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Research Article

The Separation Properties of Binary Topological Spaces

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In the present study, we introduce some new separation axioms for binary topological spaces. This new idea gives the notion of generalized binary (T_0, T_1, T_2, T_3, and T_4 spaces) and binary generalized semi (T_0, T_1, T_2, T_3, and T_4 spaces) using generalized binary open sets and binary generalized semi open sets to investigate their properties. We also provide adequate examples to assist and understand abstract concepts. In the similar manner, we begin researching the b-sg-T_0, b-sg-T_1, b-sg-T_2, b-sg-T_3, and b-sg-T_4 spaces in binary topological spaces. The study on the axioms is done over binary-T_0, binary-T_1, binary-T_2, binary-T_3, and binary-T_4 spaces, motivated to do the analysis of the spaces gb(b-gs)-T_0, gb(b-gs)-T_1, gb(b-gs)-T_2, gb(b-gs)-T_3, and gb(b-gs)-T_4 as well.

1. Introduction and Preliminaries

Topology is the most advanced area of pure mathematics which studies mathematical structures. Many scholars have recently analyzed the binary topology that was originally developed by Nithyanantha Jothi and Thangavelu [1]. They also investigated topological structures, displaying their many characteristics in relation to binary topological spaces. In 2011, Nithyanantha Jothi and Thangavelu [1, 2] introduced $\mathscr{BT}$ from X to Y. The authors explored the ideas of binary closed, binary closure, binary interior, binary continuity, base, and subbase of a $\mathscr{BTS}$. In 2012, the authors [3] introduced the concept of binary-T_0, binary-T_1, binary-T_2, binary-T_3, and binary-T_4 spaces. The binary points $(x_1, y_1), (x_2, y_2) \in X \times Y$ are distinct if $x_1 \neq x_2, y_1 \neq y_2$. In a $\mathscr{BTS}(X, Y, \mathscr{M})$, a subset (A, B) is said to be binary semi open [4] if there exists a binary open set (U, V) so that $(U, V) \subseteq (A, B) \subseteq b\text{-cl}(U, V)$, in which $b\text{-cl}(U, V)$ denotes the binary closure of (U, V) in (X, Y). The complement of a binary semi open set is called binary semi closed, and a subset (A, B) of (X, Y) is said to be generalized binary closed [5] if $b\text{-cl}(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open. The complement of generalized binary closed set is called generalized binary open. Izadi et al. and Kosari et al. [6, 7] tried to transform quartic Diophantine equations into cubic elliptic curves in 2021, and Shao et al. [8] introduced some extensions of Fejér-divergences in 2022. Recently, Sathishmohan et al. [9] proposed the idea of b-gs(b-sg)-closed sets in $\mathscr{BTS}$. Consequently, they [10] introduced the concept of bs-T_0, bs-T_1, bs-T_2, bs-T_3, and bs-T_4 spaces. This work introduces and identifies the basic features of the gb(b-gs)-T_0, gb(b-gs)-T_1, gb(b-gs)-T_2, gb(b-gs)-T_3, and gb(b-gs)-T_4 spaces in $\mathscr{BTS}$. The analysis ended up with b-sg-T_0, b-sg-T_1, b-sg-T_2, b-sg-T_3, and b-sg-T_4 spaces in $\mathscr{BTS}$ with various illustrations to demonstrate the behaviour of these new classes of functions.

In the present examination, we use the following symbols: $\mathscr{TS}$, $\mathscr{GSO}$, $\mathscr{GSC}$, $\mathscr{SGC}$, $\mathscr{SGO}$, $\mathscr{BTS}$, $\mathscr{BC}$, $\mathscr{BO}$, $\mathscr{GBO}$, $\mathscr{GBC}$, $\mathscr{BGSC}$, $\mathscr{BGSO}$, $\mathscr{BSGC}$, and $\mathscr{BSGO}$ (topological spaces, binary topological spaces, generalized semi open set, generalized semi closed set semi generalized closed set, semi generalized open set, binary closed set, binary open set, generalized binary open, generalized binary closed, binary generalized semi closed set, binary generalized semi open set, binary semi generalized closed set, and binary semi generalized open set).

2. gb(b-gs)-T_0, gb(b-gs)-T_1, and gb(b-gs)-T_2 Spaces

We define the concept of gb(b-gs)-T_0, gb(b-gs)-T_1, and gb(b-gs)-T_2 spaces and explore some of their characterizations in the study.

Definition 1. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a generalized binary-T_0 (briefly, gb-T_0) if for any two jointly distinct points $(k_1,l_1),(k_2,l_2)\in\mathscr{K}\times\mathscr{L}$, there exists $\mathscr{GBO}(\mathscr{H},\mathscr{J})$ such that exactly one of the following holds:

(i) $(k_1,l_1)\in(\mathscr{H},\mathscr{J}),(k_2,l_2)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J})$

(ii) $(k_1,l_1)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J}),(k_2,l_2)\in(\mathscr{H},\mathscr{J})$

Definition 2. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a generalized binary-T_1 (briefly, gb-T_1) if for every two jointly distinct points $(k_1,l_1),(k_2,l_2)\in\mathscr{K}\times\mathscr{L}$ with $k_1\neq x_2,y_1\neq y_2$, there exists $\mathscr{GBO}(\mathscr{H},\mathscr{J})$ and $(\mathscr{Q},\mathscr{W})$ with $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$ such that $(k_2,l_2)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J})$, $(k_1,l_1)\in(\mathscr{K}-\mathscr{Q},\mathscr{L}-\mathscr{W})$.

Definition 3. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a generalized binary-T_2 (briefly, gb-T_2) if for any two jointly distinct points $(k_1,l_1),(k_2,l_2)\in\mathscr{K}\times\mathscr{L}$, with $x_1\neq x_2,y_1\neq y_2$, there exists disjoint $\mathscr{GBO}(\mathscr{H},\mathscr{J})$ and $(\mathscr{Q},\mathscr{W})$ such that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$.

Definition 4. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a binary generalized semi-T_0 (briefly, b-gs-T_0) if for any two jointly distinct points $(k_1,l_1),(k_2,l_2)\in\mathscr{K}\times\mathscr{L}$, there exists $\mathscr{BGSO}(\mathscr{H},\mathscr{J})$ such that exactly one of the following holds:

(i) $(k_1,l_1)\in(\mathscr{H},\mathscr{J}),(k_2,l_2)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J})$

(ii) $(k_1,l_1)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J}),(k_2,l_2)\in(\mathscr{H},\mathscr{J})$

Definition 5. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a binary generalized semi-T_1 (briefly, b-gs-T_1) if for every two jointly distinct points $(k_1,l_1),(k_2,l_2)\in\mathscr{K}\times\mathscr{L}$ with $k_1\neq k_2,l_1\neq l_2$, there exists $\mathscr{BGSO}(\mathscr{H},\mathscr{J})$ and $(\mathscr{Q},\mathscr{W})$ with $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$ such that $(k_2,l_2)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J}),(k_1,l_1)\in(\mathscr{K}-\mathscr{Q},\mathscr{L}-\mathscr{W})$.

Definition 6. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a binary generalized semi-T_2 (briefly, b-gs-T_2) if for any two jointly distinct points $(k_1,l_1),(k_2,l_2)\in\mathscr{K}\times\mathscr{L}$, with $k_1\neq k_2,l_1\neq l_2$, there exists disjoint $\mathscr{BGSO}(\mathscr{H},\mathscr{J})$ and $(\mathscr{Q},\mathscr{W})$ such that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$.

Theorem 7. *Let $(\mathscr{K},\mathscr{L},\mathscr{M})$ be a $\mathscr{BTS}$; then, for*

(1) every b-T_0 space is gb(b-gs)-T_0 space

(2) every b-T_1 space is gb(b-gs)-T_1 space

(3) every b-T_2 space is gb(b-gs)-T_2 space.

(4) every gb(b-gs)-T_1 space is gb(b-gs)-T_0 space

(5) every gb(b-gs)-T_2 space is gb(b-gs)-T_0 space

(6) every gb(b-gs)-T_2 space is gb(b-gs)-T_1 space

Proof.

(1) Let $(\mathscr{K},\mathscr{L})$ be a b-T_0 space and (k_1,l_1) and (k_2,l_2) be a two distinct points of $(\mathscr{K},\mathscr{L})$; as $(\mathscr{K},\mathscr{L})$ is b-T_0 space, there exists $\mathscr{BO}(\mathscr{H},\mathscr{J})$ such that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J})$. Even before every $\mathscr{BO}$ is gb(b-gs)-open and ergo $(\mathscr{H},\mathscr{J})$ is gb(b-gs)-open set such that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J})$, change has occurred $(\mathscr{K},\mathscr{L})$ is gb(b-gs)-T_0 space

(2) Proof of (2) to (6) is obvious

□

Example 1. Let $\mathscr{K}=\{\xi,\varpi\}$, $\mathscr{L}=\{\xi,\varpi,\varrho\}$. Clearly, $\mathscr{M}=\{(\phi,\phi),(\phi,\{\xi\}),(\{\xi\},\{\xi\}),(\{\varpi\},\{\xi\}),(\mathscr{K},\{\xi\}),(\mathscr{K},\mathscr{L})\}$ is a $\mathscr{BT}$ from $\mathscr{K}$ to $\mathscr{L}$.

(1) Let $(k_1,l_1)=(\{\varpi\},\{\xi\})$ and $(k_2,l_2)=(\{\xi\},\{\varrho\})$, $(k_1,l_1),(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$; there exists gb(b-gs)-open set $(\mathscr{H},\mathscr{J})=(\{\varpi\},\{\xi,\varpi\})$; then, it is gb(b-gs)-$T_0$ space but not b-T_0 space

(2) Let $(\mathscr{H},\mathscr{J})=(\{\xi\},\{\xi,\varpi\})$ and $(\mathscr{Q},\mathscr{W})=(\{\varpi\},\{\xi,\varrho\})$. Assume $(k_1,l_1)=(\{\xi\},\{\varpi\})$ and $(k_2,l_2)=(\{\varpi\},\{\varrho\})$, $(k_1,l_1),(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$; then, it is clear that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$, $(k_2,l_2)\notin(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$, $(k_1,l_1)\notin(\mathscr{Q},\mathscr{W})$. After that, we can declare it to be gb(b-gs)-T_1 space but not b-T_1 space

(3) Let $(\mathscr{H},\mathscr{J})=(\{\xi\},\{\varpi,\varrho\})$ and $(\mathscr{Q},\mathscr{W})=(\{\varpi\},\{\xi\})$. Suppose $(k_1,l_1)=(\{\xi\},\{\varrho\})$ and $(k_2,l_2)=(\{\varpi\},\{\xi\})$, $(k_1,l_1),(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$; then, it is clear that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$, $(k_2,l_2)\notin(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$, $(k_1,l_1)\notin(\mathscr{Q},\mathscr{W})$. Thereafter, we can formally declare it to be gb(b-gs)-T_2 space but not b-T_2 space

(4) Let $(\mathscr{H},\mathscr{J})=(\{\varpi\},\{\varpi,\varrho\})$ and $(\mathscr{Q},\mathscr{W})=(\{\xi\},\{\xi,\varpi\})$. Assume $(k_1,l_1)=(\{\varpi\},\{\varpi\})$ and $(k_2,l_2)=(\{\xi\},\{\xi\})$, $(k_1,l_1),(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$; then, it is clear that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$, $(k_2,l_2)\notin(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$, $(k_1,l_1)\notin(\mathscr{Q},\mathscr{W})$. Once that is done, we can proclaim it to be gb(b-gs)-T_0 space but not gb(b-gs)-T_1 space

(5) Let $(k_1,l_1)=(\{\xi\},\{\varpi\})$ and $(k_2,l_2)=(\{\varpi\},\{\xi\})$. Suppose $(\mathscr{H},\mathscr{J})=(\{\xi\},\{\varpi\})$ and $(\mathscr{Q},\mathscr{W})=(\{\varpi\},\{\varrho\})$, $(k_1,l_1),(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$; then, it is clear that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and (k_2,l_2)

$\in (\mathcal{Q}, \mathcal{W})$. When that is finished, we can declare it to be gb(b-gs)-T_0 space but not gb(b-gs)-T_2 space

(6) Let $(\mathcal{H}, \mathcal{J}) = (\{\xi\}, \{\xi, \varrho\})$ and $(\mathcal{Q}, \mathcal{W}) = (\{\varpi\}, \{\xi, \varpi\})$. Consider $(k_1, l_1) = (\{\xi\}, \{\varrho\})$ and $(k_2, l_2) = (\{\varpi\}, \{\varpi\})$, $(k_1, l_1), (k_2, l_2) \in (\mathcal{K}, \mathcal{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; then, it is clear that $(k_1, l_1) \in (\mathcal{H}, \mathcal{J})$, $(k_2, l_2) \notin (\mathcal{H}, \mathcal{J})$ and $(k_2, l_2) \in (\mathcal{Q}, \mathcal{W})$, $(k_1, l_1) \notin (\mathcal{Q}, \mathcal{W})$. We may formally proclaim it after it is done to be gb(b-gs)-T_1 space but not gb(b-gs)-T_2 space

Theorem 8. *A $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is a b-gs-T_1 space if and only if every binary point is $\mathcal{BGSC}$.*

Proof. Consider that $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is a b-gs-T_1. Let $(k, l) \in \mathcal{K} \times \mathcal{L}$. Let $(\{k\}, \{l\}) \in \mathcal{P}(\mathcal{K}) \times \mathcal{P}(\mathcal{L})$. By demonstrating it, $(\{k\}, \{l\})$ is $\mathcal{BGSC}$. It appears likely to depict this. $(\mathcal{K} - \{k\}, \mathcal{L} - \{l\})$ is $\mathcal{BGSO}$. Let $(\mathcal{H}, \mathcal{J}) \in (\mathcal{K} - \{k\}, \mathcal{L} - \{l\})$. This indicates that $h \in \mathcal{K} - \{k\}$ and $j \in \mathcal{L} - \{l\}$. Ergo $h \neq k$ and $j \neq l$. That is, $(\mathcal{H}, \mathcal{J})$ and $(\mathcal{K}, \mathcal{L})$ are jointly distinct binary points of $\mathcal{K} \times \mathcal{L}$. Even before $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is b-gs-T_1, there exists $\mathcal{BGSO}(\mathcal{H}, \mathcal{J})$ and $(\mathcal{Q}, \mathcal{W})$, $(\mathcal{H}, \mathcal{J}) \in (\mathcal{H}, \mathcal{J})$ and $(\mathcal{K}, \mathcal{L}) \in (\mathcal{Q}, \mathcal{W})$, such that $(\mathcal{H}, \mathcal{J}) \in (\mathcal{K} - \mathcal{Q}, \mathcal{L} - \mathcal{W})$ and $(\mathcal{K}, \mathcal{L}) \in (\mathcal{K} - \mathcal{H}, \mathcal{L} - \mathcal{J})$, although $(\mathcal{H}, \mathcal{J}) \subseteq (\mathcal{K} - \{k\}, \mathcal{L} - \{l\})$. Hence, $(\mathcal{K} - \{k\}, \mathcal{L} - \{l\})$ is a binary neighbourhood of $(\mathcal{H}, \mathcal{J})$. This implies that $(\{k\}, \{l\})$ is $\mathcal{BGSC}$.

Conversely, assume that $(\{k\}, \{l\})$ is $\mathcal{BGSC}$, for every $(\mathcal{K}, \mathcal{L}) \in \mathcal{K} \times \mathcal{L}$. Let (k_1, l_1) and $(k_2, l_2) \in \mathcal{K} \times \mathcal{L}$ with $k_1 \neq k_2$, $l_1 \neq l_2$. Therefore, $(k_2, l_2) \in (\mathcal{K} - \{k_1\}, \mathcal{L} - \{l_1\})$ and $(\mathcal{K} - \{k_1\}, \mathcal{L} - \{l_1\})$ is $\mathcal{BGSO}$. Also, $(k_1, l_1) \in (\mathcal{K} - \{k_2\}, \mathcal{L} - \{l_2\})$ and $(\mathcal{K} - \{k_2\}, \mathcal{L} - \{l_2\})$ is $\mathcal{BGSO}$. This shows that $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is a b-gs-T_1. □

Theorem 9. *If a $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is a b-gs-T_0, then $(\mathcal{K}, \mathcal{M}_{\mathcal{K}})$ is gs-T_0 and $(\mathcal{L}, \mathcal{M}_{\mathcal{L}})$ is gs-T_0.*

Proof. Even before $(\mathcal{M})$ is a $\mathcal{BT}$ from $\mathcal{K}$ to $\mathcal{L}$, we have $(\mathcal{M}_{\mathcal{K}}) = \{\mathcal{H} \subseteq \mathcal{K} : (\mathcal{H}, \mathcal{J}) \in (\mathcal{M}) \text{for some } \mathcal{J} \subseteq \mathcal{L}\}$ as a topology on $\mathcal{K}$ and $(\mathcal{M}_{\mathcal{L}}) = \{\mathcal{J} \subseteq \mathcal{L} : (\mathcal{H}, \mathcal{J}) \in (\mathcal{M}) \text{for some } \mathcal{H} \subseteq \mathcal{K}\}$ as a topology on $\mathcal{L}$. Let $(k_1, k_2) \in \mathcal{K}$ and $(l_1, l_2) \in \mathcal{L}$ with $k_1 \neq k_2$, $l_1 \neq l_2$. Even before $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is b-gs-T_0, there exists $\mathcal{GSO}(\mathcal{H}, \mathcal{J})$ such that either $(k_1, l_1) \in (\mathcal{H}, \mathcal{J})$, $(k_2, l_2) \in (\mathcal{K} - \mathcal{H}, \mathcal{L} - \mathcal{J})$ or $(k_1, l_1) \in (\mathcal{K} - \mathcal{H}, \mathcal{L} - \mathcal{J})$, $(k_2, l_2) \in (\mathcal{H}, \mathcal{J})$. This implies that either $k_1 \in \mathcal{H}$, $k_2 \in \mathcal{K} - \mathcal{H}$, $l_1 \in \mathcal{J}$, $l_2 \in \mathcal{L} - \mathcal{J}$ or $k_1 \in \mathcal{K} - \mathcal{H}$, $k_2 \in \mathcal{H}$, $l_1 \in \mathcal{L} - \mathcal{J}$, $l_2 \in \mathcal{J}$. This implies that $(\mathcal{K}, \mathcal{M}_{\mathcal{K}})$ is gs-T_0 and $(\mathcal{L}, \mathcal{M}_{\mathcal{L}})$ is gs-T_0. □

Theorem 10. *If a $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \tau_{\mathcal{M}(\mathcal{K})} \times \sigma_{\mathcal{M}(\mathcal{L})})$ is a b-gs-T_0, then the $\mathcal{TS}$ $(\mathcal{K}, \tau)$ and $(\mathcal{L}, \sigma)$ are gs-T_0.*

Proof. Suppose that $(\mathcal{K}, \mathcal{L}, \tau_{\mathcal{M}(\mathcal{K})} \times \sigma_{\mathcal{M}(\mathcal{L})})$ is b-gs-T_0. Let $(k_1, k_2) \in \mathcal{K}$ and $(l_1, l_2) \in \mathcal{L}$ with $k_1 \neq k_2$, $l_1 \neq l_2$. Even before $(\mathcal{K}, \mathcal{L}, \tau_{\mathcal{M}(\mathcal{K})} \times \sigma_{\mathcal{M}(\mathcal{L})})$ is b-gs-T_0, there exists $(\mathcal{H}, \mathcal{J}) \in \tau_{\mathcal{M}(\mathcal{K})} \times \sigma_{\mathcal{M}(\mathcal{L})}$ such that either $(k_1, l_1) \in (\mathcal{H}, \mathcal{J})$, $(k_2, l_2) \in (\mathcal{K} - \mathcal{H}, \mathcal{L} - \mathcal{J})$ or $(k_1, l_1) \in (\mathcal{K} - \mathcal{H}, \mathcal{L} - \mathcal{J})$, $(k_2, l_2) \in (\mathcal{H}, \mathcal{J})$. This implies that either $k_1 \in \mathcal{H}$, $k_2 \in \mathcal{K} - \mathcal{H}$, $l_1 \in \mathcal{J}$, $l_2 \in \mathcal{L} - \mathcal{J}$ or $k_1 \in \mathcal{K} - \mathcal{H}$, $k_2 \in \mathcal{H}$, $l_1 \in \mathcal{L} - \mathcal{J}$, $l_2 \in \mathcal{H}$. This implies that either $k_1 \in \mathcal{H}$, $k_2 \in \mathcal{K} - \mathcal{H}$ or $k_1 \in \mathcal{K} - \mathcal{H}$, $k_2 \in \mathcal{H}$ and $l_1 \in \mathcal{J}$, $l_2 \in \mathcal{L} - \mathcal{J}$ or $l_1 \in \mathcal{L} - \mathcal{J}$, $l_2 \in \mathcal{J}$. Even before $(\mathcal{H}, \mathcal{J}) \in \tau_{\mathcal{M}(\mathcal{K})} \times \sigma_{\mathcal{M}(\mathcal{L})}$, we have $\mathcal{H} \in \tau$ and $\mathcal{J} \in \sigma$. Change has occurred; $(\mathcal{K}, \tau)$ and $(\mathcal{L}, \sigma)$ are gs-T_0. □

Theorem 11. *If a $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is a b-gs-T_1, then $(\mathcal{K}, \mathcal{M}_{\mathcal{K}})$ is gs-T_1 and $(\mathcal{L}, \mathcal{M}_{\mathcal{L}})$ is gs-T_1.*

Proof. Even before $(\mathcal{M})$ is a $\mathcal{BT}$ from $\mathcal{K}$ to $\mathcal{L}$, we have $(\mathcal{M}_{\mathcal{K}}) = \{\mathcal{H} \subseteq \mathcal{K} : (\mathcal{H}, \mathcal{J}) \in (\mathcal{M}) \text{for some } \mathcal{J} \subseteq \mathcal{L}\}$ as a topology on $\mathcal{K}$ and $(\mathcal{M}_{\mathcal{L}}) = \{\mathcal{J} \subseteq \mathcal{L} : (\mathcal{H}, \mathcal{J}) \in (\mathcal{M}) \text{for some } \mathcal{H} \subseteq \mathcal{K}\}$ as a topology on $\mathcal{L}$. Let $(k_1, k_2) \in \mathcal{K}$ and $(l_1, l_2) \in \mathcal{L}$ with $k_1 \neq k_2$, $l_1 \neq l_2$. Even before $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is b-gs-T_1, there exists $\mathcal{BGSO}(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}_2, \mathcal{G}_2)$ with $(k_1, l_1) \in (\mathcal{F}_1, \mathcal{G}_1)$, $(k_2, l_2) \in (\mathcal{F}_2, \mathcal{G}_2)$, such that $(k_1, l_1) \in (\mathcal{K} - \mathcal{F}_2, \mathcal{L} - \mathcal{G}_2)$, $(k_2, l_2) \in (\mathcal{K} - \mathcal{F}_1, \mathcal{L} - \mathcal{G}_1)$. This implies that $k_1 \in \mathcal{F}_1$, $l_2 \in \mathcal{F}_2$ and $l_1 \in \mathcal{G}_1$, $l_2 \in \mathcal{G}_2$ such that $k_1 \in \mathcal{K} - \mathcal{F}_2$, $k_2 \in \mathcal{K} - \mathcal{F}_1$ and $l_1 \in \mathcal{L} - \mathcal{G}_2$, $l_2 \in \mathcal{L} - \mathcal{G}_1$. Hence, $(\mathcal{K}, \mathcal{M}_{\mathcal{K}})$ is gs-T_1 and $(\mathcal{L}, \mathcal{M}_{\mathcal{L}})$ is gs-T_1. □

Theorem 12. *A $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is a*

(1) gb-T_1 space if and only if every binary point is $\mathcal{GBC}$

(2) gb-T_0, then $(\mathcal{K}, \mathcal{M}_{\mathcal{K}})$ is g-T_0 and $(\mathcal{L}, \mathcal{M}_{\mathcal{L}})$ is g-T_0

(3) If a $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \tau_{\mathcal{M}(\mathcal{K})} \times \sigma_{\mathcal{M}(\mathcal{L})})$ is a gb-T_0, then the $\mathcal{TS}$ $(\mathcal{K}, \tau)$ and $(\mathcal{L}, \sigma)$ are gb-T_0

(4) gb-T_1, then $(\mathcal{K}, \mathcal{M}_{\mathcal{K}})$ is g-T_1 and $(\mathcal{L}, \mathcal{M}_{\mathcal{L}})$ is g-T_1

Proof. Proof of (1) to (4) follows from Definitions 1, 2, and 3 and Theorems 8, 9, 10, and 11. □

3. gb(b-gs)-T_3 and gb(b-gs)-T_4 Spaces

We use gb(b-gs)-open sets to create gb(b-gs)-T_3 and gb(b-gs)-T_4 spaces and examine some of their characteristics in this section.

Definition 13. A $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is called a generalized binary-T_3 (briefly, gb-T_3) or generalized binary regular if $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is gb-T_1 and for every $(\mathcal{K}, \mathcal{L}) \in \mathcal{K} \times \mathcal{L}$ and every $\mathcal{GBC}(\mathcal{H}, \mathcal{J}) \subseteq \mathcal{K} \times \mathcal{L}$ such that $(\mathcal{K}, \mathcal{L}) \in (\mathcal{K} - \mathcal{H}, \mathcal{L} - \mathcal{J})$, there exists jointly disjoint $\mathcal{GBOS}$ $(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}_2, \mathcal{G}_2)$ such that $(\mathcal{K}, \mathcal{L}) \in (\mathcal{F}_1, \mathcal{G}_1)$, $(\mathcal{H}, \mathcal{J}) \subseteq (\mathcal{F}_2, \mathcal{G}_2)$.

Definition 14. A $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is called a generalized binary-T_4 (briefly, gb-T_4) or generalized binary normal if $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is gb-T_1 and for every pair of jointly disjoint $\mathcal{GBC}(\mathcal{H}_1, \mathcal{J}_1), (\mathcal{H}_2, \mathcal{J}_2)$, there exists jointly disjoint $\mathcal{GBOS}(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}_2, \mathcal{G}_2)$ such that $(\mathcal{H}_1, \mathcal{J}_1) \subseteq (\mathcal{F}_1, \mathcal{G}_1)$ and $(\mathcal{H}_2, \mathcal{J}_2) \subseteq (\mathcal{F}_2, \mathcal{G}_2)$.

Definition 15. A $\mathcal{BTS}(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is called a b-gs-T_3 or b-gs regular if $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is b-gs-T_1 and for every $(\mathcal{K}, \mathcal{L}) \in \mathcal{K} \times \mathcal{L}$ and every $\mathcal{BGSC}$ set $(\mathcal{H}, \mathcal{J}) \subseteq \mathcal{K} \times \mathcal{L}$ such that

$(\mathscr{K},\mathscr{L}) \in(\mathscr{K}-\mathscr{H},\mathscr{L}-\mathscr{J})$, there exists jointly disjoint $\mathscr{BGSO}(\mathscr{F}_1,\mathscr{G}_1),(\mathscr{F}_2,\mathscr{G}_2)$ such that $(\mathscr{K},\mathscr{L}) \in(\mathscr{F}_1,\mathscr{G}_1)$, $(\mathscr{H},\mathscr{J})\subseteq(\mathscr{F}_2,\mathscr{G}_2)$.

Definition 16. A $\mathscr{BTS}(\mathscr{K},\mathscr{L},\mathscr{M})$ is called a b-gs-T_4 *or* b-gs normal if $(\mathscr{K},\mathscr{L},\mathscr{M})$ is b-gs-T_1 and for every pair of jointly disjoint $\mathscr{BGSC}(\mathscr{H}_1,\mathscr{J}_1),(\mathscr{H}_2,\mathscr{J}_2)$, there exists jointly disjoint $\mathscr{BGSO}(\mathscr{F}_1,\mathscr{G}_1),(\mathscr{F}_2,\mathscr{G}_2)$ such that $(\mathscr{H}_1,\mathscr{J}_1)\subseteq(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{H}_2,\mathscr{J}_2)\subseteq(\mathscr{F}_2,\mathscr{G}_2)$.

Theorem 17.

(1) *Every binary-T_3 is gb(b-gs)-regular space*

(2) *Every gb(b-gs)-T_3 is gb(b-gs)-T_0 space*

(3) *Every gb(b-gs)-T_3 is b-gs-T_2 space*

(4) *Every binary-T_4 is gb(b-gs)-T_4*

(5) *Every gb(b-gs)-T_4 is gb(b-gs)-T_3*

Proof.

(1) Let $(\mathscr{K},\mathscr{L})$ be a binary regular and $(\mathscr{H},\mathscr{J})$ be a $\mathscr{BC}$ not containing $(\mathscr{K},\mathscr{L})$ which implies $(\mathscr{H},\mathscr{J})$ to be a $\mathscr{BGSC}$ set not containing $(\mathscr{K},\mathscr{L})$. As $(\mathscr{K},\mathscr{L})$ is b-gs regular, there exists jointly disjoint $\mathscr{BGSO}$ $(\mathscr{F}_1,\mathscr{G}_1),(\mathscr{F}_2,\mathscr{G}_2)$ such that $(\mathscr{K},\mathscr{L}) \in(\mathscr{F}_1,\mathscr{G}_1)$, $(\mathscr{H},\mathscr{J})\subseteq(\mathscr{F}_2,\mathscr{G}_2)$. Hence, $(\mathscr{K},\mathscr{L})$ is b-gs regular

(2) Proof of (2) to (5) is obvious

□

Example 2. From Example 1,

(1) let $(\mathscr{H},\mathscr{J})=(\{\varpi\},\{\varpi\})$, $(k,l)=(\phi,\{\varrho\})$, $(\mathscr{F}_1,\mathscr{G}_1)=(\{\xi\},\{\varrho\})$, and $(\mathscr{F}_2,\mathscr{G}_2)=(\{\varpi\},\{\xi,\varpi\})$; that is why it is gb(b-gs)-T_3 space but not b-T_3 space

(2) let $(k_1,l_1)=(\phi,\{\varpi\})$ and $(k_2,l_2)=(\{\xi\},\{\varrho\})$, $(k_1,l_1),(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$; there exists $\mathscr{BGSO}(\mathscr{H},\mathscr{J})=(\{\varpi\},\{\xi,\varpi\})$ and $(\mathscr{F}_1,\mathscr{G}_1)=(\{\varpi\},\{\varpi\})$, $(\mathscr{F}_2,\mathscr{G}_2)=(\phi,\{\xi,\varrho\})$ because of this gb(b-gs)-T_0 space but not gb(b-gs)-T_3

(3) let $(k_1,l_1)=(\{\xi\},\{\varrho\})$ and $(k_2,l_2)=(\phi,\{\xi\})$. Let $(\mathscr{F}_1,\mathscr{G}_1)=(\{\xi\},\{\varpi\})$ and $(\mathscr{F}_2,\mathscr{G}_2)=(\{\varpi\},\{\varrho\})$, (k_1,l_1),$(k_2,l_2)\in(\mathscr{K},\mathscr{L})$ and $(k_1,l_1)\neq(k_2,l_2)$, and $(\mathscr{H},\mathscr{J})=(\mathscr{K},\{\varrho\})$ and $(\mathscr{Q},\mathscr{W})=(\phi,\{\xi,\varpi\})$; then, it is clear that $(k_1,l_1)\in(\mathscr{H},\mathscr{J})$ and $(k_2,l_2)\in(\mathscr{Q},\mathscr{W})$. After that, we can declare it to be gb(b-gs)-T_2 space but not gb(b-gs)-T_3 space

(4) let $(\mathscr{H}_1,\mathscr{J}_1)=(\phi,\{\varrho\})$, $(\mathscr{H}_2,\mathscr{J}_2)=(\{\xi\},\{\varpi\})$, $(\mathscr{F}_1,\mathscr{G}_1)=(\{\varpi\},\{\varrho\})$, and $(\mathscr{F}_2,\mathscr{G}_2)=(\{\xi\},\{\xi,\varpi\})$. Otherwise, we might officially declare it to be gb(b-gs)-T_4 space but not binary-T_4 space

(5) let $(k,l)=(\{\xi\},\phi)$ and $(\mathscr{H}_1,\mathscr{J}_1)=(\{\varpi\},\{\varpi\})$, $(\mathscr{H}_2,\mathscr{J}_2)=(\{\xi\},\{\xi,\varpi\})$, $(\mathscr{F}_1,\mathscr{G}_1)=(\{\xi\},\{\varrho\})$, and $(\mathscr{F}_2,\mathscr{G}_2)=(\{\varpi\},\{\xi,\varpi\})$. Thereafter, we can formally declare it to be gb(b-gs)-T_3 space but not gb(b-gs)-T_4 space

Theorem 18. *Let $\mathscr{TS}$ $(\mathscr{K},\tau)$ and $(\mathscr{L},\sigma)$ to be gs-T_3 spaces if and only if $\mathscr{BTS}(\mathscr{K},\mathscr{L},\tau_{\mathscr{M}(\mathscr{K})}\times\sigma_{\mathscr{M}(\mathscr{L})})$ is called a b-gs-T_3.*

Proof. Suppose $(\mathscr{K},\tau)$ and $(\mathscr{L},\sigma)$ are gs-T_3 spaces. Let $(\mathscr{K},\mathscr{L}) \in\mathscr{K}\times\mathscr{L}$ and $(\mathscr{H},\mathscr{J})\subseteq\mathscr{K}\times\mathscr{L}$ be a $\mathscr{BGSC}$ $(\mathscr{K},\mathscr{L})\in(\mathscr{K}-\mathscr{H}\times\mathscr{L}-\mathscr{J})$. Therefore, $k\in\mathscr{K}$, $l\in\mathscr{L}$ and $\mathscr{H}\subseteq\mathscr{K}$, $\mathscr{J}\subseteq\mathscr{L}$. Even before $(\mathscr{K},\tau)$ is gs-T_3, there exists disjoint $\mathscr{F}_1,\mathscr{F}_2\in\tau$, k$\in\mathscr{F}_1$, and $\mathscr{H}\subseteq\mathscr{F}_2$. Also, even before $(\mathscr{L},\sigma)$ is gs-T_3, there exists disjoint $\mathscr{G}_1,\mathscr{G}_2\in\sigma$, l$\in\mathscr{G}_1$, and $\mathscr{J}\subseteq\mathscr{G}_2$. This implies that $(\mathscr{K},\mathscr{L})\in(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{H},\mathscr{J})\in(\mathscr{F}_2,\mathscr{G}_2)$. Even before $\mathscr{F}_1$ and $\mathscr{F}_2$ are disjoint $\mathscr{GSO}$, we have $\mathscr{F}_1\cap\mathscr{F}_2=\phi$. Also even before $\mathscr{G}_1$ and $\mathscr{G}_2$ are disjoint $\mathscr{GSO}$, we have $\mathscr{G}_1\cap\mathscr{G}_2=\phi$. Thus, $(\mathscr{F}_1\cap\mathscr{F}_2,\mathscr{G}_1\cap\mathscr{G}_2)=(\phi,\phi)$. Ergo $(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{F}_2,\mathscr{G}_2)$ are disjoint $\mathscr{BGSO}$. This implies that $(\mathscr{K},\mathscr{L},\tau_{\mathscr{M}(\mathscr{K})}\times\sigma_{\mathscr{M}(\mathscr{L})})$ is b-gs-T_3.

Conversely, assume that $(\mathscr{K},\mathscr{L},\tau_{\mathscr{M}(\mathscr{K})}\times\sigma_{\mathscr{M}(\mathscr{L})})$ is b-gs-T_3. Let $k\in\mathscr{K}$ and $\mathscr{H}$ be a $\mathscr{GSC}$ subset of $(\mathscr{K},\tau)$. Let $l\in\mathscr{L}$ and $\mathscr{J}$ be a subset of $(\mathscr{L},\sigma)$. Therefore, $(\mathscr{K},\mathscr{L})\in\mathscr{K}\times\mathscr{L}$ and $(\mathscr{H},\mathscr{J})$ is $\mathscr{BGSC}$ in $(\mathscr{K},\mathscr{L},\tau_{\mathscr{M}(\mathscr{K})}\times\sigma_{\mathscr{M}(\mathscr{L})})$. Even before $(\mathscr{K},\mathscr{L},\tau_{\mathscr{M}(\mathscr{K})}\times\sigma_{\mathscr{M}(\mathscr{L})})$ is b-gs-T_3, there exists disjoint $\mathscr{GSO}(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{F}_2,\mathscr{G}_2)$ such that $(\mathscr{K},\mathscr{L})\in(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{H},\mathscr{J})\subseteq(\mathscr{F}_2,\mathscr{G}_2)$. Ergo $k\in\mathscr{F}_1$ and $\mathscr{H}\subseteq\mathscr{F}_2$, $l\in\mathscr{G}_1$ and $\mathscr{J}\subseteq\mathscr{G}_2$. This consistently shows that $(\mathscr{K},\tau)$ and $(\mathscr{L},\sigma)$ are gs-T_3 spaces. □

Theorem 19. *A $\mathscr{BGSC}$ subspace of a b-gs normal space is b-gs normal.*

Proof. Let $(\mathscr{D},\mathscr{E})$ be a $\mathscr{BGSC}$ subspace of a b-gs normal space. Let $(\mathscr{H}_1,\mathscr{J}_1)$ and $(\mathscr{H}_2,\mathscr{J}_2)$ be disjoint $\mathscr{BGSC}$ subset of $(\mathscr{D},\mathscr{E})$. Even before $(\mathscr{D},\mathscr{E})$ is $\mathscr{BGSC}$ in $(\mathscr{K},\mathscr{L})$, $(\mathscr{H}_1,\mathscr{J}_1)$ and $(\mathscr{H}_2,\mathscr{J}_2)$ are $\mathscr{BGSC}$ in $(\mathscr{K},\mathscr{L})$. Even before $(\mathscr{K},\mathscr{L})$ is b-gs normal, there exists disjoint $\mathscr{BGSO}$ $(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{F}_2,\mathscr{G}_2)$ in $(\mathscr{K},\mathscr{L})$, such that $(\mathscr{H}_1,\mathscr{J}_1)\subseteq(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{H}_2,\mathscr{J}_2)\subseteq(\mathscr{F}_2,\mathscr{G}_2)$. Even before $(\mathscr{D},\mathscr{E})$ contains both $(\mathscr{H}_1,\mathscr{J}_1)$ and $(\mathscr{H}_2,\mathscr{J}_2)$, we have $(\mathscr{H}_1,\mathscr{J}_1)\subseteq(\mathscr{D},\mathscr{E})\cap(\mathscr{F}_1,\mathscr{G}_1)$, $(\mathscr{H}_2,\mathscr{J}_2)\subseteq(\mathscr{D},\mathscr{E})\cap(\mathscr{F}_2,\mathscr{G}_2)$, and $((\mathscr{D},\mathscr{E})\cap(\mathscr{F}_1,\mathscr{G}_1))\cap(\mathscr{D},\mathscr{E})\cap(\mathscr{F}_2,\mathscr{G}_2)=(\phi,\phi)$. Even before $(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{F}_2,\mathscr{G}_2)$ are $\mathscr{BGSO}$ in $(\mathscr{K},\mathscr{L})$, $(\mathscr{D},\mathscr{E})\cap(\mathscr{F}_1,\mathscr{G}_1)$ and $(\mathscr{D},\mathscr{E})\cap(\mathscr{F}_2,\mathscr{G}_2)$ are $\mathscr{BGSO}$ in $(\mathscr{D},\mathscr{E})$. Thus in the subspace $(\mathscr{D},\mathscr{E})$, we have disjoint $\mathscr{BGSO}((\mathscr{D},\mathscr{E})\cap(\mathscr{F}_1,\mathscr{G}_1))$ containing $(\mathscr{H}_1,\mathscr{J}_1)$ and $((\mathscr{D},\mathscr{E})\cap(\mathscr{F}_2,\mathscr{G}_2))$ containing $(\mathscr{H}_2,\mathscr{J}_2)$. Ergo the subspace $(\mathscr{D},\mathscr{E})$ is b-gs normal. □

Theorem 20. *Let $\mathscr{TS}$ $(\mathscr{K},\tau)$ and $(\mathscr{L},\sigma)$ be gs-T_4 spaces iff the $\mathscr{BTS}(\mathscr{K},\mathscr{L},\tau_{\mathscr{M}(\mathscr{K})}\times\sigma_{\mathscr{M}(\mathscr{L})})$ is called a b-gs-T_4.*

Proof. Suppose $(\mathscr{K},\tau)$ and $(\mathscr{L},\sigma)$ are gs-T_4 spaces. $(\mathscr{H}_1,\mathscr{J}_1)$ and $(\mathscr{H}_2,\mathscr{J}_2)$ are disjoint pair of $\mathscr{BGSC}$ in $(\mathscr{K},\mathscr{L},\mathscr{M})$. Then, $\mathscr{H}_1,\mathscr{H}_2$ are disjoint $\mathscr{GSC}$ in $(\mathscr{K},\tau)$ and $\mathscr{J}_1,\mathscr{J}_2$ are disjoint $\mathscr{GSC}$ in $(\mathscr{L},\sigma)$. Even before $(\mathscr{K},\tau)$ is gs-T_4, there exists disjoint $\mathscr{GSO}$ in $\mathscr{F}_1,\mathscr{F}_2\in\tau$, $\mathscr{H}_1\subseteq\mathscr{F}_1$, and

$\mathscr{H}_2 \subseteq \mathscr{F}_2$. Also, even before $(\mathscr{L}, \sigma)$ is gs-T_4, there exists disjoint $\mathscr{G}_1, \mathscr{G}_2 \in \sigma$, $\mathscr{J}_1 \subseteq \mathscr{G}_1$, and $\mathscr{J}_2 \subseteq \mathscr{G}_2$. This implies that $(\mathscr{H}_1, \mathscr{J}_1) \subseteq (\mathscr{F}_1, \mathscr{G}_1)$ and $(\mathscr{H}_2, \mathscr{J}_2) \subseteq (\mathscr{F}_2, \mathscr{G}_2)$. Even before $\mathscr{F}_1$ and $\mathscr{F}_2$ are disjoint $\mathscr{GSO}$, we have $\mathscr{F}_1 \cap \mathscr{F}_2 = \phi$. Also even before $\mathscr{G}_1$ and $\mathscr{G}_2$ are disjoint $\mathscr{GSO}$, we have $\mathscr{G}_1 \cap \mathscr{G}_2 = \phi$. Thus, $(\mathscr{F}_1 \cap \mathscr{F}_2, \mathscr{G}_1 \cap \mathscr{G}_2) = (\phi, \phi)$. Hence, $(\mathscr{F}_1, \mathscr{G}_1)$ and $(\mathscr{F}_2, \mathscr{G}_2)$ are disjoint $\mathscr{BGSO}$. This implies that $(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is a b-gs-T_4.

Conversely, assume that $(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is b-gs-T_4. Let $\mathscr{H}_1, \mathscr{H}_2$ be disjoint $\mathscr{GSC}$ in $(\mathscr{K}, \tau)$ and $\mathscr{J}_1, \mathscr{J}_2$ be disjoint $\mathscr{GSC}$ in $(\mathscr{L}, \sigma)$. Then, $(\mathscr{H}_1, \mathscr{J}_1)$ and $(\mathscr{H}_2, \mathscr{J}_2)$ are $\mathscr{BGSC}$ in $(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$. Even before $(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is b-gs-T_4, there exists disjoint $\mathscr{BGSO}$ $(\mathscr{F}_1, \mathscr{G}_1)$ and $(\mathscr{F}_2, \mathscr{G}_2)$ such that $(\mathscr{H}_1, \mathscr{J}_1) \subseteq (\mathscr{F}_1, \mathscr{G}_1)$ and $(\mathscr{H}_2, \mathscr{J}_2) \subseteq (\mathscr{F}_2, \mathscr{G}_2)$. That is, $\mathscr{H}_1 \subseteq \mathscr{F}_1$, $\mathscr{H}_2 \subseteq \mathscr{F}_2$ and $\mathscr{J}_1 \subseteq \mathscr{G}_1$, $\mathscr{J}_2 \subseteq \mathscr{G}_2$. Hence, $(\mathscr{K}, \tau)$ and $(\mathscr{L}, \sigma)$ are gs-T_4 spaces □

Theorem 21.

(1) Let $\mathscr{TS}$ $(\mathscr{K}, \tau)$ and $(\mathscr{L}, \sigma)$ be g-T_3 spaces iff $\mathscr{BTS}$ $(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is called a gb-T_3

(2) A $\mathscr{GBC}$ subspace of a gb-normal space is gb-normal

(3) Let $\mathscr{TS}$ $(\mathscr{K}, \tau)$ and $(\mathscr{L}, \sigma)$ be g-T_4 spaces iff $\mathscr{BTS}$ $(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is called a gb-T_4

Proof. Proof of (1) to (3) follows from Definitions 13 and 14 and Theorems 18, 19, and 20. □

4. bsg-T_0, bsg-T_1, and bsg-T_2 Spaces

The aspects of b-sg-T_0, b-sg-T_1, and b-sg-T_2 spaces are established, and some of their corresponding characterizations are studied in the section.

Definition 22. A $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is called a binary semi generalized-T_0 (briefly, bsg-T_0) if for any two jointly distinct points $(k_1, l_1), (k_2, l_2) \in \mathscr{K} \times \mathscr{L}$, there exists $\mathscr{BSGO}(\mathscr{H}, \mathscr{J})$ such that exactly one of the following holds:

(i) $(k_1, l_1) \in (\mathscr{H}, \mathscr{J}), (k_2, l_2) \in (\mathscr{K} - \mathscr{H}, \mathscr{L} - \mathscr{J})$

(ii) $(k_1, l_1) \in (\mathscr{K} - \mathscr{H}, \mathscr{L} - \mathscr{J}), (k_2, l_2) \in (\mathscr{H}, \mathscr{J})$

Definition 23. A $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is called a binary semi generalized-T_1 (briefly, bsg-T_1) if for every two jointly distinct points $(k_1, l_1), (k_2, l_2) \in \mathscr{K} \times \mathscr{L}$ with $x_1 \neq x_2, y_1 \neq y_2$, there exists $\mathscr{BSGO}(\mathscr{H}, \mathscr{J})$ and $(\mathscr{Q}, \mathscr{W})$ with $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$ such that $(k_2, l_2) \in (\mathscr{K} - \mathscr{H}, \mathscr{L} - \mathscr{J}), (k_1, l_1) \in (\mathscr{K} - \mathscr{Q}, \mathscr{L} - \mathscr{W})$.

Definition 24. A $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is called a binary semi generalized-T_2 (briefly, bsg-T_2) if for every two jointly distinct points $(k_1, l_1), (k_2, l_2) \in \mathscr{K} \times \mathscr{L}$, with $k_1 \neq k_2, l_1 \neq l_2$, there exists disjoint $\mathscr{BSGO}(\mathscr{H}, \mathscr{J})$ and $(\mathscr{Q}, \mathscr{W})$ such that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$.

Theorem 25. *Let $(\mathscr{K}, \mathscr{L}, \mathscr{M})$ be a $\mathscr{BTS}$; then, for*

(1) every binary-T_0 is b-sg-T_0

(2) every binary-T_1 is b-sg-T_1

(3) every binary-T_2 is b-sg-T_2

(4) every b-sg-T_1 is b-sg-T_0

(5) every b-sg-T_2 is b-sg-T_0

(6) every b-sg-T_2 is b-sg-T_1

Proof.

(1) Let $(\mathscr{K}, \mathscr{L})$ be a binary-T_0 space and (k_1, l_1) and (k_2, l_2) be a two distinct points of $(\mathscr{K}, \mathscr{L})$; as $(\mathscr{K}, \mathscr{L})$ is binary-T_0 space, there exists $\mathscr{BOS}$ $(\mathscr{H}, \mathscr{J})$ such that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{K} - \mathscr{H}, \mathscr{L} - \mathscr{J})$. Even before every $\mathscr{BOS}$ is $\mathscr{BSGO}$ and ergo $(\mathscr{H}, \mathscr{J})$ is $\mathscr{BSGO}$ such that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{K} - \mathscr{H}, \mathscr{L} - \mathscr{J})$, ergo $(\mathscr{K}, \mathscr{L})$ is b-sg-T_0 space

(2) Proof of (2) to (6) is obvious

□

Example 3. Let $\mathscr{K} = \{\xi, \varpi\}$ and $\mathscr{L} = \{\xi, \varpi, \varrho\}$. Clearly, $\mathscr{M} = \{(\phi, \phi), (\{\varpi\}, \{\xi\}), (\phi, \{\varpi, \varrho\}), (\{\varpi\}, \mathscr{L}), (\mathscr{K}, \mathscr{L})\}$ is a $\mathscr{BT}$ from $\mathscr{K}$ to $\mathscr{L}$.

(1) Let $(k_1, l_1) = (\{\varpi\}, \{\xi\})$ and $(k_2, l_2) = (\{\xi\}, \{\varrho\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; there exists b-sg-open set $(\mathscr{H}, \mathscr{J}) = (\{\varpi\}, \{\xi, \varpi\})$; then, it is b-sg-$T_0$ space but not b-T_0 space

(2) Let $(\mathscr{H}, \mathscr{J}) = (\{\xi\}, \{\xi, \varpi\})$ and $(\mathscr{Q}, \mathscr{W}) = (\{\varpi\}, \{\xi, \varrho\})$. Let $(k_1, l_1) = (\{\xi\}, \{\varpi\})$ and $(k_2, l_2) = (\{\varpi\}, \{\varrho\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; then, it is clear that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$, $(k_2, l_2) \notin (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$, $(k_1, l_1) \notin (\mathscr{Q}, \mathscr{W})$. Then, we can say that it is b-sg-T_1 space but not binary-T_1 space

(3) Let $(\mathscr{H}, \mathscr{J}) = (\{\varpi\}, \{\xi, \varpi\})$ and $(\mathscr{Q}, \mathscr{W}) = (\phi, \{\varrho\})$. Let $(k_1, l_1) = (\{\varpi\}, \{\varpi\})$ and $(k_2, l_2) = (\phi, \{\varrho\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; then, it is clear that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$, $(k_2, l_2) \notin (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$, $(k_1, l_1) \notin (\mathscr{Q}, \mathscr{W})$. Then, we might declare it to be b-sg-T_2 space but not binary-T_2 space

(4) Let $(\mathscr{H}, \mathscr{J}) = (\{\varpi\}, \{\varpi, \varrho\})$ and $(\mathscr{Q}, \mathscr{W}) = (\{\xi\}, \mathscr{L})$. Let $(k_1, l_1) = (\{\varpi\}, \{\xi\})$ and $(k_2, l_2) = (\{\xi\}, \phi)$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; then, it is clear that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$, $(k_2, l_2) \notin (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$, $(k_1, l_1) \notin (\mathscr{Q}, \mathscr{W})$. Afterward, we could declare it to be b-sg-T_0 space but not b-sg-T_1 space

(5) Let $(k_1, l_1) = (\phi, \{\xi\})$ and $(k_2, l_2) = (\{\xi\}, \{\varrho\})$. Let $(\mathscr{H}, \mathscr{J}) = (\{\varpi\}, \{\xi\})$ and $(\mathscr{Q}, \mathscr{W}) = (\phi, \{\varpi, \varrho\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; then, it is clear that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$. We might then proclaim it to be b-sg-T_0 space but not b-sg-T_2 space

(6) Let $(\mathscr{H}, \mathscr{J}) = (\{\xi\}, \{\varpi, \varrho\})$ and $(\mathscr{Q}, \mathscr{W}) = (\{\varpi\}, \{\xi, \varpi\})$. Let $(k_1, l_1) = (\{\xi\}, \{\varrho\})$ and $(k_2, l_2) = (\{\varpi\}, \{\xi\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; then, it is clear that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$, $(k_2, l_2) \notin (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$, $(k_1, l_1) \notin (\mathscr{Q}, \mathscr{W})$. Then, we can say that it is b-sg-T_1 space but not b-sg-T_2 space

Theorem 26. *A $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is a*

(1) b-sg-T_1 space if and only if every binary point is $\mathscr{BSGC}$

(2) b-sg-T_0, then $(\mathscr{K}, \mathscr{M}_{\mathscr{K}})$ is sg-T_0 and $(\mathscr{L}, \mathscr{M}_{\mathscr{L}})$ is sg-T_0

(3) If a $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is called a b-sg-T_0, then the $\mathscr{TS}$ $(\mathscr{K}, \tau)$ and $(\mathscr{L}, \sigma)$ are sg-T_0

(4) b-sg-T_1, then $(\mathscr{K}, \mathscr{M}_{\mathscr{K}})$ is sg-T_1 and $(\mathscr{L}, \mathscr{M}_{\mathscr{L}})$ is sg-T_1

Proof. Proof of (1) to (4) follows from Definitions 22, 23, and 24 and Theorems 8, 9, 10, and 11. □

5. b-sg-T_3 and b-sg-T_4 Spaces

The initiation of binary semi-T_3 and semi-T_4 spaces by utilizing b-sg open sets and their properties are examined in this segment.

Definition 27. A $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is called a b-sg-T_3 or b-sg regular if $(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is b-sg-T_1 and for every $(\mathscr{K}, \mathscr{L}) \in \mathscr{K} \times \mathscr{L}$ and every $\mathscr{BSGC}(\mathscr{H}, \mathscr{J}) \subseteq \mathscr{K} \times \mathscr{L}$ such that $(\mathscr{K}, \mathscr{L}) \in (\mathscr{K} - \mathscr{H}, \mathscr{L} - \mathscr{J})$, there exists jointly disjoint $\mathscr{BSGO}$ $(\mathscr{F}_1, \mathscr{G}_1), (\mathscr{F}_2, \mathscr{G}_2)$ such that $(\mathscr{K}, \mathscr{L}) \in (\mathscr{F}_1, \mathscr{G}_1)$, $(\mathscr{H}, \mathscr{J}) \subseteq (\mathscr{F}_2, \mathscr{G}_2)$.

Definition 28. A $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is called a b-sg-T_4 or b-sg normal if $(\mathscr{K}, \mathscr{L}, \mathscr{M})$ is b-sg-T_1 and for every pair of jointly disjoint $\mathscr{BSGC}(\mathscr{H}_1, \mathscr{J}_1), (A_2, B_2)$, there exists jointly disjoint $\mathscr{BSGO}(\mathscr{F}_1, \mathscr{G}_1), (\mathscr{F}_2, \mathscr{G}_2)$ such that $(\mathscr{H}_1, \mathscr{J}_1) \subseteq (\mathscr{F}_1, \mathscr{G}_1)$ and $(\mathscr{H}_2, \mathscr{J}_2) \subseteq (\mathscr{F}_2, \mathscr{G}_2)$.

Theorem 29.

(1) Every binary T_3 is b-sg T_3

(2) Every b-sg T_3 is b-sg-T_0 space

(3) Every b-sg T_3 is b-sg-T_2 space

(4) Every binary normal space is b-sg T_4

(5) Every b-sg normal space is b-sg T_3

Proof. Proof of (1) to (5) follows from Definitions 27 and 28 and Theorem 17. □

Example 4. From Example 3,

(1) let $(\mathscr{H}, \mathscr{J}) = (\{\xi\}, \{\varrho\})$, $(k, l) = (\phi, \{\xi\})$, $(\mathscr{F}_1, \mathscr{G}_1) = (\{\varpi\}, \{\xi\})$, and $(\mathscr{F}_2, \mathscr{G}_2) = (\{\xi\}, \{\varpi, \varrho\})$; then, it is b-sg-$T_3$ space but not b-T_3 space

(2) let $(k_1, l_1) = (\phi, \{\varpi\})$ and $(k_2, l_2) = (\{\xi\}, \{\varrho\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$; there exists $\mathscr{BGSO}(\mathscr{H}, \mathscr{J}) = (\{\varpi\}, \{\xi, \varpi\})$ and $(\mathscr{F}_1, \mathscr{G}_1) = (\{\varpi\}, \{\varpi\})$, $(\mathscr{F}_2, \mathscr{G}_2) = (\phi, \{\xi, \varrho\})$; then, it is b-sg-$T_0$ space but not b-sg-T_3

(3) let $(k_1, l_1) = (\{\xi\}, \{\xi\})$ and $(k_2, l_2) = (\phi, \{\varrho\})$. Let $(\mathscr{F}_1, \mathscr{G}_1) = (\{\varpi\}, \{\xi\})$ and $(\mathscr{F}_2, \mathscr{G}_2) = (\{\xi\}, \{\varpi, \varrho\})$, $(k_1, l_1), (k_2, l_2) \in (\mathscr{K}, \mathscr{L})$ and $(k_1, l_1) \neq (k_2, l_2)$, and $(\mathscr{H}, \mathscr{J}) = (\mathscr{K}, \{\xi\})$ and $(\mathscr{Q}, \mathscr{W}) = (\phi, \{\varpi, \varrho\})$; then, it is clear that $(k_1, l_1) \in (\mathscr{H}, \mathscr{J})$ and $(k_2, l_2) \in (\mathscr{Q}, \mathscr{W})$. Then, we can say that it is b-sg-T_2 space but not b-sg-T_3 space

(4) let $(\mathscr{H}_1, \mathscr{J}_1) = (\phi, \{\xi\})$, $(\mathscr{H}_2, \mathscr{J}_2) = (\{\xi\}, \{\varpi\})$, $(\mathscr{F}_1, \mathscr{G}_1) = (\{\varpi\}, \{\xi\})$, and $(\mathscr{F}_2, \mathscr{G}_2) = (\{\xi\}, \{\varpi, \varrho\})$; then, it is b-sg-$T_4$ space but not binary-T_4 space

(5) let $(k, l) = (\{\xi\}, \{\varpi\})$ and $(\mathscr{H}_1, \mathscr{J}_1) = (\phi, \{\xi\})$, $(\mathscr{H}_2, \mathscr{J}_2) = (\{\varpi\}, \phi)$, $(\mathscr{F}_1, \mathscr{G}_1) = (\{\xi\}, \{\varpi, \varrho\})$, and $(\mathscr{F}_2, \mathscr{G}_2) = (\{\varpi\}, \{\xi\})$. Then, we can say that it is b-sg-T_3 space but not b-sg-T_4 space

Theorem 30.

(1) Let $\mathscr{TS}$ $(\mathscr{K}, \tau)$ and $(\mathscr{L}, \sigma)$ be sg-T_3 spaces if and only if $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(Y)})$ is called a b-sg-T_3

(2) A $\mathscr{BSGC}$ subspace of a b-sg normal space is b-sg normal

(3) Let $\mathscr{TS}$ $(\mathscr{K}, \tau)$ and $(\mathscr{L}, \sigma)$ be sg-T_4 spaces if and only if $\mathscr{BTS}(\mathscr{K}, \mathscr{L}, \tau_{\mathscr{M}(\mathscr{K})} \times \sigma_{\mathscr{M}(\mathscr{L})})$ is called a b-sg-T_4

Proof. Proof of (1) to (3) follows from Definitions 27 and 28 and Theorems 18, 19, and 20. □

6. Conclusion

We defined a few separation axioms in binary topological spaces with respect to binary points of a binary topological spaces, compared their characteristics with those of the existing spaces, and established a few theorems in the paper. The separation axioms, namely, g(gs)-T_0, g(gs)-T_1, g(gs)-T_2, g(gs)-T_3, and g(gs)-T_4, are extended to $\mathscr{BTS}$. The perceived result is gb(b-gs)-$T_4 \Rightarrow$ gb(b-gs)-$T_3 \Rightarrow$ gb(b-gs)-$T_2 \Rightarrow$ gb(b-gs)-$T_1 \Rightarrow$ gb(b-gs)-T_0. Eventually, we identified sg-T_0, sg-T_1, sg-T_2, sg-T_3, and sg-T_4 spaces extended to

$\mathscr{BTS}$. In our future work, we will extend these structures to infer various results such as binary urysohn space and binary tychonoff space in binary topological spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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