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Research Article

Fixed Point Results in Fuzzy Strong Controlled Metric Spaces with an Application to the Domain Words

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In this manuscript, we introduce the notions of fuzzy strong controlled metric spaces, fuzzy strong controlled quasi-metric spaces, and non-Archimedean fuzzy strong controlled quasi-metric spaces and generalize the famous Banach contraction principle. We prove several fixed point results in the context of non-Archimedean fuzzy strong controlled quasi-metric space. Furthermore, we use our main result to obtain the existence of a solution for a recurrence problem linked with the study of Quicksort algorithms.

1. Introduction and Preliminaries

In 1965, Zadeh [1] introduced the notion of fuzzy sets. The term "fuzzy" appears to be highly common and prevalent in modern research linked to the logical and set-theoretical aspects of mathematics. We believe that the primary cause of this rapid change is simple to comprehend. The surrounding world is full of uncertainty, the information we obtain from the environment, the notions we use and the data resulting from our observation or measurement are, in general, vague, and incorrect. Because of this, each explicit representation of the world's reality or a portion of it is, in each instance, merely an estimate and an idealization of the real situation. Fuzzy concepts, such as fuzzy sets, fuzzy orderings, fuzzy languages, etc., make it possible to deal with and explore the aforementioned level of uncertainty in a mathematical and formal manner.

In 1988, Grabiec [2] proved a famous fuzzy version of the Banach contraction principle by employing the notion of a fuzzy metric space in the sense of Ivan Kramosil [3]. Although Grabiec's fixed point theorem has the drawback of not being applicable to the fuzzy metric induced by the Euclidean metric on $\mathbb{R}$, it is nevertheless useful (for more detail, see [4, 5]). Rakić et al. [6] proved several fixed-point theorems in the context of fuzzy b-metric spaces. As an important result, they gave a sufficient condition for a sequence to be Cauchy in a fuzzy b-metric space and they simplified the proofs of many fixed-point theorems in fuzzy b-metric spaces with the well-known contraction conditions. Mecheraoui et al. [7] proved several interesting fixed-point results in the context of E-fuzzy metric spaces. Moussaoui et al. [8] established several fixed-point results for contraction mappings via admissible functions and FZ-simulation functions in the context of fuzzy metric spaces. Zhou et al. [9] proved several fixed-point results for contraction mappings in the sense of non-Archimedean fuzzy metric spaces. Recently, Kanwal et al. [10] have established the notion of fuzzy strong b-metric spaces and generalized a fuzzy version of the Banach contraction principle. Sezen [11] presented a generalized version of Banach contraction principle in the context of controlled fuzzy metric spaces. Ishtiaq et al. [12] and Farhan et al. [13] used controlled function in generalization of metric spaces and proved several fixed point results with applications. Al-Omeri et al. [14] introduced (Φ, Ψ)-weak contractions in neutrosophic cone metric spaces and established several fixed point theorems. Al-Omeri et al. [15] and Al-Omeri [16] introduced several contraction mappings and topological structures in generalized spaces and derived some interesting results to find the fixed point for contraction mappings. Ghareeb and Al-Omeri [17] introduced new degrees for functions in (L, M)-fuzzy topological spaces based on (L, M)-fuzzy semiopen and (L, M)-fuzzy preopen operators. Batul et al. [18] examined several fuzzy fixed point results of fuzzy mappings on b-metric spaces. Mohammadi et al. [19] proved some fixed point results for generalized fuzzy contractive mappings in fuzzy metric spaces with application to integral equations. Rezaee et al. [20] worked on JS-Presic contractive mappings in extended modular S-metric spaces and extended fuzzy S-metric spaces.

We aim to extend the fuzzy version of the Banach contraction principle in the context of fuzzy strong controlled (fsc) metric spaces, fuzzy strong controlled quasi-metric spaces and non-Archimedean fuzzy strong controlled quasi-metric spaces. In fact, we prove results in the broader setting of non-Archimedean fuzzy strong controlled quasi-metric spaces, because in this case, measuring the distance between two words $\varkappa$ and ς automatically shows whether $\varkappa$ is a prefix of ς or not. Finally, we will use our approaches to show that some recurrence equations related to the complexity analysis of Quicksort algorithms have a solution (and that it is unique) (see [21–23]).

Kanwal et al. [10] established the following definition: Consider $\mathfrak{V} \neq \phi$ as an arbitrary set, $*$ is a continuous t-norm (Ct-norm), $g \geq 1$, and $\aleph$ is a fuzzy set (F-set) on $\mathfrak{V} \times \mathfrak{V} \times (0, +\infty)$. It is said to be a fsc-metric if it verifies for all $\acute{\omega}, \kappa, N \in \mathfrak{V}$ and $\varpi, k \geq 0$,

(i) $\aleph(\acute{\omega}, \kappa, 0) = 0$;
(ii) $\aleph(\acute{\omega}, \kappa, \varpi) = 1$if and only if $\acute{\omega} = \kappa$;
(iii) $\aleph(\acute{\omega}, \kappa, \varpi) = \aleph(\kappa, \acute{\omega}, \varpi)$;
(iv) $\aleph(\acute{\omega}, \kappa, \varpi) \times \aleph(\kappa, \mathrm{N}, k) \leq \aleph(\acute{\omega}, \mathrm{N}, \varpi + g.k)$;
(v) $\aleph(\acute{\omega}, \kappa, \cdot) : [0, +\infty) \longrightarrow [0, 1]$ is left continuous.

Then, $(\mathfrak{V}, \aleph, \times, g)$ is known as a fuzzy strong b-metric space.

2. Main Results

In this section, several new concepts and fixed-point results are demonstrated.

Definition 1. Consider $\mathfrak{V} \neq \phi$ is an arbitrary set, $*$ is a Ct-norm, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and $\aleph$ is a F-set on $\mathfrak{V} \times \mathfrak{V} \times (0, +\infty)$. It is said to be a fsc-metric if it verifies for all $\acute{\omega}, \kappa, N \in \mathfrak{V}$ and $\varpi, k \geq 0$,

(i) $\aleph(\acute{\omega}, \kappa, \varpi) \geq 0$;
(ii) $\aleph(\acute{\omega}, \kappa, \varpi) = 1$ if and only if $\acute{\omega} = \kappa$;
(iii) $\aleph(\acute{\omega}, \kappa, \varpi) = \aleph(\kappa, \acute{\omega}, \varpi)$;
(iv) $\aleph(\acute{\omega}, \kappa, \varpi) \times \aleph(\kappa, \mathrm{N}, k) \leq \aleph(\acute{\omega}, N, \varpi + \zeta(\acute{\omega}, N).k)$;
(v) $\aleph(\acute{\omega}, \kappa, \cdot) : [0, +\infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{\varpi \longrightarrow +\infty} \aleph(\acute{\omega}, \kappa, \varpi) = 1$.

Then $(\mathfrak{V}, \aleph, \times, \zeta)$ is known as fsc-metric space.

Remark 1. If we take $\zeta(\acute{\omega}, \kappa) = g \geq 1$, then any fsc-metric space is a fuzzy strong b-metric space.

Proposition 1. Assume $\mathfrak{V} = \mathbb{R}$ and $\zeta : \mathbb{R} \times \mathbb{R} \longrightarrow (1, +\infty)$ is defined by $\zeta(\acute{\omega}, \kappa) = 1 + |\acute{\omega} + \kappa|$.

Let $\aleph : \mathfrak{V} \times \mathfrak{V} \times [0, +\infty) \longrightarrow [0, 1]$ be defined by the following:

$$\aleph(\acute{\omega}, \kappa, \varpi) = \frac{\alpha^\tau \varpi^\tau}{\alpha^\tau \varpi^\tau + |\acute{\omega} - \kappa|^\rho}, \tag{1}$$

for all $\alpha, \tau > 0, \varpi \geq 0, \rho \in \mathbb{N}$ and $\acute{\omega}, \kappa \in \mathfrak{V}$. Then $(\mathfrak{V}, \aleph, \times, \zeta)$ is a fsc-metric space with product and minimum Ct-norms.

Proposition 2. Let $\mathfrak{V} = \mathbb{R}$ *and* $\zeta : \mathbb{R} \times \mathbb{R} \longrightarrow (1, +\infty)$ defined by $\zeta(\acute{\omega}, \kappa) = 1 + \acute{\omega}^2 + \kappa^2$.

Let $\aleph : \mathfrak{V} \times \mathfrak{V} \times [0, +\infty) \longrightarrow [0, 1]$ defined by the following:

$$\aleph(\acute{\omega}, \kappa, \varpi) = \left[e^{\frac{|\acute{\omega} - \kappa|^\rho}{\alpha^\tau \varpi^\tau}} \right]^{-1}, \tag{2}$$

for all $\alpha, \tau > 0, \varpi \geq 0$ and $\acute{\omega}, \kappa \in \mathfrak{V}$. Then $(\mathfrak{V}, \aleph, \times, \zeta)$ is a fsc-metric space with product and minimum Ct-norms.

Example 2.1. Let $\mathfrak{V} = \mathbb{R}^+$ and $f : \mathfrak{V} \longrightarrow \mathbb{R}^+$ be a one-to-one function. Assume a continuous and increasing function $g : \mathbb{R}^+ \longrightarrow (0, +\infty)$, fix $\alpha, \beta > 0$ and define $\aleph$ by the following:

$$\aleph(\acute{\omega}, \kappa, \varpi) = \left[\frac{(\min\{f(\acute{\omega}), f(\kappa)\})^\alpha + g(\varpi)}{(\max\{f(\acute{\omega}), f(\kappa)\})^\alpha + g(\varpi)} \right]^\beta. \tag{3}$$

Then, $(\mathfrak{V}, \aleph, \times, \zeta)$ is a fsc-metric space with product Ct-norm and $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ is defined by the following:

$$\zeta(\acute{\omega}, \kappa) = \left\{ \begin{array}{c} 1, \text{if } \acute{\omega} = \kappa, \\ \max\{1 + \acute{\omega}, 1 + \kappa\}, \text{if otherwise.} \end{array} \right\} \tag{4}$$

Proof. We examine only triangular inequality. Let $f(\acute{\omega}) \leq f(N)$, we have three cases:

(1) $f(\acute{\omega}) \leq f(\kappa) \leq f(N)$,
(2) $f(\acute{\omega}) \leq f(N) \leq f(\kappa)$,
(3) $f(\kappa) \leq f(\acute{\omega}) \leq f(N)$.

Now, if we put the following:

$$\begin{aligned} &\aleph(\acute{\omega}, N, \varpi + \zeta \cdot k) \\ &\quad = \left[\frac{f(\acute{\omega})^\alpha + g(\varpi + \zeta.k)}{f(\kappa)^\alpha + g(\varpi + \zeta.k)} \right]^\beta \cdot \left[\frac{f(\kappa)^\alpha + g(\varpi + \zeta.k)}{f(N)^\alpha + g(\varpi + \zeta.k)} \right]^\beta. \end{aligned} \tag{5}$$

Then, it is easy to examine the above three cases of inequality

$$\aleph(\acute{\omega}, \kappa, \varpi) \times \aleph(\kappa, N, k) \leq \aleph(\acute{\omega}, N, \varpi + \zeta.k), \tag{6}$$

satisfied, since g is increasing.

The proof in case $f(\acute{\omega}) > f(N)$ is similar. □

Definition 2. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a fsc-metric space.

(i) Suppose $\{\acute{\omega}_\tau\}$ is a sequence in $\mathfrak{V}$. The sequence $\{\acute{\omega}_\tau\}$ is said to be convergent to $\acute{\omega}$ if

$$\lim_{\tau \longrightarrow +\infty} \aleph(\acute{\omega}_\tau, \acute{\omega}, \varpi) = 1 \text{ for all } \varpi > 0. \tag{7}$$

(ii) We say that a sequence $\{\acute{\omega}_\tau\}$ is Cauchy if for each $\varpi > 0$, and any $\varepsilon \in (0, 1)$, there exists a natural number N such that $\aleph(\acute{\omega}_\tau, \acute{\omega}_\rho, \varpi) > 1 - \varepsilon$ for all $\tau, \rho > N$.

(iii) A fsc-metric space is known as a complete space if every Cauchy sequence is convergent in $\mathfrak{V}$.

We will utilize continuous fsc-metric space in the next study.

Theorem 1. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \kappa, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{8}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$, we deduce

$$\lim_{\tau \longrightarrow +\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau \longrightarrow +\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{9}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Proof. Assume $\acute{\omega}_0 \in \mathfrak{V}$ is an arbitrary point and $\{\acute{\omega}_\tau\}$ be a sequence in $\mathfrak{V}$, so that

$$\acute{\omega}_\tau = \mathcal{Q}\acute{\omega}_{\tau-1} = \mathcal{Q}^\tau \acute{\omega}_0 \text{ for all } \tau \in \mathbb{N}. \tag{10}$$

Now,

$$\begin{aligned}
&\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) \\
&\quad = \aleph(\mathcal{Q}^\tau \acute{\omega}_0, \mathfrak{R}^{\tau+1}\acute{\omega}_0, k\varpi) \geq \aleph(\mathcal{Q}^{\tau-1}\acute{\omega}_0, \mathcal{Q}^\tau \acute{\omega}_0, \varpi) \\
&\quad = \aleph(\acute{\omega}_{\tau-1}, \acute{\omega}_\tau, \varpi) = \aleph(\mathcal{Q}^{\tau-1}\acute{\omega}_0, \mathcal{Q}^\tau \acute{\omega}_0, \varpi) \\
&\quad \geq \aleph\left(\mathcal{Q}^{\tau-2}\acute{\omega}_0, \mathcal{Q}^{\tau-1}\acute{\omega}_0, \frac{\varpi}{k}\right) \\
&\quad = \aleph\left(\acute{\omega}_{\tau-2}, \acute{\omega}_{\tau-1}, \frac{\varpi}{k}\right) \geq \cdots \geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right).
\end{aligned} \tag{11}$$

That is,

$$\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) \geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right), \tag{12}$$

for each $\tau \in \mathbb{N}$ and $\varpi \geq 0$. Thus, for any integer $\rho > 0$ by utilizing triangular inequality, we deduce the following:

$$\begin{aligned}
\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, k\varpi) &\geq \aleph\left(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho}, \frac{\varpi}{2\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})}\right) \\
&\geq \aleph\left(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+2}, \frac{\varpi}{4\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})}\right) \times \aleph\left(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+3}, \frac{\varpi}{8\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})}\right) \\
&\times \aleph\left(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho}, \frac{\varpi}{8\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho})}\right) \\
&\vdots \\
&\times \aleph\left(\acute{\omega}_{\tau+\rho-1}, \acute{\omega}_{\tau+\rho}, \frac{\varpi}{2^{\rho-1}\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho}) \cdots \zeta(\acute{\omega}_{\tau+\rho-1}, \acute{\omega}_{\tau+\rho})}\right).
\end{aligned} \tag{13}$$

By utilizing Equations (12) and (13), we deduce the following:

$$\begin{aligned}
\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, \varpi) &\geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{2k^\tau}\right) \times \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{2^2\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})k^{\tau+1}}\right) \\
&\times \cdots \times \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{2^{\rho-1}\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho}) \cdots \zeta(\acute{\omega}_{\tau+\rho-1}, \acute{\omega}_{\tau+\rho})k^{\tau+\rho-1}}\right).
\end{aligned} \tag{14}$$

As $\tau \longrightarrow +\infty$ and $k^\tau \longrightarrow 0$ implies that $\frac{\varpi}{2k^\tau} \longrightarrow +\infty$, so by utilizing the definition of fsc-metric space, we get the following:

$$\aleph\left(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, \varpi\right) \geq 1 \times 1 \times 1 \times \cdots \times 1 = 1. \quad (15)$$

Thus, $\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, \varpi) \geq 1$ and this implies that $\{\acute{\omega}_\tau\}$ is a Cauchy sequence. Given $\mathfrak{V}$ is complete, so there exists κ in $\mathfrak{V}$ such that $\lim_{\tau\longrightarrow+\infty} \acute{\omega}_\tau = \acute{\omega}$.

Using triangular inequality

$$\begin{aligned}\aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) &\geq \aleph\left(\acute{\omega}, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}_{\tau+1}, \mathcal{Q}\acute{\omega}, \frac{\varpi}{2\zeta(\acute{\omega}, \mathcal{Q}\acute{\omega})}\right)\\ &\geq \aleph\left(\acute{\omega}, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\mathcal{Q}\acute{\omega}_\tau, \mathcal{Q}\acute{\omega}, \frac{\varpi}{2\zeta(\acute{\omega}, \mathcal{Q}\acute{\omega})}\right), \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi)\\ &\geq \aleph\left(\acute{\omega}, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}_\tau, \acute{\omega}, \frac{\varpi}{2\zeta(\acute{\omega}, \mathcal{Q}\acute{\omega})}\right).\end{aligned} \quad (16)$$

As $\tau \longrightarrow +\infty$, we get the following:

$$\aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) \geq \aleph\left(\acute{\omega}, \acute{\omega}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}, \acute{\omega}, \frac{\varpi}{2\zeta(\acute{\omega}, \mathcal{Q}\acute{\omega})k}\right) = 1 \times 1. \quad (17)$$

That is, $\aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) \geq 1$. So, $\mathcal{Q}\acute{\omega} = \acute{\omega}$.

Uniqueness: Let $\acute{\omega}$ and $\acute{\omega}^*$ be two fixed points of the operator $\mathcal{Q}$; then $\mathcal{Q}\acute{\omega} = \acute{\omega}$ and $\mathcal{Q}\acute{\omega}^* = \acute{\omega}^*$ hence,

$$\aleph(\mathcal{Q}\kappa, \kappa, \varpi) = 1 \text{ and } \aleph(\mathcal{Q}\kappa^*, \kappa^*, \varpi) = 1. \quad (18)$$

Then,

$$\begin{aligned}\aleph(\acute{\omega}, \acute{\omega}^*, \varpi) &= \aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\acute{\omega}^*, \varpi) \geq \aleph\left(\acute{\omega}, \acute{\omega}^*, \frac{\varpi}{k}\right)\\ &\geq \aleph\left(\acute{\omega}, \acute{\omega}^*, \frac{\varpi}{k^2}\right) \geq \cdots \geq \aleph\left(\acute{\omega}, \acute{\omega}^*, \frac{\varpi}{k^\tau}\right),\end{aligned} \quad (19)$$

for all $\tau \in \mathbb{N}$. By taking limit as $\tau \longrightarrow +\infty$ in the preceding inequality we get $\aleph(\acute{\omega}, \acute{\omega}^*, \varpi) = 1$ for all $\varpi > 0$, hence $\acute{\omega} = \acute{\omega}^*$. □

Example 2.2. Let $\mathfrak{V} = [0, 1]$ and $\zeta : \mathbb{R} \times \mathbb{R} \longrightarrow [1, +\infty)$ be defined by $\zeta(\acute{\omega}, \kappa) = 1 + |\acute{\omega} + \kappa|$.

Let $\aleph : \mathfrak{V} \times \mathfrak{V} \times [0, +\infty) \longrightarrow [0, 1]$ be defined by the following:

$$\aleph(\acute{\omega}, \kappa, \varpi) = \frac{\alpha^\tau \varpi^\tau}{\alpha^\tau \varpi^\tau + |\acute{\omega} - \kappa|^\rho}, \quad (20)$$

for all $\alpha, \tau > 0, \varpi \geq 0$ and $\acute{\omega}, \kappa \in \mathfrak{V}$. Then $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-metric space with product t-norm. Let $\mathcal{Q}(\varkappa) = \frac{\varkappa}{2}$, then

$$\begin{aligned}\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) &= \frac{k\alpha^\tau \varpi^\tau}{k\alpha^\tau \varpi^\tau + \left|\frac{\acute{\omega}}{2} - \frac{\kappa}{2}\right|^\rho} = \frac{2^\rho k\alpha^\tau \varpi^\tau}{2^\rho k\alpha^\tau \varpi^\tau + |\acute{\omega} - \kappa|^\rho}.\\ &\geq \frac{\alpha^\tau \varpi^\tau}{\alpha^\tau \varpi^\tau + |\acute{\omega} - \kappa|^\rho} = \aleph(\acute{\omega}, \kappa, \varpi)\end{aligned} \quad (21)$$

That is,

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \kappa, \varpi). \quad (22)$$

So, $\mathcal{Q}$ has a unique fixed point 0.

Theorem 2. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) \times \aleph(\kappa, \mathcal{Q}\kappa, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \quad (23)$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$,

$$\lim_{\tau\longrightarrow+\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau\longrightarrow+\infty} \zeta(\kappa, \acute{\omega}_\tau), \quad (24)$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Proof. Assume $\acute{\omega}_0 \in \mathfrak{V}$ is an arbitrary point and $\{\acute{\omega}_\tau\}$ be a sequence in $\mathfrak{V}$, so that

$$\acute{\omega}_\tau = \mathcal{Q}\,\acute{\omega}_{\tau-1} = \mathcal{Q}^\tau \acute{\omega}_0 \text{ for all } \tau \in \mathbb{N}. \quad (25)$$

Now,

$$\begin{aligned}\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) &= \aleph(\mathcal{Q}\acute{\omega}_{\tau-1}, \mathcal{Q}\acute{\omega}_\tau, k\varpi) \geq \aleph(\acute{\omega}_{\tau-1}, \mathcal{Q}\acute{\omega}_{\tau-1}, \varpi)\\ &\times \aleph(\acute{\omega}_\tau, \mathcal{Q}\acute{\omega}_\tau, \varpi) = \aleph(\acute{\omega}_{\tau-1}, \acute{\omega}_\tau, \varpi) \times \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \varpi).\end{aligned} \quad (26)$$

Since, $\aleph(\acute{\omega}, \kappa, \varpi)$ is strictly increasing and $k\varpi < \varpi$, we cannot write the following:

$$\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) \geq \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \varpi). \quad (27)$$

Therefore,

$$\begin{aligned}\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) &\geq \aleph(\acute{\omega}_{\tau-1}, \acute{\omega}_\tau, \varpi) = \aleph(\mathcal{Q}\acute{\omega}_{\tau-2}, \mathcal{Q}\acute{\omega}_{\tau-1}, \varpi) \geq \aleph\left(\acute{\omega}_{\tau-2}, \mathcal{Q}\acute{\omega}_{\tau-2}, \frac{\varpi}{k}\right) \times \aleph\left(\acute{\omega}_{\tau-1}, \mathcal{Q}\acute{\omega}_{\tau-1}, \frac{\varpi}{k}\right)\\ &\geq \aleph\left(\acute{\omega}_{\tau-2}, \acute{\omega}_{\tau-1}, \frac{\varpi}{k}\right) \times \aleph\left(\acute{\omega}_{\tau-1}, \acute{\omega}_\tau, \frac{\varpi}{k}\right) \geq \cdots \geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right), \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) \geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right).\end{aligned} \quad (28)$$

For every $\tau \in \mathbb{N}$ and $\varpi \geq 0$. Thus, for any integer $\rho > 0$ and by utilizing triangular inequality, we deduce the following:

$$\begin{aligned}\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, k\varpi) &\geq \aleph\left(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho}, \frac{\varpi}{2\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})}\right) \geq \aleph\left(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \frac{\varpi}{2}\right) \times \aleph\left(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+2}, \frac{\varpi}{4\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})}\right) \\ &\times \aleph\left(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+3}, \frac{\varpi}{8\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})}\right) \times \aleph\left(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho}, \frac{\varpi}{8\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho})}\right) \\ &\vdots \\ &\times \aleph\left(\acute{\omega}_{\tau+\rho-1}, \acute{\omega}_{\tau+\rho}, \frac{\varpi}{2^{\rho-1}\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho}) \cdots \zeta(\acute{\omega}_{\tau+\rho-1}, \acute{\omega}_{\tau+\rho})}\right).\end{aligned} \tag{29}$$

By utilizing Equations (28) and (29), we deduce the following:

$$\begin{aligned}\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, \varpi) &\geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{2k^\tau}\right) \times \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{2^2\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})k^{\tau+1}}\right) \times \cdots \\ &\times \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{2^{\rho-1}\zeta(\acute{\omega}_{\tau+1}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+2}, \acute{\omega}_{\tau+\rho})\zeta(\acute{\omega}_{\tau+3}, \acute{\omega}_{\tau+\rho}) \cdots \zeta(\acute{\omega}_{\tau+\rho-1}, \acute{\omega}_{\tau+\rho})k^{\tau+\rho-1}}\right).\end{aligned} \tag{30}$$

Taking as $\tau \longrightarrow +\infty, k^\tau \longrightarrow 0,$ this implies that $\frac{\varpi}{2k^\tau} \longrightarrow +\infty$, so by utilizing definition of a fsc-metric space, we get the following:

$$\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, \varpi) \geq 1 \times 1 \times 1 \times \cdots \times 1 = 1. \tag{31}$$

Thus, $\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+\rho}, \varpi) \geq 1$ and this implies that $\{\acute{\omega}_\tau\}$ is a Cauchy sequence. Given $\mathfrak{V}$ is complete and so, there exists κ in $\mathfrak{V}$ such that $\lim_{\tau\longrightarrow+\infty} \acute{\omega}_\tau = \acute{\omega}$.

Now, utilizing the contractive condition,

$$\begin{aligned}\aleph(\mathcal{Q}\acute{\omega}_\tau, \mathcal{Q}\acute{\omega}, k\varpi) &\geq \aleph(\acute{\omega}_\tau, \mathcal{Q}\acute{\omega}_\tau, \varpi) \times \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) \\ &\geq \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, \varpi) \times \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi),\end{aligned} \tag{32}$$

As $\tau \longrightarrow +\infty,$ we have the following:

$$\begin{aligned}\aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, k\varpi) &\geq \aleph(\acute{\omega}, \acute{\omega}, \varpi) \times \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) \\ &= 1 \times \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi), \\ \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, k\varpi) &\geq \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi),\end{aligned} \tag{33}$$

which is a contradiction. Hence, $\mathcal{Q}\acute{\omega} = \acute{\omega}$. So, $\acute{\omega}$ is a fixed point of $\mathcal{Q}$.

Uniqueness: Let $\acute{\omega}$ and $\acute{\omega}^*$ be two fixed points of $\mathcal{Q}$. So, $\mathcal{Q}\acute{\omega} = \acute{\omega}$ and $\mathcal{Q}\acute{\omega}^* = \acute{\omega}^*$, then

$$\begin{aligned}\aleph(\acute{\omega}, \acute{\omega}^*, \varpi) &= \aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\acute{\omega}^*, \varpi) \\ &\geq \aleph\left(\acute{\omega}, \mathcal{Q}\acute{\omega}, \frac{\varpi}{k}\right) \times \aleph\left(\acute{\omega}^*, \mathcal{Q}\acute{\omega}^*, \frac{\varpi}{k}\right) = 1 \times 1.\end{aligned} \tag{34}$$

That is, $\aleph(\acute{\omega}, \acute{\omega}^*, \varpi) = 1$. Hence, $\acute{\omega} = \acute{\omega}^*$. □

Corollary 1. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying.

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \mathcal{Q}\kappa, \varpi) \times \aleph(\kappa, \mathcal{Q}\acute{\omega}, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{35}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V},$ we obtain the following:

$$\lim_{\tau\longrightarrow+\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau\longrightarrow+\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{36}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Proof. Immediate from Theorem 2. □

Definition 3. Suppose $h : \mathfrak{V} \longrightarrow \mathfrak{V}$ and $O(\acute{\omega}_0) = \{\acute{\omega}_0, h\acute{\omega}_0, h^2\acute{\omega}_0, \cdots\}$ for some $\acute{\omega}_0 \in \mathfrak{V}$ is an orbit of h. A function $T : \mathfrak{V} \longrightarrow \mathbb{R}$ is known as h-orbitally lower semi continuous at $\nu \in \mathfrak{V}$ if for $\{\acute{\omega}_\tau\} \subset O(\acute{\omega}_0)$ such that $\acute{\omega}_\tau \longrightarrow \nu,$ then we obtain $T(\nu) \geq \lim_{\tau\longrightarrow+\infty} \sup T(\acute{\omega}_\tau)$.

Example 2.3. Let $\mathfrak{V} = [0, 1]$ and $h : \mathfrak{V} \longrightarrow \mathfrak{V}$ defined by $h(\varkappa) = \frac{\varkappa}{2}$. Pick an element $\acute{\omega}_0 = \frac{1}{2}$ in $\mathfrak{V}$, then we obtain the following:

$$O(\acute{\omega}_0) = O\left(\frac{1}{2}\right) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots\right\}, \tag{37}$$

Observe that for any sequence $\{\acute{\omega}_\tau\} \subset O(\frac{1}{2})$, we examine $\acute{\omega}_\tau \longrightarrow 0$. Let $T : \mathfrak{V} \longrightarrow \mathbb{R}$ be defined by the following:

$$T(\varkappa) = \begin{cases} 1, \text{ if } \varkappa = 0 \\ \dfrac{3\varkappa + \sqrt{\varkappa}}{3}, \text{ if } 0 < \varkappa \leq 1. \end{cases} \tag{38}$$

Now, $T(0) = 3$ and $\acute{\omega}_\tau \longrightarrow \nu = 0$ implies that

$$T(0) = 1 > 0 = \lim_{\tau \longrightarrow +\infty} \sup T(\acute{\omega}_\tau) = \lim_{\tau \longrightarrow +\infty} \sup \frac{3\acute{\omega}_\tau + \sqrt{\acute{\omega}_\tau}}{3}, \tag{39}$$

which implies that h is orbital lower semicontinuous.

Theorem 3. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(h\acute{\omega}, h^2\acute{\omega}, k\varpi) \geq \aleph(\acute{\omega}, h\acute{\omega}, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1), \tag{40}$$

for every $\acute{\omega} \in O(\acute{\omega})$ and $\varpi > 0$, *where* $0 < k < 1$. *Then* $h^\tau \acute{\omega}_0 \longrightarrow \nu$. *Furthermore,* ν *is a fixed point* of h *if and only if* $T\acute{\omega} = \aleph(\acute{\omega}, h\acute{\omega}, \varpi)$ *is* h *is orbital lower semi continuous at* ν.

Proof. Assume $\acute{\omega}_0 \in \mathfrak{V}$ is an arbitrary point and $\{\acute{\omega}_\tau\}$ be a sequence in $\mathfrak{V}$, so that

$$\acute{\omega}_\tau = h\acute{\omega}_{\tau-1} = h\acute{\omega}_0 \text{ for all } \tau \in \mathbb{N}, \tag{41}$$

Now,

$$\begin{aligned} \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) &= \aleph(h^\tau \acute{\omega}_0, h^{\tau+1}\acute{\omega}_0, k\varpi) \\ &\geq \aleph(h^{\tau-1}\acute{\omega}_0, h^\tau \acute{\omega}_0, \varpi) \\ &= \aleph(\acute{\omega}_{\tau-1}, \acute{\omega}_\tau, \varpi) = \aleph(h^{\tau-1}\acute{\omega}_0, h^\tau \acute{\omega}_0, \varpi) \\ &\geq \aleph\left(h^{\tau-2}\acute{\omega}_0, h^{\tau-1}\acute{\omega}_0, \frac{\varpi}{k}\right) \\ &= \aleph\left(\acute{\omega}_{\tau-2}, \acute{\omega}_{\tau-1}, \frac{\varpi}{k}\right) \geq \cdots \geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right), \end{aligned} \tag{42}$$

That is,

$$\aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) \geq \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right). \tag{43}$$

Same manners of Theorem 1, we get $\{\acute{\omega}_\tau\}$ is a Cauchy sequence. From the completeness of $\mathfrak{V}$, we have $\acute{\omega}_\tau \longrightarrow \nu$. Suppose that T is orbitally lower semicontinuous at $\nu \in \mathfrak{V}$, then we obtain the following:

$$\begin{aligned} \aleph(\nu, h\nu, k\varpi) &\geq \lim_{\tau \longrightarrow +\infty} \sup \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi) \\ &\geq \lim_{\tau \longrightarrow +\infty} \sup \aleph\left(\acute{\omega}_0, \acute{\omega}_1, \frac{\varpi}{k^{\tau-1}}\right) = 1. \end{aligned} \tag{44}$$

Conversely, suppose $h\nu = \nu$ and $\acute{\omega}_\tau \subset O(\acute{\omega})$ with $\acute{\omega}_\tau \longrightarrow \nu$, then we have the following:

$$T(\nu) = \aleph(\nu, h\nu, k\varpi) \geq \lim_{\tau \longrightarrow +\infty} \sup T(\acute{\omega}_\tau) = \aleph(\acute{\omega}_\tau, \acute{\omega}_{\tau+1}, k\varpi). \tag{45}$$

□

Corollary 2. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ be a complete fsb-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(h\acute{\omega}, h^2\acute{\omega}, k\varpi) \geq \aleph(\acute{\omega}, h\acute{\omega}, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1), \tag{46}$$

for every $\acute{\omega} \in O(\acute{\omega})$, $\varpi > 0$, *where* $0 < k < 1$. *Then* $h^\tau \acute{\omega}_0 \longrightarrow \varpi$. *Furthermore,* ϖ *is a fixed point of* h *if and only if* $T\acute{\omega} = \aleph(\acute{\omega}, h\acute{\omega}, \varpi)$ *is* h *is orbital lower semi continuous at* ν.

Proof. Immediate from Theorem 3. □

Definition 4. Consider $\mathfrak{V} \neq \phi$ be an arbitrary set, $*$ be a Ct-norm, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and $\aleph$ be a F-set on $\mathfrak{V} \times \mathfrak{V} \times (0, +\infty)$. It is said to be a fsc-quasi-metric if it verifies for all $\acute{\omega}, \kappa, \mathrm{N} \in \mathfrak{V}$ and $\varpi, k \geq 0$,

(i) $\aleph(\acute{\omega}, \kappa, 0) = 0$;
(ii) $\aleph(\acute{\omega}, \kappa, \varpi) = \aleph(\kappa, \acute{\omega}, \varpi) = 1$ if and only if $\acute{\omega} = \kappa$;
(iii) $\aleph(\acute{\omega}, \kappa, \varpi) \times \aleph(\kappa, N, k) \leq \aleph(\acute{\omega}, N, \varpi + \zeta(\acute{\omega}, N).k)$;
(iv) $\aleph(\acute{\omega}, \kappa, \cdot) : [0, +\infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{\varpi \longrightarrow +\infty} \aleph(\acute{\omega}, \kappa, \varpi) = 1$.

Then $(\mathfrak{V}, \aleph, \times, \zeta)$ is known as a fsc-quasi-metric space.

Remark 2. Every fsc-quasi-metric space $(\mathfrak{V}, \aleph, \times, \zeta)$ is nondecreasing.

Remark 3. If $(\mathfrak{V}, \aleph, \times, \zeta)$ is a fsc-quasi-metric space, then $(\mathfrak{V}, \aleph^{-1}, \times, \zeta)$ is also a fsc-quasi-metric space, where $\aleph^{-1}$ is a fuzzy set defined by $\aleph^{-1}(\varkappa, \varsigma, \varpi) = \aleph(\varsigma, \varkappa, \varpi)$. Moreover, we denote fuzzy set $\aleph^i$ by $\aleph^i(\varkappa, \varsigma, \varpi) = \min\{\aleph(\varkappa, \varsigma, \varpi), \aleph^{-1}(\varkappa, \varsigma, \varpi)\}$, then $(\mathfrak{V}, \aleph^i, \times, \zeta)$ is a fsc-metric space.

Theorem 4. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ be a complete fsc-quasi-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \kappa, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{47}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$, we deduce

$$\lim_{\tau \longrightarrow +\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau \longrightarrow +\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{48}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Theorem 5. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-quasi-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(\mathcal{Q}\acute{\omega}, \Re\kappa, k\varpi) \geq \aleph(\acute{\omega}, \mathcal{Q}\acute{\omega}, \varpi) \times \aleph(\kappa, \mathcal{Q}\kappa, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{49}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$, we deduce the following:

$$\lim_{\tau \longrightarrow +\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau \longrightarrow +\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{50}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Theorem 6. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-quasi-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \mathcal{Q}\kappa, \varpi) \times \aleph(\kappa, \mathcal{Q}\acute{\omega}, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{51}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$, we deduce the following:

$$\lim_{\tau \longrightarrow +\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau \longrightarrow +\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{52}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Theorem 7. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a complete fsc-quasi-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifying

$$\aleph(h\acute{\omega}, h^2\acute{\omega}, k\varpi) \geq \aleph(\acute{\omega}, h\acute{\omega}, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1), \tag{53}$$

for every $\acute{\omega} \in O(\acute{\omega}), \varpi > 0,$ where $0 < k < 1$. Then $h^\tau \acute{\omega}_0 \longrightarrow \varpi$. Furthermore, ϖ is a fixed point of h if and only if $T\acute{\omega} = \aleph(\acute{\omega}, h\acute{\omega}, \varpi)$ is h orbital lower semi continuous at v.

Definition 5. If $(\mathfrak{V}, \aleph, \times, \zeta)$ is a fsc-quasi-metric space, then it is known as bicomplete fsc-quasi-metric space if $(\mathfrak{V}, \aleph^i, \times, \zeta)$ is complete.

Theorem 8. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a bicomplete fsc-quasi-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a mapping verifies

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \kappa, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{54}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$, we deduce the following:

$$\lim_{\tau \longrightarrow +\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau \longrightarrow +\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{55}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

Proof. Immediate if we take $\aleph^i(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph^i(\acute{\omega}, \kappa, \varpi)$ and proceeding on the lines of Theorem 1. □

Definition 6. A fsc-quasi-metric space $(\mathfrak{V}, \aleph, \times, \zeta)$ such that

$$\aleph(\varkappa, \varsigma, \varpi + \zeta(\acute{\omega}, \mathrm{N}).k) \geq \min \{\aleph(\varkappa, z, \varpi), \aleph(z, \varsigma, k)\}, \tag{56}$$

for all $\varkappa, \varsigma, z \in \mathfrak{V}$ and $\varpi > 0$ is known as a non-Archimedean fsc-quasi-metric space.

Theorem 9. Suppose $(\mathfrak{V}, \aleph, \times, \zeta)$ is a bicomplete non-Archimedean fsc-quasi-metric space, $\zeta : \mathfrak{V} \times \mathfrak{V} \longrightarrow (1, +\infty)$ and let $\mathcal{Q} : \mathfrak{V} \longrightarrow \mathfrak{V}$ is a mapping verifying

$$\aleph(\mathcal{Q}\acute{\omega}, \mathcal{Q}\kappa, k\varpi) \geq \aleph(\acute{\omega}, \kappa, \varpi) \text{ for all } \acute{\omega}, \kappa \in \mathfrak{V} \text{ and } k \in (0, 1). \tag{57}$$

Also, suppose that for each $\acute{\omega} \in \mathfrak{V}$, we deduce the following:

$$\lim_{\tau \longrightarrow +\infty} \zeta(\acute{\omega}_\tau, \kappa) \text{ and } \lim_{\tau \longrightarrow +\infty} \zeta(\kappa, \acute{\omega}_\tau), \tag{58}$$

exists and are finite. Then $\mathcal{Q}$ has a unique fixed point in $\mathfrak{V}$.

3. Quicksort Algorithm

Let τ be the size of the input and $\mathscr{E}(\tau)$ be the average (anticipated value) of the number of times the algorithm performs the fundamental operation for an input size of ρ for a given algorithm. Now we look at the quicksort algorithm, which was established by Hoare [22] (for more details, see [21]). Quicksort performs the sort by dividing the array into partitions and then recursively sorting each partition.

Average-case time complexity.

The basic operation compares $S[i]$ to pivot items in a partition.

The number of items in the array S determines the size of the input.

We suppose that there is no reason to believe the numbers in the array are in any particular order and that the value of the pivot point provided by partition might be any integer from 1 to τ. This study would be invalid if there were cause to believe the different distributions. When every conceivable ordering is sorted the same number of times, the average achieved is the average sorting time. The following recurrence gives the average-case time complexity in this case:

$$\mathcal{E}(\tau) = \sum_{\rho=1}^{\tau} \frac{1}{\tau}[\mathcal{E}(\rho-1) + \mathcal{E}(\tau-\rho)] + \tau - 1, \quad (59)$$

$\frac{1}{\tau}$ = probability pivot point is ρ,

$[\mathcal{E}(\rho-1) + \mathcal{E}(\tau-\rho)]$ = average time to sort sub arrays when pivot point is p,

$\tau - 1$ = times of partition.

Therefore,

$$\sum_{\rho=1}^{\tau}[\mathcal{E}(\rho-1) + \mathcal{E}(\tau-\rho)] = 2\sum_{\rho=1}^{\tau}\mathcal{E}(\rho-1). \quad (60)$$

Combination of Equations (59) and (60) yields

$$\mathcal{E}(\tau) = \frac{2}{\tau}\sum_{\rho=1}^{\tau}\mathcal{E}(\rho-1) + (\tau-1), \quad (61)$$

multiplying τ on both sides, we get the following:

$$\tau\mathcal{E}(\tau) = 2\sum_{\rho=1}^{\tau}\mathcal{E}(\rho-1) + \tau(\tau-1). \quad (62)$$

Utilizing Equation (62) to $\tau - 1$ yields

$$(\tau-1)\mathcal{E}(\tau-1) = 2\sum_{\rho=1}^{\tau-1}\mathcal{E}(\rho-1) + (\tau-1)(\tau-2). \quad (63)$$

Subtracting Equation (62) from Equation (63) gives

$$\tau\mathcal{E}(\tau) - (\tau-1)\mathcal{E}(\tau-1) = 2\mathcal{E}(\tau-1) + 2(\tau-1), \quad (64)$$

which yields

$$\mathcal{E}(1) = 0, \mathcal{E}(\tau) = \frac{2(\tau-1)}{\tau} + \frac{\tau+1}{\tau}\mathcal{E}(\tau-1), \tau \geq 2. \quad (65)$$

4. Application to Domain Words

Suppose a nonempty alphabet $\sum$ and assume the set of all finite and infinite sequences (words) over $\sum$, that is $\sum_{+\infty}$. Where we assume that $\phi \in \sum_{+\infty}$. The prefix order on $\sum_{+\infty}$ denoted by $\sqsubseteq$, i.e., $\varkappa \sqsubseteq \varsigma \Longleftrightarrow \varkappa$ is a prefix of ς. For every $\varkappa \in \sum_{+\infty}$ defined by $l(\varkappa)$, the length of $\varkappa$. That is, $l(\varkappa) \in [1, +\infty)$ whenever $\varkappa \neq \phi$ and $l(\phi) = 0$. For every $\varkappa, \varsigma \in \sum_{+\infty}$, suppose $\varkappa \sqcap \varsigma$ be the common prefix of $\varkappa$ and ς. Thus the function $d_{\sqsubseteq}$ defined on $\sum_{+\infty} \times \sum_{+\infty}$ by the following:

$$\begin{aligned} d_{\sqsubseteq}(\varkappa,\varsigma) &= 0, \text{ if } \varkappa \sqsubseteq \varsigma \\ d_{\sqsubseteq}(\varkappa,\varsigma) &= 2^{-l(\varkappa\sqcap\varsigma)}, \text{ if otherwise,} \end{aligned} \quad (66)$$

is a quasi-metric on $\sum_{+\infty}$. We take the convention $2^{-\infty} = 0$.

Let

$$\begin{aligned} &\aleph^{d_{\sqsubseteq 1}}(\varkappa,\varsigma,0) = 0, \text{for all } \varkappa,\varsigma \in \sum_{+\infty}, \\ &\aleph^{d_{\sqsubseteq 1}}(\varkappa,\varsigma,\varpi) = 1, \text{if } \varkappa \sqsubseteq \varsigma, \\ &\aleph^{d_{\sqsubseteq 1}}(\varkappa,\varsigma,\varpi) = 1 - 2^{-l(\varkappa\sqcap\varsigma)}, \\ &\text{if } \varkappa \text{ is not a prefix of } \varsigma \text{ and } \varpi \in (0,1], \\ &\aleph^{d_{\sqsubseteq 1}}(\varkappa,\varsigma,\varpi) = 1, \text{if } \varkappa \text{ is not a prefix of } \varsigma \text{ and } \varpi > 1. \end{aligned} \quad (67)$$

Remark 4. $(\sum_{+\infty}, \aleph^{d_{\sqsubseteq 1}}, \times, \zeta)$ is a bicomplete non-Archimedean fsc-quasi-metric space with minimum Ct-norm and $\zeta : \sum_{+\infty} \times \sum_{+\infty} \longrightarrow (1, +\infty)$ defined by $\zeta(\varkappa,\varsigma) = 1 + |\varkappa| + |\varsigma|$.

Let $\aleph^{d_{\sqsubseteq}}(\varkappa,\varsigma,\varpi)$ be defined as follows:

$$\aleph^{d_{\sqsubseteq}}(\varkappa,\varsigma,\varpi) = \begin{cases} 0, \text{if } \varpi = 0, \text{for all } \varkappa,\varsigma \in \sum_{+\infty}, \\ \dfrac{\varpi}{\varpi + 2^{-l(\varkappa\sqcap\varsigma)}}, \text{if } \varkappa \text{ is not a prefix of } \varsigma \text{ and } \varpi > 0, \\ 1, \text{if } \varkappa \sqsubseteq \varsigma \text{ and } \varpi > 0. \end{cases} \quad (68)$$

Remark 5. $(\sum_{+\infty}, \aleph^{d_{\sqsubseteq}}, \times, \zeta)$ is a bicomplete non-Archimedean fsc-quasi-metric space with minimum Ct-norm and $\zeta : \sum_{+\infty} \times \sum_{+\infty} \longrightarrow (1, +\infty)$ defined by $\zeta(\varkappa,\varsigma) = 1 + |\varkappa| + |\varsigma|$.

Next, for complexity analysis of the quicksort algorithm, utilize Theorem 9. The below recurrence equation

$$\mathcal{E}(1) = 0, \mathcal{E}(\tau) = \frac{2(\tau-1)}{\tau} + \frac{\tau+1}{\tau}\mathcal{E}(\tau-1), \tau \geq 2, \quad (69)$$

is examined in the average case analysis of the quicksort algorithm. Assume as an alphabet $\sum = (0, +\infty)$. We associate $\mathcal{E}$ with the functional $\Psi : \sum_{+\infty} \longrightarrow \sum_{+\infty}$ defined by the following:

$$(\Psi(\varkappa))_1 = \mathcal{E}(1) \text{ and } (\Psi(\varkappa))_\tau = \frac{2(\tau-1)}{\tau} + \frac{\tau+1}{\tau}\varkappa(\tau-1), \tau \geq 2. \quad (70)$$

If $\varkappa \in \sum_{+\infty}$ has length $\tau < +\infty$, we write $\varkappa := \varkappa_1\varkappa_2 \cdots \varkappa_\tau$ otherwise we write $\varkappa := \varkappa_1\varkappa_2 \cdots$. Now we show that Ψ satisfies Theorem 9 on $(\sum_{+\infty}, \aleph^{d_{\sqsubseteq}}, \times, \zeta)$ with contraction constant $\frac{1}{2}$. From construction, we obtain $l(\Psi(\varkappa)) = l(\varkappa) + 1$ for all $\varkappa, \varsigma \in \sum_{+\infty}$ (in particular, $l(\Psi(\varkappa)) = +\infty$ whenever $l(\varkappa) = +\infty$). Furthermore, it is obvious that $\varkappa \sqsubseteq \varsigma$ if and only if $\Psi(\varkappa) \sqsubseteq \Psi(\varsigma)$ and $\Psi(\varkappa\sqcap\varsigma) \sqsubseteq \Psi(\varkappa)\sqcap\Psi(\varsigma)$. Therefore, $l(\Psi(\varkappa)\sqcap\Psi(\varsigma)) \leq l(\Psi(\varkappa))\sqcap l(\Psi(\varsigma))$. From the following:

$$\aleph^{d_{\sqsubseteq}}\left(\Psi(\varkappa), \Psi(\varsigma), \frac{\varpi}{2}\right) = \aleph^{d_{\sqsubseteq}}(\varkappa,\varsigma,\varpi) = 1, \quad (71)$$

and if $\varkappa$ is not a prefix of ς, then

$$\begin{aligned} \aleph^{d_{\sqsubseteq}}\left(\Psi(\varkappa), \Psi(\varsigma), \frac{\varpi}{2}\right) &= \frac{\frac{\varpi}{2}}{\frac{\varpi}{2} + 2^{-l(\Psi(\varkappa)\sqcap\Psi(\varsigma))}} \geq \frac{\frac{\varpi}{2}}{\frac{\varpi}{2} + 2^{-l(\Psi(\varkappa\sqcap\varsigma))}}, \\ &= \frac{\frac{\varpi}{2}}{\frac{\varpi}{2} + 2^{-(l(\varkappa\sqcap\varsigma)+1)}} = \frac{\varpi}{\varpi + 2^{-l(\varkappa\sqcap\varsigma)}} = \aleph^{d_{\sqsubseteq}}(\varkappa,\varsigma,\varpi), \end{aligned} \quad (72)$$

for all $\varpi > 0$. That is, $\aleph^{d_{\sqsubseteq}}(\Psi(\varkappa), \Psi(\varsigma), \frac{\varpi}{2}) \geq \aleph^{d_{\sqsubseteq}}(\varkappa,\varsigma,\varpi)$. That is, Ψ has a unique fixed point $z = z_1z_2z_3\cdots$, which is a unique solution for the recurrence equation $\mathcal{E}$, i.e., $z_1 = 0$ and

$$z_\tau = \frac{2(\tau - 1)}{\tau} + \frac{\tau + 1}{\tau} z_{\tau-1},\ \tau \geq 2. \tag{73}$$

Remark 6. The above procedure can also be used with product Ct-norm instead of minimum Ct-norm.

5. Conclusion

In this manuscript, we established fuzzy strong controlled metric spaces, fuzzy strong controlled quasi-metric spaces, and non-Archimedean fuzzy strong controlled quasi-metric spaces and generalized the famous Banach contraction principle. In fact, we proved our findings in the broader setting of non-Archimedean fuzzy strong controlled quasi-metric spaces, because in this case, measuring the distance between two words $\varkappa$ and ς automatically shows whether $\varkappa$ is a prefix of ς or not. Finally, we utilized our approaches to show that some recurrence equations related to the complexity analysis of the quicksort algorithms have a solution (and that it is unique). In future, we will work on generalizations of fuzzy metric spaces and fixed point results for new types of contraction mappings.

Data Availability

On request, the data used to support the findings of this study can be obtained from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This article was written equally by all contributors. The final manuscript was read and approved by all of the authors who contributed equally to this work.

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