t,x)|^2dx\geq\|Q\|_2^2. \tag{107}$$

For every $t\in[0,T)$, one can easily see that the function $g(y):=\int_{|x-y|\leq s(t)}|\varphi(t,x)|^2dx$ is continuous on $y\in\mathbb{R}^N$ and $\lim_{|y|\to\infty}g(y)=0$. Therefore, for every $t\in[0,T)$, there exists a function $x(t)\in\mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leq s(t)}|\varphi(t,x)|^2dx=\int_{|x-x(t)|\leq s(t)}|\varphi(t,x)|^2dx. \tag{108}$$

Thus, it follows from Equations (107) and (108) that (101) holds true. □

Remark 23. According to Theorem 22, we know that the blow-up solutions of Equation (1) must have a lower L^2-bound, i.e., $\|\varphi_0\|_2\geq\|Q(x)\|_2$, which on the contrary, indicates that Theorem 8 holds true.

By Theorem 22, we can immediately obtain the conclusion below.

Corollary 24. *Let $\varphi(t,x)$ be a solution of Equation (1) that blows up in finite time T. Then for all $l>0$, there exists $x(t)\in\mathbb{R}^N$ for $t<T$ such that*

$$\liminf_{t\to T}\int_{B(x(t),l)}|\varphi(t,x)|^2dx\geq\int Q^2dx, \tag{109}$$

where $Q(x)$ is the ground state solution of Equation (14) and $B(x(t),l)=\{x\in\mathbb{R}^N\,||x-x(t)|\leq l\}$.

Theorem 25. *Assume that $N-2\leq\alpha<N$ and $p=1+(2+\alpha)/N$. Let $\varphi_0\in\Sigma$ and $\varphi(t,x)$ be the corresponding solution of problem Equation (1) that blows up in finite time T with $\|\varphi_0\|_2=\|Q(x)\|_2$. Then*

(i) *(Location of L^2-concentration point) there exists $x_0\in\mathbb{R}^N$ such that*

$$\lim_{t\to T}x(t)=x_0, \text{ and } |\varphi(t,x)|^2\to\|Q\|_2^2\delta_{x=x_0} \text{ in the distribution sense as } t\to T, \tag{110}$$

where $Q(x)$ is the ground state solution of Equation (14).

(ii) *(Blow-up rate) There exists a positive constant $C>0$ such that*

$$\|\nabla\varphi(t)\|_2 \geq \frac{C}{T-t}, \quad \forall t\in[0,T). \tag{111}$$

Proof.

(i) According to Equation (7) and $\|\varphi_0\|_2=\|Q(x)\|_2$, for $t<T$, we have

$$\|\varphi\|_2=\|\varphi_0\|_2=\|Q(x)\|_2. \tag{112}$$

On the other hand, from Theorem 22 and Corollary 24, for all $l>0$, one has that

$$\begin{aligned}\|Q(x)\|_2^2 &\leq \liminf_{t\to T}\int_{|x-x(t)|\leq l}|\varphi(t,x)|^2dx\\ &\leq \liminf_{t\to T}\int|\varphi(t,x)|^2dx \leq \|\varphi_0\|_2^2.\end{aligned} \tag{113}$$

It is distinct that Equations (112) and (113) deduce

$$\liminf_{t\to T}\int_{|x-x(t)|<l}|\varphi(t,x)|^2dx=\|Q(x)\|_2^2, \tag{114}$$

which implies that

$$|\varphi(t,x+x(t))|^2\to\|Q(x)\|_2^2\delta_{x=0},\ \text{in the distribution sense as } t\to T. \tag{115}$$

Next, we will prove that there exists $x_0\in\mathbb{R}^N$ such that

$$|\varphi(t,x)|^2\to\|Q(x)\|_2^2\delta_{x=x_0}\ \text{in the distribution sense as } t\to T. \tag{116}$$

In fact, for any real-valued function $\theta(x)$ defined on $\mathbb{R}^N$ and any real number β, from Equations (11) and (7), one can estimate

$$\begin{aligned}E\left(e^{i\beta\theta(x)}\varphi\right) &= \frac{1}{2}\int\left(\left|\nabla\left(e^{i\beta\theta(x)}\varphi\right)\right|^2+\sum_{j=1}^k\nu_j^2x_j^2\left|e^{i\beta\theta(x)}\varphi\right|^2-\frac{1}{p}\left(I_\alpha*\left|e^{i\beta\theta(x)}\varphi\right|^p\right)\left|e^{i\beta\theta(x)}\varphi\right|^p\right)dx\\ &\geq \frac{1}{2}\int\left|\nabla\left(e^{i\beta\theta(x)}\varphi\right)\right|^2dx-\frac{1}{2p}\int\left(I_\alpha*\left|e^{i\beta\theta(x)}\varphi\right|^p\right)\left|e^{i\beta\theta(x)}\varphi\right|^pdx\\ &\geq \frac{1}{2}\int\left|\nabla\left(e^{i\beta\theta(x)}\varphi\right)\right|^2dx\left(1-\frac{\left(\int|\varphi|^2dx\right)^{p-1}}{\left(\int|Q(x)|^2dx\right)^{p-1}}\right)\\ &= 0.\end{aligned} \tag{117}$$

Therefore, for any $\beta\in\mathbb{R}$, we infer from Equation (8) that

$$\begin{aligned}0\leq E\left(e^{i\beta\theta(x)}\varphi\right) &= \frac{1}{2}\int\left(\left|\nabla\left(e^{i\beta\theta(x)}\varphi\right)\right|^2+\sum_{j=1}^k\nu_j^2x_j^2\left|e^{i\beta\theta(x)}\varphi\right|^2-\frac{1}{p}\left(I_\alpha*\left|e^{i\beta\theta(x)}\varphi\right|^p\right)\left|e^{i\beta\theta(x)}\varphi\right|^p\right)dx\\ &= \frac{1}{2}\beta^2\int|\nabla\theta(x)\cdot\varphi|^2dx+\beta Im\int\nabla\theta(x)\cdot\nabla\varphi\cdot\overline{\varphi}dx+E(\varphi)\\ &= \frac{1}{2}\beta^2\int|\nabla\theta(x)|^2|\varphi|^2dx+\beta Im\int\nabla\theta(x)\cdot\nabla\varphi\cdot\overline{\varphi}dx+E(\varphi_0),\end{aligned} \tag{118}$$

which implies that

$$\left|\mathrm{Im}\int\nabla\theta(x)\cdot\nabla\varphi\cdot\overline{\varphi}dx\right|\leq\left[2E(\varphi_0)\int|\nabla\theta|^2|\varphi|^2dx\right]^{\frac{1}{2}}. \tag{119}$$

Then, choosing $\theta_a(x)=x_j$ for $j=1,2,\cdots,N$ in Equation (119), using Equations (1), (119), and (8), we derive

$$\begin{aligned}\left|\frac{d}{dt}\int|\varphi(t,x)|^2x_jdx\right| &= \left|2\mathrm{Im}\int i\varphi_t\cdot\overline{\varphi}\cdot x_jdx\right| \\ &= \left|2\mathrm{Im}\int\left(-\Delta\varphi+\sum_{j=1}^{k}\nu_j^2x_j^2\varphi\right.\right. \\ &\quad \left.-\lambda(I_\alpha*|\varphi|^p)|\varphi|^{p-2})\overline{\varphi}\cdot x_jdx\right| \\ &= \left|2\mathrm{Im}\int-\Delta\varphi\cdot\overline{\varphi}\cdot x_jdx\right| \\ &= \left|2\mathrm{Im}\int\nabla\varphi\cdot\overline{\varphi}\cdot\nabla x_jdx\right| \\ &\le 2\left(2E(\varphi_0)\int|\varphi_0|^2dx\right)^{\frac{1}{2}}=C.\end{aligned} \tag{120}$$

Taking any two sequences $\{t_n\}_{n=1}^{\infty}$, $\{t_m\}_{m=1}^{\infty}\subset[0,T)$ such that $\lim_{n\to\infty}t_n=\lim_{m\to\infty}t_m=T$. Therefore, for all $j=1,2,\ldots,N$, we deduce from the inequality Equation (120) that

$$\begin{aligned}&\left|\int|\varphi(t_n,x)|^2x_jdx-\int|\varphi(t_m,x)|^2x_jdx\right| \\ &\le\int_{t_m}^{t_n}\left|\frac{d}{dt}\int|\varphi(t,x)|^2x_jdx\right|dt \\ &\le C|t_n-t_m|\to0 \ as\ n,\ m\to\infty,\end{aligned} \tag{121}$$

which implies that

$$\lim_{t\to T}\int|\varphi(t,x)|^2x_jdx\ \text{exists for any}\ j=1,2,\ldots,N. \tag{122}$$

In other words,

$$\lim_{t\to T}\int|\varphi(t,x)|^2xdx\ \text{exists}. \tag{123}$$

Set $x_0=\frac{\lim_{t\to T}\int|\varphi(t,x)|^2xdx}{\|Q(x)\|_2^2}$, then $x_0\in\mathbb{R}^N$ and we obtain

$$\lim_{t\to T}\int|\varphi(t,x)|^2xdx=x_0\|Q(x)\|_2^2. \tag{124}$$

On the other hand, we infer from Equation (22) that

$$\frac{d^2}{dt^2}\int|x|^2|\varphi(t,x)|^2dx=16E(\varphi_0)-16\int\sum_{j=1}^{k}\nu_j^2x_j^2|\varphi|^2dx<16E(\varphi_0). \tag{125}$$

Thus, for any $t\in[0,T)$, there exists a constant $c_1>0$ such that

$$\int|x|^2|\varphi(t,x)|^2dx\le c_1. \tag{126}$$

Hence, we deduce that

$$\begin{aligned}\int|x|^2|\varphi(t,x+x(t))|^2dx &\le 2\int|x+x(t)|^2|\varphi(t,x+x(t))|^2dx \\ &\quad+2\int|x(t)|^2|\varphi(t,x+x(t))|^2dx \\ &\le 2c_1+2|x(t)|^2\|\varphi_0\|_2^2 \\ &=2c_1+2|x(t)|^2\|Q(x)\|_2^2.\end{aligned} \tag{127}$$

From Equation (115), it follows that

$$\begin{aligned}\limsup_{t\to T}|x(t)|^2\|Q(x)\|_2^2 &= \limsup_{t\to T}\int_{|x|<1}|x+x(t)|^2|\varphi(t,x+x(t))|^2dx \\ &\le\int|x|^2|\varphi(t,x)|^2dx\le c_0.\end{aligned} \tag{128}$$

From Equation (128), one can estimate

$$\limsup_{t\to T}|x(t)|\le\frac{\sqrt{c_1}}{\|Q(x)\|_2}. \tag{129}$$

Combined Equation (127) with Equation (129), we have

$$\limsup_{t\to T}\int|x|^2|\varphi(t,x+x(t))|^2dx\le C, \tag{130}$$

where $C=4c_1$. Thus, for any $l_0>0$, one has

$$\begin{aligned}&\limsup_{t\to T}\int_{|x|\ge l_0}l_0|x||\varphi(t,x+x(t))|^2dx \\ &\le\limsup_{t\to T}\int_{|x|\ge l_0}|x|^2|\varphi(t,x+x(t))|^2dx\le C.\end{aligned} \tag{131}$$

Therefore, for any $\varepsilon>0$, there exists a large enough $l_0=l_0(\varepsilon)>0$ such that

$$\limsup_{t\to T}\left|\int_{|x|\ge l_0}x|\varphi(t,x+x(t))|^2dx\right|\le\frac{C}{l_0}<\varepsilon. \tag{132}$$

Then, using Equations (132) and (115), we infer that for any $\varepsilon>0$

$$\begin{aligned}\limsup_{t\to T}\left|\int x|\varphi(t,x)|^2dx - x(t)\|Q(x)\|_2^2\right| &= \limsup_{t\to T}\left|\int x|\varphi(t,x)|^2dx - x(t)\int|\varphi(t,x)|^2dx\right|\\ &= \limsup_{t\to T}\left|\int \varphi|(t,x)|^2(x-x(t))dx\right|\\ &= \limsup_{t\to T}\left|\int \varphi|(t,x+x(t))|^2xdx\right|\\ &\le \limsup_{t\to T}\left|\int_{|x|\le l_0}|\varphi(t,x+x(t))|^2xdx\right| + \varepsilon\\ &= \varepsilon.\end{aligned} \tag{133}$$

It follows from Equations (124) and (133) that

$$\lim_{t\to T}x(t) = x_0,\ \text{and}\ \limsup_{t\to T}\int x|\varphi(t,x)|^2dx = x_0\|Q(x)\|_2^2. \tag{134}$$

Therefore, there exists $x_0 \in \mathbb{R}^N$ (see Equation (124)) such that

$$|\varphi(t,x)|^2 \to \|Q(x)\|_2^2\delta_{x=x_0}\ \text{in the distribution sense as}\ t\to T. \tag{135}$$

Thus, we know that Equation (110) holds true.

(ii) Taking $z(x)\in C_0^\infty(\mathbb{R}^N)$ is a nonnegative radial function such that

$$z(x) = z(|x|) = |x|^2,\ \mathit{if}\ |x|<1\ \text{and}\ |\nabla z(x)|^2 \le Cz(x). \tag{136}$$

For $h>0$, we define that $z_h(x)=h^2z(x/h)$ and $f_h(t)=\int z_h(x-x_0)|\varphi(t,\ x)|^2dx$ with x_0 define by Equation (124) (see also Equation (135)). From Equation (119), for every $t\in[0,T)$, we derive

$$\begin{aligned}\left|\frac{d}{dt}f_h(t)\right| &= \frac{d}{dt}\left|\int\varphi|(t,x)|^2z_h(x-x_0)dx\right|\\ &= \left|2\mathrm{Im}\int\nabla\varphi\cdot\overline{\varphi}\cdot\nabla z_h(x-x_0)dx\right|\\ &\le 2\left(2E(\varphi_0)\int|\varphi(t,x)|^2|\nabla z_h(x-x_0)|^2dx\right)^{\frac{1}{2}}\\ &\le C\sqrt{f_h(t)},\end{aligned} \tag{137}$$

which implies that

$$\left|\frac{d}{dt}\sqrt{f_h(t)}\right| \le C. \tag{138}$$

By integrating on both sides, one has that

$$\left|\sqrt{f_h(t)} - \sqrt{f_h(t^*)}\right| \le C|t-t^*|. \tag{139}$$

It is clear that Equation (110) implies

$$f_h(t^*) \to \|Q(x)\|_2z_h(0)\ \ as\ t^*\to T, \tag{140}$$

where $\|Q(x)\|_2z_h(0)=0$. Thus, by letting $t^*\to T$ in Equation (139), we deduce that

$$f_h(t) \le C(T-t)^2. \tag{141}$$

Fix $t\in[0,T)$ and let $h\to\infty$, we obtain

$$\int|\varphi(t,x)|^2|x-x_0|^2dx \le C(T-t)^2. \tag{142}$$

It follows from the uncertainty principle and the above inequality that

$$\begin{aligned}\|\varphi_0\|_2^2 &= \int|\varphi(t,x)|^2dx\\ &= -\frac{2}{N}Re\int\nabla\varphi\cdot\overline{\varphi}\cdot(x-x_0)dx\\ &\le C\left(\int|\varphi(t,x)|^2|x-x_0|^2dx\right)^{\frac{1}{2}}\left(\int|\nabla\varphi|^2dx\right)^{\frac{1}{2}}\\ &\le \tilde{C}(T-t)\|\nabla\varphi(t)\|_2,\end{aligned} \tag{143}$$

which means that

$$\|\nabla\varphi(t)\|_2 \ge \frac{\widetilde{C}}{T-t},\ \mathit{for}\ \forall\, t\in[0,T). \tag{144}$$

Therefore, the whole proof is completed. □

Remark 26.

(i) For Equation (1) without harmonic confinement, Feng and Yuan [22] derived the similar mass concentration properties of blow-up solutions and dynamical properties of the L^2-minimal blow-up solutions in the L^2-critical case (see Theorems 1.4 and 1.5 by Feng and Yuan [22]). Theorems 22 and 25 in our present paper extend the corresponding conclusions of Feng and Yuan [22] to the Schrödinger–Hartree equation with anisotropic partial/whole harmonic confinement.

(ii) As we know, the characterisation of the blow-up solutions with minimal mass depends strongly on the uniqueness of the ground state of Equation (14). However, in the general case $2 \leq p < (N+\alpha)/(N-2)$ and $0 < \alpha < N$, the uniqueness of the ground state of Equation (14) is still open, so we cannot obtain the limiting profile of the minimal mass blow-up solutions to initial-value problem Equation (1) at the moment, except for some special cases discussed by Miao et al. [20] and Genev and Venkov [21].

Data Availability

No underlying data were collected or produced in this study.

Disclosure

A preprint has previously been published (Min Gong, Hui Jian, Sharp threshold of global existence and mass concentration for the Schrödinger–Hartree equation with anisotropic harmonic confinement, 2022), and a reference to the preprint has been included in the reference list, see [29]. The present manuscript is an improved version.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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