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Research Article

Some Conditions of Non-Blow-Up of Generalized Inviscid Surface Quasigeostrophic Equation

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In this paper, we survey some non-blow-up results for the following generalized modified inviscid surface quasigeostrophic equation (GSQG) $\begin{cases} \theta_t + u\cdot\nabla\theta = 0, \\ u = \nabla^{\perp}\psi, \\ -\Lambda^{\beta}\psi = \theta, \\ \theta(x,0) = \theta_0(x). \end{cases}$. This is a generalized surface quasigeostrophic equation (GSQG) with the velocity field u related to the scalar θ by $u = -\nabla^{\perp}\Lambda^{-\beta}\theta$, where $1 \le \beta \le 2$. We prove that there is no finite-time singularity if the level set of generalized quasigeostrophic equation does not have a hyperbolic saddle, and the angle of opening of the saddle can go to zero at most as an exponential decay. Moreover, we give some conditions that rule out the formation of sharp fronts for generalized inviscid surface quasigeostrophic equation, and we obtain some estimates on the formation of semiuniform fronts. These results greatly weaken the geometrical assumptions which rule out the collapse of a simple hyperbolic saddle in finite time.

1. Introduction

In this paper, we consider the generalized inviscid quasigeostrophic equation:

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad (x,t)\in\mathbb{R}^2\times(0,+\infty) \\ u = \nabla^{\perp}\psi, \\ -\Lambda^{\beta}\psi = \theta, \\ \theta(x,0) = \theta_0(x). \end{cases}, \tag{1}$$

where $\theta:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a scalar function, representing the temperature, and $u(x,t)$ is the velocity field of fluid with incompressible condition $\nabla\cdot u = 0$, where $1 \le \beta \le 2$ is a parameter. When $\beta = 2$, (1) becomes 2D Euler vorticity equation while (1) with $\beta = 1$ is the surface quasigeostrophic equation. Clearly, (1) bridges the 2D Euler equation and the surface quasigeostrophic equation. Except in the case when $\beta = 0$, the global regularity issue for (1) remains open. The gradient $\nabla = (\partial_{x_1}, \partial_{x_2})$, $\Lambda^{\alpha} = (-\Delta)^{\alpha/2}$ and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$, $\mathscr{R}^{\perp}\theta = (-\mathscr{R}_2\theta, \mathscr{R}_1\theta)$ with $\mathscr{R}_i$, $i = 1, 2$, for the Riesz transform defined by

$$\mathscr{R}_i(\theta)(x,t) = \frac{1}{2\pi} P.V.\int_{\mathbb{R}^2} \frac{(x_i - y_i)\theta(y,t)}{|x-y|^3}\,dy. \tag{2}$$

Obviously, we have

$$u = -\nabla^{\perp}\Lambda^{-\beta}\theta. \tag{3}$$

This model of generalized 2D Euler/SQG equations had been proposed by Pierrehumbert et al. in 1994 in [1]; then, this model was studied by Chae et al. in [2], and they obtained a regularity criterion in terms of the norm of θ in the Hölder space $C^{2-\beta}(R^2)$ to the generalized inviscid surface quasigeostrophic equations. Then, they researched the global regularity in [3] for a class of generalized equations that the velocity field u is determined by the active scalar θ through $\mathscr{R}\Lambda^{-1}P(\Lambda)\theta$ where $P(\Lambda)$ represents a family of Fourier multiplier operators. Subsequently, they obtained the global existence of a weak solution in [2] by Galerkin method and the local existence of patch-type solutions for the inviscid model. Kiselev et al. in [4, 5] studied the patch dynamics on the whole plane and on the half-plane for modified surface quasigeostrophic equations involving a parameter α that appears in the power of the kernel in their Biot-Savart laws, and they established local-in-time regularity for $\alpha \in (0, 1/2)$ on the whole plane and finite-time singularity for all small $\alpha > 0$. It is hoped that this research sheds some light on the global regularity issue concerning the Euler equation and surface quasigeostrophic equation. However, in this paper, we pay more attention to the generalized modified quasigeostrophic equations without dissipative term, and we give some conditions of non-blow-up of the generalized inviscid surface quasigeostrophic equation.

The rest of this paper is organized as follows. Section 2 is devoted to the main results. Section 3 details the proof of these theorems and corollaries. Furthermore, throughout this paper, we use C to denote the positive constants which may vary from line to line.

2. Main Results

Similar to [6], firstly, we give a definition of a hyperbolic saddle.

Definition 1. A simple hyperbolic saddle in a neighborhood U of the origin is the set of curves $\rho = \text{constant}$ where

$$\rho = (y_1\alpha(t) + y_2)(y_1\beta(t) - y_2), \tag{4}$$

where y_1, y_2 are nonlinear time-dependent coordinate change

$$\begin{aligned} y_1 &= F_1(x_1, x_2, t), \\ y_2 &= F_2(x_1, x_2, t), \end{aligned} \tag{5}$$

with $\alpha(t), \beta(t) \in C^1([0, T^*))$, $F_i \in C^2(\bar{U} \times (0, T^*])$, $|\alpha|, |\beta| \leq C$, $\alpha(t) + \beta(t) \geq 0$, $|\det \partial F_i/\partial x_j| \geq c > 0$, and $x \in U, t \in [0, T^*]$.

Remark 2. Set (4) represents the hyperbola in (y_1, y_2)-coordinate. In particular, we get the asymptote for a hyperbola when $\rho = 0$. The angle of opening of the saddle is $\gamma(t) \simeq \alpha(t) + \beta(t)$ when $|\alpha(t)|, |\beta(t)| \leq C$, while the angle of opening of the saddle is $\gamma(t) \simeq 1/\alpha(t) + 1/\beta(t)$ when $|\alpha(t)|, |\beta(t)| \geq C$ (see also [7]).

The possible singularity is due to $\gamma(t)$ becoming zero in finite time. The following theorems will show that this is not possible, and $\gamma(t)$ can go to zero at most as an exponential decay.

Theorem 3. *Suppose $\theta(x_1, x_2, t)$ is a smooth solution of (1) defined for $0 \leq t < T_*$, $(x_1, x_2) \in \mathbb{R}^2$. Assume that θ is a constant along the hyperbolas defined in Definition 1 for $0 \leq t < T_*$, and for each, fixed t that θ is not a constant on any disc in U, and then, $\lim_{t \longrightarrow T_*} \gamma(t)$ exists and is not 0.*

Corollary 4. *Let $\theta(x, t)$ be as in Theorem 3, let $\xi = \nabla^{\perp}\theta/|\nabla^{\perp}\theta|$, and assume $|\nabla\xi| < C$ on $(\mathbb{R}^2 \setminus U) \times [0, T_*]$. Then, $\theta(x, t)$ continues to some solution of (1) on $\mathbb{R}^2 \times [0, T_* + \varepsilon]$, for some $\varepsilon > 0$.*

Theorem 5. *Let $\theta(x_1, x_2, t)$, α, β, U and F_i be as in Definition 1 but with $T_* = +\infty$. Assume that the C^2 seminorms of F_i are bounded for all time $0 \leq t \leq +\infty$. Then*

$$|\log \gamma| \leq C_1 t + C_2, \tag{6}$$

for all t, where C_1 and C_2 are constants.

Corollary 6. *Let $\theta(x, t)$ be as in Theorem 5, let $\xi = \nabla^{\perp}\theta/|\nabla^{\perp}\theta|$, and assume $|\nabla\xi| < \phi(t)$ on $(\mathbb{R}^2 \setminus U) \times [0, T_*]$, then*

$$\left\|\nabla^{\perp}\theta\right\|_{L^\infty} \leq \left[2\int_0^T C(C + \phi(t))dt + \left\|\nabla^{\perp}\theta(0)\right\|_{L^\infty}^2\right]^{1/2}. \tag{7}$$

Moreover, we give a condition that rules out the formation of sharp fronts for generalized surface quasigeostrophic equations and obtains estimates on the formation of semiuniform fronts.

Theorem 7. *For a generalized modified inviscid quasigeostrophic equations with a semiuniform front, if $1 < \beta < 2$, the thickness $\delta(t)$ satisfies:*

$$\delta(t) > e^{-(C_1 t + C_2)}, \quad \forall t \in [0, T), \tag{8}$$

where the constants C_1 and C_2 depend only on the length of the front, the semiuniformity constant, the initial thickness $\delta(0)$, and the norm of the initial datum $L^1 \cap L^\infty$.

3. Proofs of Theorems and Corollaries

3.1. Proof of Theorems 3 and 5. Suppose $\theta(x, t)$ be a smooth solution of (1) defined for $0 \leq t < T_*$, $(x_1, x_2) \in \mathbb{R}^2$. If for $0 \leq t < T_*$ that θ is constant along the curves $\rho = \text{constant}$, where $\rho = \rho(x_1, x_2, t)$, we denote $\theta(x, t) = \widetilde{\theta}(\rho, t)$ for a function $\widetilde{\theta}$, and $\psi(x, t) = \tilde{\psi}(\rho, \sigma, t)$. Besides, we have

$$\frac{\nabla^{\perp}\theta}{|\nabla^{\perp}\theta|} = \frac{(\eta_1, \eta_2)}{\sqrt{\eta_1^2 + \eta_2^2}}. \tag{9}$$

In the following analysis, we introduce a new set of variables (ρ, σ) to analyze the level set. The first variable ρ is defined in Definition 1, and the σ is from the following identities

$$\frac{\partial x_1}{\partial \sigma} = -\frac{\partial \rho}{\partial x_2}, \frac{\partial x_2}{\partial \sigma} = \frac{\partial \rho}{\partial x_1}, \tag{10}$$

where we write $\widetilde{\phi}(x)$ for $\phi(\rho(x))$, and $\phi(\rho(x))$ is the intersection of the bisector $B(y_1, y_2, t) = 0$ of the angle δ with $\rho > 0$. We write the stream function ψ and system (1) in terms of a time-independent change of variables

$$(x_1, x_2) \longrightarrow (\rho(x_1, x_2, t), \sigma(x_1, x_2, t)). \tag{11}$$

By performing the change of variables in the first equation of (1), we get

$$\begin{aligned} u \cdot \nabla_x \widetilde{\theta} &= \frac{\partial \widetilde{\theta}}{\partial \rho}(u \cdot \nabla_x \rho) = -\frac{\partial \widetilde{\theta}}{\partial \rho}\left(\frac{\partial \psi}{\partial x_2}\frac{\partial \rho}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \rho}{\partial x_2}\right) \\ &= -\frac{\partial \widetilde{\theta}}{\partial \rho}\left(\frac{\partial \psi}{\partial x_2}\frac{\partial x_2}{\partial \sigma} + \frac{\partial \psi}{\partial x_1}\frac{\partial x_1}{\partial \sigma}\right) = -\frac{\partial \widetilde{\theta}}{\partial \rho}\frac{\partial \psi}{\partial \sigma}. \end{aligned} \tag{12}$$

Since $\theta(x, t) = \widetilde{\theta}(\rho, t)$, we have

$$\frac{d\widetilde{\theta}}{dt} = \frac{\partial \widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{\theta}}{\partial \rho}\partial_t \rho. \tag{13}$$

By $\widetilde{\theta}_t + u \cdot \nabla_x \widetilde{\theta} = 0$, we have

$$\frac{d\widetilde{\theta}}{dt} + u \cdot \nabla_x \widetilde{\theta} = 0. \tag{14}$$

Substitute (12) and (13) into (14), we get

$$\frac{\partial \widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{\theta}}{\partial \rho}\partial_t \rho + \left(-\frac{\partial \widetilde{\theta}}{\partial \rho}\frac{\partial \psi}{\partial \sigma}\right) = 0. \tag{15}$$

Thus

$$\frac{\partial \widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{\theta}}{\partial \rho}\left(\frac{\partial \rho}{\partial t} - \frac{\partial \psi}{\partial \sigma}\right) = 0. \tag{16}$$

Since $\partial\widetilde{\theta}/\partial t$ and $\partial\widetilde{\theta}/\partial \rho$ are independent of σ, we easily deduce that

$$\frac{\partial \psi}{\partial \sigma} = \frac{\partial \rho}{\partial t} + G_1(\rho, t). \tag{17}$$

Furthermore, the integration with respect to σ of (17) gives

$$\psi(\rho, \sigma, t) = G_1(\rho, t)\sigma + \int_0^\sigma \frac{\partial \rho}{\partial t} d\sigma + G_2(\rho, t). \tag{18}$$

It is obvious that $G_2(\rho, t) = \psi(\rho, 0, t)$. We obtain a new expression for the stream function. The expression (18) for the velocity stream function in terms of the new variable (ρ, σ) will be used in following part to obtain an estimate on the angle of the saddle.

The main strategy is to estimate the difference of the value of the stream function at a point p that lies in the branch of the saddle $y_2 = \beta(t)y_1$ with the value of the stream function at a point q that lies in the other branch $y_2 = -\alpha(t)y_1$. Both point p and point q have the same y_1 coordinate. We need two expressions of the stream function; one was derived from the equality $\psi = -\Lambda^{-\beta}\theta$, and the other one was derived from the change of variables done in the above analysis. To prove both theorems, we give a lemma.

Firstly, we need the key estimate of the difference of the value of the stream function at two different points that are close to each other that is obtained by the stream function as follows:

$$\psi(x, t) = -\int_{\mathbb{R}^2} \frac{\theta(y)}{|y - x|^{2-\beta}} dy, \tag{19}$$

because of the fact $\psi = -(-\Delta)^{-\beta/2}\theta$. Similar results for the 2D Euler equations have been announced in [8], and the similar phenomenon has been noticed in 2D quasigeostrophic thermal active scalar in [9, 10].

Lemma 8. *Let θ be a solution of (1), ψ be given by the equality $\psi = -(-\Delta)^{-\beta/2}\theta = -\Lambda^{-\beta}\theta$, and p and q be defined as before, and if $1 < \beta < 2$, then, we have*

$$|\psi(p) - \psi(q)| \leq k\gamma, \tag{20}$$

where k satisfies $0 \leq c \leq k \leq C$, and c and C are constants, and $|p - q| \sim \gamma$.

Proof. We evaluate ψ at the point $p = (\rho, \sigma_1)$ and $q = (\rho, \sigma_2)$ with $\sigma_1 \neq \sigma_2$; thus, we

$$I = \psi(p) - \psi(q) = \int_{\mathbb{R}^2} \theta(y)\left(\frac{1}{|y - p|^{2-\beta}} - \frac{1}{|y - q|^{2-\beta}}\right) dy. \tag{21}$$

If we denote $\tau = |p - q|$. We split the integral I:

$$\begin{aligned} I(x) = &\left(\int_{|y-p|\leq 2\tau} \theta(y)\left(\frac{1}{|y - p|^{2-\beta}} - \frac{1}{|y - q|^{2-\beta}}\right) dy\right. \\ &+ \int_{2\tau<|y-p|\leq k} \theta(y)\left(\frac{1}{|y - p|^{2-\beta}} - \frac{1}{|y - q|^{2-\beta}}\right) dy \\ &\left.+ \int_{k<|y-p|} \theta(y)\left(\frac{1}{|y - p|^{2-\beta}} - \frac{1}{|y - q|^{2-\beta}}\right) dy\right) \\ = &I_1 + I_2 + I_3, \end{aligned} \tag{22}$$

where k is a fixed number.

Next, we estimate every term in (22), respectively.

$$\begin{aligned}|I_1| &\le \|\theta\|_{L^\infty}\int_{|y-p|\le 2\tau}\left|\frac{1}{|y-p|^{2-\beta}}-\frac{1}{|y-q|^{2-\beta}}\right|dy\\ &\le C\int_{|y-p|\le 2\tau}\left|\frac{1}{|y-p|^{2-\beta}}+\frac{1}{|y-q|^{2-\beta}}\right|dy\\ &\le C\int_{|y-p|\le 3\tau}\left|\frac{1}{|y-p|^{2-\beta}}\right|dy\le\frac{C}{\beta}(3\tau)^\beta\le C\tau^\beta.\end{aligned}\tag{23}$$

If we choose s to be a point in the line between p and q, then, $|y-p|\le 2|y-s|$, and we estimate I_2 by

$$\begin{aligned}|I_2| &\le C\tau\int_{2\tau<|y-p|\le k}\max_s\left|\nabla\left(\frac{1}{|y-s|^{2-\beta}}\right)\right|dy\\ &\le C\tau\int_{2\tau<|y-p|\le k}(2-\beta)\max_s|y-s|^{\beta-3}dy\\ &\le C\tau(2-\beta)\int_{2\tau<|y-p|\le k}\max_s|y-s|^{\beta-3}dy\\ &\le C\tau\left(k^{\beta-1}-c\tau^\beta\right)\le C\tau.\end{aligned}\tag{24}$$

For the third term I_3, we have

$$|I_3|\le C\tau,\tag{25}$$

where we have used the fact that the norm $\|\theta\|_{L^2}$ is conserved for all time. Collecting (23), (24), and (25), we have the needed result in (20). □

Then, we need to divide it into two cases to prove the main theorem. However, we only deal with the case $|\alpha|,|\beta|\le C$, and for the case $|\alpha|,|\beta|>C$, we can use a similar method to prove the result, and we omit it here.

Assume $|\alpha|,|\beta|\le C$. In this case, the angle of the saddle is $r(t)=\alpha(t)+\beta(t)$. We take two points p and q lying in the same level set but in different arms. Using the identity (4), we define

$$\tilde{p}(y_1,t)=(y_1,\beta(t)y_1),\tilde{q}(y_1,t)=(y_1,-\alpha(t)y_1).\tag{26}$$

Then, we take the limit approaching points p and q, respectively. Then, necessarily, $\rho\longrightarrow 0$ and σ grow logarithmically. Then, we have the following lemma.

Lemma 9. *Under the assumptions in Theorem 3, $|\alpha|,|\beta|\le C$, let ψ be given by expression (18) and (p,q) defined as before, then*

$$\begin{aligned}S_1=\psi(p)-\psi(q)&=\frac{d\alpha}{dt}\int_0^{y_1}\frac{\tilde{y}_1}{D(\tilde{q}(\tilde{y}_1,t))}d\tilde{y}_1\\ &\quad+\frac{d\beta}{dt}\int_0^{y_1}\frac{\tilde{y}_1}{D(\tilde{q}(\tilde{y}_1,t))}d\tilde{y}_1+O(\gamma),\end{aligned}\tag{27}$$

where $D=\det|\partial F_i/\partial x_j|$.

Proof. We evaluate the stream function ψ at the points $p_1=(\rho,\sigma_1)$ and $q_1=(\rho,\sigma_2)$ with $\sigma_1=\sigma_2$. According to (18), we have

$$\psi(q_1)-\psi(p_1)=G_1(\rho,t)(\sigma_2-\sigma_1)+\int_{\sigma_1}^{\sigma_2}\frac{\partial\rho}{\partial t}d\sigma.\tag{28}$$

Next, we take the limit when $p_1\longrightarrow p$ and $q_1\longrightarrow q$, and it means $\rho\longrightarrow 0$.

By (17), we have

$$\begin{aligned}G_1(\rho,t)&=\frac{\partial\psi}{\partial\sigma}-\frac{\partial\rho}{\partial t}=\frac{\partial\psi}{\partial x_1}\frac{\partial x_1}{\partial\sigma}+\frac{\partial\psi}{\partial x_2}\frac{\partial x_2}{\partial\sigma}-\frac{\partial\rho}{\partial t}\\ &=-\frac{\partial\psi}{\partial x_1}\frac{\partial\rho}{\partial x_2}+\frac{\partial\psi}{\partial x_2}\frac{\partial\rho}{\partial x_1}-\frac{\partial\rho}{\partial t}=-u\cdot\nabla\rho-\frac{\partial\rho}{\partial t},\end{aligned}\tag{29}$$

where we have used (10).

For a fixed $t\in[0,T_*]$, we can estimate $G_1(\rho,t)$ by

$$|G_1(\rho,t)|\le|u|\cdot|\nabla\rho|+\left|\frac{\partial\rho}{\partial t}\right|,\tag{30}$$

where

$$\frac{\partial\rho}{\partial x_i}=\frac{\partial\rho}{\partial y_1}\frac{\partial y_1}{\partial x_i}+\frac{\partial\rho}{\partial y_2}\frac{\partial y_2}{\partial x_i},\tag{31}$$

$$\frac{\partial\rho}{\partial t}=\frac{\partial\rho}{\partial y_1}\frac{\partial y_1}{\partial t}+\frac{\partial\rho}{\partial y_2}\frac{\partial y_2}{\partial t}+\frac{d(\delta-\beta)}{dt}y_1y_2+\frac{d(\beta\delta)}{dt}y_1^2.\tag{32}$$

Considering the definition of $\rho=\rho(y_1,y_2,t), y=F(x_1,x_2,t)$, and $F_i\in C^2(\bar{U}\times[0,T_*])$, we have

$$\begin{aligned}\frac{\partial\rho}{\partial x_i}&\le|y|(\text{const}),\\ \frac{\partial\rho}{\partial t}&\le|y|(\text{const}).\end{aligned}\tag{33}$$

Now, we derive an estimate for the velocity $u(x)$:

$$u(x,t)=\int_{\mathbb{R}^2}\frac{y^\perp\theta(x+y,t)}{|y|^{4-\beta}}dy.\tag{34}$$

We consider $\varepsilon>0$, and we define $N(\varepsilon)$ to be a smooth nonnegative function such that $N(\varepsilon)=1$ for $0\le\varepsilon\le 1$, and $N(\varepsilon)=0$ for $\varepsilon\ge 2$. Then, we have

$$\begin{aligned}u(x)&=\int_{|y|\le 2\varepsilon}N\left(\frac{|y|}{\varepsilon}\right)\frac{y^\perp\theta(x+y)}{|y|^{4-\beta}}dy\\ &\quad+\int_{|y|\ge\varepsilon}\left(1-N\left(\frac{|y|}{\varepsilon}\right)\right)\frac{y^\perp\theta(x+y)}{|y|^{4-\beta}}dy=I_1+I_2.\end{aligned}\tag{35}$$

For the first term, by integrating by parts, we have

$$\begin{aligned}
|I_1| &= \int_{|y|\le 2\varepsilon} N\left(\frac{|y|}{\varepsilon}\right)\left(\frac{1}{2-\beta}\right)\left(\nabla^{\perp}|y|^{-(2-\beta)}\right)\theta(x+y)dy \\
&\le \frac{1}{2-\beta}\int_{|y|\le 2\varepsilon}|y|^{-(2-\beta)}\nabla^{\perp}\left(N\left(\frac{|y|}{\varepsilon}\right)\theta(x+y)\right)dy \\
&\le \frac{1}{2-\beta}\int_{|y|\le 2\varepsilon}|y|^{-(2-\beta)}\nabla^{\perp}\left(N\left(\frac{|y|}{\varepsilon}\right)\right)\theta(x+y)dy \\
&\quad + \frac{1}{2-\beta}\int_{|y|\le 2\varepsilon}|y|^{-(2-\beta)}\nabla^{\perp}\theta(x+y)N\left(\frac{|y|}{\varepsilon}\right)dy \\
&\le \frac{1}{2-\beta}\|\theta(x+y)\|_{L^\infty}\int_{|y|\le 2\varepsilon}y^{-(2-\beta)}dy \\
&\quad + \left\|\nabla^{\perp}\theta(x+y)\right\|_{L^\infty}\frac{1}{2-\beta}\int_{|y|\le 2\varepsilon}|y|^{-(2-\beta)}N\left(\frac{|y|}{\varepsilon}\right)dy \\
&\le \frac{1}{\beta(2-\beta)}\|\theta_0\|_{L^\infty}\varepsilon^{\beta} + \left\|\nabla^{\perp}\theta\right\|_{L^\infty}\frac{1}{\beta(2-\beta)}\varepsilon^{\beta} \\
&\le C\frac{1}{\beta(2-\beta)}\varepsilon^{\beta} + \frac{1}{\beta(2-\beta)}\left\|\nabla^{\perp}\theta\right\|_{L^\infty}\varepsilon^{\beta}.
\end{aligned} \tag{36}$$

For the second term, we have

$$\begin{aligned}
|I_2| &\le \int_{|y|\ge\varepsilon}\frac{\theta(x+y)}{|y|^{3-\beta}}dy = \int_{\varepsilon\le|y|<k}\frac{\theta(x+y)}{|y|^{3-\beta}}dy \\
&\quad + \int_{k\le|y|}\frac{\theta(x+y)}{|y|^{3-\beta}}dy \\
&\le \|\theta\|_{L^\infty}\int_{\varepsilon\le|y|<k}\frac{1}{|y|^{3-\beta}}dy + \int_{k\le|y|}\frac{\theta(x+y)}{k^{\beta+1}}dy \\
&\le C_1\|\theta\|_{L^\infty}\frac{1}{\beta-1}\varepsilon^{\beta-1} + C_2\|\theta\|_{L^2} \\
&\le \frac{C_1}{\beta-1}\|\theta_0\|_{L^\infty}k^{\beta-1} + C_2\|\theta_0\|_{L^2}.
\end{aligned} \tag{37}$$

Therefore, we have

$$\begin{aligned}
|u(x)| &\le \frac{1}{\beta(2-\beta)}\varepsilon^{\beta} + \frac{1}{\beta(2-\beta)}\left\|\nabla^{\perp}\theta\right\|_{L^\infty}\varepsilon^{\beta} \\
&\quad + \frac{C_1}{\beta-1}\|\theta_0\|_{L^\infty}k^{\beta-1} + C_2.
\end{aligned} \tag{38}$$

Take $\varepsilon = \|\nabla^{\perp}\theta\|_{L^\infty}^{-1/\beta}$, and then, u is bounded by $\|\nabla^{\perp}\theta\|_{L^\infty}^{-1}$, assuming that $\|\nabla^{\perp}\theta\|_{L^\infty}^{-1} \ge M$, where M is a constant. Therefore

$$|G_1(\rho,t)| \le |y|(\text{const}). \tag{39}$$

From the definition of ρ, we know that $y^2 \sim \rho$ when we approach the origin along the bisector B with $\rho > 0$. Therefore, $|G_1|$ is at most $\rho^{1/2}$ when $\rho \longrightarrow 0$. This implies that

$$\lim_{\rho\longrightarrow 0} G_1(\rho,t)\cdot(\sigma_2-\sigma_1) = 0. \tag{40}$$

For the second term, firstly, we define

$$\Gamma = \{(y_1,y_2)\colon \rho = \text{const}\}. \tag{41}$$

By the change of variables, we have the following expression for $\partial y_i/\partial\sigma$,

$$\begin{aligned}
\frac{\partial y_i}{\partial\sigma} &= \frac{\partial y_i}{\partial x_1}\frac{\partial x_1}{\partial\sigma} + \frac{\partial y_i}{\partial x_2}\frac{\partial x_2}{\partial\sigma} = -\frac{\partial y_i}{\partial x_1}\frac{\partial\rho}{\partial x_2} + \frac{\partial y_i}{\partial x_2}\frac{\partial\rho}{\partial x_1} \\
&= -\frac{\partial y_i}{\partial x_1}\left(\frac{\partial\rho}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial\rho}{\partial y_2}\frac{\partial y_2}{\partial x_2}\right) \\
&\quad + \frac{\partial y_i}{\partial x_2}\left(\frac{\partial\rho}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial\rho}{\partial y_2}\frac{\partial y_2}{\partial x_1}\right),
\end{aligned} \tag{42}$$

and then by the above identify, (42) we have

$$\begin{aligned}
&\frac{\partial y_1}{\partial\sigma} = -D\frac{\partial\rho}{\partial y_2},\ \frac{\partial y_2}{\partial\sigma} = D\frac{\partial\rho}{\partial y_1}, \\
&\frac{\partial y_1}{\partial\sigma}\frac{d\sigma}{dy_1} = 1,\ \frac{\partial y_2}{\partial\sigma}\frac{d\sigma}{dy_2} = 1, \text{on } \Gamma.
\end{aligned} \tag{43}$$

□

Then, we utilize (31), (32), and (42) to estimate the last term $\int_{\sigma_1}^{\sigma_2}(\partial\rho/\partial t)d\sigma$ in (28); it is similar to the surface quasi-geostrophic equation in [7], and we omit it here.

If we take $q = (0,0)$ in Lemma 9, we obtain

$$S_2 = \psi(p) - \psi(q) = \frac{d\beta}{dt}\int_0^{y_1}\frac{\tilde{y}_1}{D(\tilde{p}(\tilde{y}_1,t))}d\tilde{y}_1 + A(x_1,x_2,t), \tag{44}$$

where $D = \det|\partial F_i/\partial x_j|$, and $A(x_1,x_2,t)$ is a bounded function for all t.

If we denote

$$\begin{aligned}
M(p) &= \int_0^{y_1}\frac{\tilde{y}_1}{D(\tilde{p}(\tilde{y}_1,t))}d\tilde{y}_1, \\
M(q) &= \int_0^{y_1}\frac{\tilde{y}_1}{D(\tilde{q}(\tilde{y}_1,t))}d\tilde{y}_1,
\end{aligned} \tag{45}$$

then, (44) becomes $S_2 = (d\beta/dt)M(p) - A(x_1,x_2,t)$; therefore, (27) in Lemma 9 becomes

$$\begin{aligned}
S_1 &= \frac{d\beta}{dt}M(p) + \frac{d\alpha}{dt}M(q) + O(\gamma) \\
&= \left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right)M(p) + \frac{d\alpha}{dt}(M(q) - M(p)) + O(\gamma).
\end{aligned} \tag{46}$$

By $|\alpha|, |\beta| \le C$, we have $\gamma \simeq \alpha + \beta$, thus $d\gamma/dt \simeq (d\alpha/dt) + (d\beta/dt)$. By the expression $M(p)$ and $M(q)$, there exists constants c and C such that $0 < c \le M(p) \le C$, $0 < c \le M(q) \le C$, and $M(p) - M(q) = O(\gamma)$, and then, we have $S_1 = (d\gamma/dt)M(p) + O(\gamma)$. By Lemma 8, we have

$$\left|\frac{d\gamma}{dt}\right| \le C\gamma, \tag{47}$$

γ is less than a small constant, and the constant C is different from line to line. The proofs of Theorems 3 and 5 follow directly from integrating (47).

3.2. Proof of Corollaries 4 and 6. Next, we prove Corollaries 4 and 6 By the fact $u(x,t) = \int_{\mathbb{R}^2} \nabla_y^\perp \theta(x+y,t)/|y|^{2-\beta} dy$, we have

$$\begin{aligned}\nabla u(x,t) &= -\nabla_x \int_{\mathbb{R}^2} \frac{\nabla_y^\perp \theta(x+y,t)}{|y|^{2-\beta}} dy \\ &= -\int_{\mathbb{R}^2} \frac{1}{|y|^{2-\beta}} \left(\nabla_y \nabla_y^\perp \theta\right)(x+y) dy,\end{aligned} \tag{48}$$

and then, we write the integral as a limit as $\varepsilon \longrightarrow 0$ of integrals on $|y| > \varepsilon$. Because the two gradients applied to θ commute, we can choose any one of them and integrate by parts. The limit of the contribution from $|y| = \varepsilon$ vanishes. In this way, we have

$$\nabla u(x,t) = -P.V. \int_{\mathbb{R}^2} (2-\beta)\left\{\widehat{y}^\perp \otimes [\nabla\theta(x+y)]\right\} \frac{dy}{|y|^{3-\beta}}, \tag{49}$$

or

$$\nabla u(x,t) = -P.V. \int_{\mathbb{R}^2} (2-\beta)\left\{\left[\nabla^\perp\theta(x+y)\right] \otimes \widehat{y}\right\} \frac{dy}{|y|^{3-\beta}}, \tag{50}$$

where $\widehat{y} = y/|y|$.

Let $\nabla^\perp\theta = A\xi$, $A = |\nabla^\perp\theta|$, by the fact

$$\begin{aligned}\frac{D\nabla^\perp\theta}{Dt} &= \nabla u \cdot \nabla^\perp\theta, \\ \frac{D|\nabla^\perp\theta|}{Dt} &= \alpha\left|\nabla^\perp\theta\right|.\end{aligned} \tag{51}$$

We have

$$\alpha(x) = \{[(\nabla u)(x)]\xi(x)\} \cdot \xi(x), \tag{52}$$

and then, we deduce the two representations of α. Utilizing (49), we have

$$\begin{aligned}\alpha(x) &= \nabla u \cdot \xi(x) \cdot \xi(x) \\ &= -P.V. \int (2-\beta)\widehat{y} \otimes \nabla\theta(x+y) \frac{1}{|y|^{3-\beta}} dy \cdot \xi(x) \cdot \xi(x) \\ &= -P.V. \int (2-\beta)[\widehat{y} \otimes \nabla\theta(x+y) \cdot \xi(x)] \cdot \xi(x) \frac{1}{|y|^{3-\beta}} dy \\ &= -P.V.(2-\beta) \int (2-\beta)\left[\widehat{y} \otimes \frac{\nabla\theta(x+y)}{|\nabla^\perp\theta(x+y)|} \cdot \xi(x)\right] \\ &\quad \cdot \xi(x)\left|\nabla^\perp\theta(x+y)\right| \frac{1}{|y|^{3-\beta}} dy = -P.V.(2-\beta) \\ &\quad \cdot \int (2-\beta)\left[\left(\widehat{y} \otimes \xi^\perp(x+y)\right) \cdot \xi\right] \cdot \xi \left|\nabla^\perp\theta(x+y) \frac{1}{|y|^{3-\beta}} dy\right. \\ &= P.V.(2-\beta) \int \xi^T\left[\widehat{y} \cdot \left(\xi^T(x+y)\right)^T \cdot \xi\right] A(x+y) \frac{1}{|y|^{3-\beta}} dy \\ &= P.V.(2-\beta) \int \left(\widehat{y}^T \cdot \xi\left(\xi^\perp(x+y)\right) \cdot \xi\right) A(x+y) \frac{1}{|y|^{3-\beta}} dy \\ &= P.V.(2-\beta) \int \left(\widehat{y}^T \cdot \xi^\perp\right)\left(\xi(x+y) \cdot \xi^\perp\right) A(x+y) \frac{1}{|y|^{3-\beta}} dy.\end{aligned} \tag{53}$$

Therefore, we obtain the first representation of $\alpha(x)$ as follows:

$$\begin{aligned}\alpha(x) &= P.V. \int_{\mathbb{R}^2} (2-\beta)\left\{\left[\widehat{y} \cdot \xi^\perp(x)\right]\left[\xi(x+y) \cdot \xi^\perp(x)\right]\right\} \\ &\quad \cdot A(x+y) \frac{dy}{|y|^{3-\beta}},\end{aligned} \tag{54}$$

By (50), we used similar estimates as above, and we have an alternative expression of $\alpha(x)$:

$$\begin{aligned}\alpha(x) &= -P.V. \int_{\mathbb{R}^2} (2-\beta)\{[\widehat{y} \cdot \xi(x)][\xi(x+y) \cdot \xi(x)]\} \\ &\quad \cdot A(x+y) \frac{dy}{|y|^{3-\beta}}.\end{aligned} \tag{55}$$

Consider $\rho > 0$, with $\chi(r)$ a smooth, nonnegative function of one positive variable $\chi(r) = 1$ for $0 \le r \le 1/2$, $\chi(r) = 0$ for $r \ge 1$.

$$\begin{aligned}\alpha(x) &= P.V. \int_{\mathbb{R}^2} \chi\left(\frac{|y|}{\rho}\right)(2-\beta)\left\{\left[\widehat{y} \cdot \xi^\perp(x)\right]\left[\xi(x+y) \cdot \xi^\perp(x)\right]\right\} \\ &\quad \cdot A(x+y) \frac{dy}{|y|^{3-\beta}} + P.V \int_{\mathbb{R}^2} \left[1 - \chi\left(\frac{|y|}{\rho}\right)\right] \\ &\quad \cdot (2-\beta)\left\{\left[\widehat{y} \cdot \xi^\perp(x)\right]\left[\xi(x+y) \cdot \xi^\perp(x)\right]\right\} \\ &\quad \cdot A(x+y) \frac{dy}{|y|^{3-\beta}} = \alpha_{\text{in}} + \alpha_{\text{out}}.\end{aligned} \tag{56}$$

For the second term, it is easy to obtain

$$|\alpha_{\text{out}}| \le C\rho^{-1}\int_{|y|\ge(1/2)\rho}|\theta(x+y)|\frac{dy}{|y|^{3-\beta}} \le C\rho^{-(\beta+2)}\|\theta\|_{L^2}C\rho^{-(\beta+2)}\|\theta_0\|_{L^2}. \tag{57}$$

Let us consider now the situation in which the direction field ξ is smooth in the ball of centre x and the radius ρ, corresponding to the smoothly directed case. We use the representation in (54), let denote $G=\sup_{|y|\le\rho}|\nabla\xi(x+y)|$, clearly

$$\left|\xi(x+y)\cdot\xi^{\perp}(x)\right| \le G|y| \quad \text{for} \quad |y|\le\rho. \tag{58}$$

Then, we have

$$|\alpha_{\text{in}}| \le G\int_{|y|\le\rho}\chi\left(\frac{|y|}{\rho}\right)A(x+y)\frac{dy}{|y|^{3-\beta}}. \tag{59}$$

By the fact $A=\xi\cdot(\nabla^{\perp}\theta)$, integrating by parts on the right on (59), we have

$$\begin{aligned}
&\int_{|y|\le\rho}\chi\left(\frac{|y|}{\rho}\right)\xi(x+y)\nabla_y^{\perp}\theta(x+y)\frac{dy}{|y|^{3-\beta}}\\
&= -\int_{|y|\le\rho}\theta(x+y)\nabla_y^{\perp}\cdot\left(\xi(x+y)\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{3-\beta}}\right)dy\\
&= -\int_{|y|\le\rho}\left(\nabla_y^{\perp}\xi(x+y)\right)\theta(x+y)\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{3-\beta}}dy\\
&\quad -\int_{|y|\le\rho}\theta(x+y)\xi(x+y)\cdot\nabla_y^{\perp}\left(\chi\left(\frac{|y|}{\rho}\right)\right)\frac{1}{|y|^{3-\beta}}dy\\
&\quad -(3-\beta)\int_{|y|\le\rho}\left(\xi(x+y)\cdot\hat{y}^{\perp}\right)\theta(x+y)\\
&\quad \cdot\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{4-\beta}}\frac{(y_1,y_2)}{|y|}dy = I_1+I_2+I_3.
\end{aligned} \tag{60}$$

We estimate every term in (60), respectively, and then, we have

$$\begin{aligned}
|I_1| &\le CG\|\theta\|_{L^\infty}\left|\int_{|y|<(1/2)\rho}\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{3-\beta}}dy\right|\\
&\le CG\|\theta\|_{L^\infty}\int_0^{2\pi}\int_0^{(1/2)\rho}\frac{1}{|y|^{3-\beta}}ydrd\alpha\\
&\le C\frac{1}{\beta-1}G\rho^{\beta-1}\|\theta\|_{L^\infty},\\
|I_2| &\le C\|\theta\|_{L^\infty}\rho^{\beta-1}.
\end{aligned} \tag{61}$$

For the third term, we write $\xi(x+y)=\xi(x)+(\xi(x+y)-\xi(x))$, and therefore

$$\begin{aligned}
|I_3| &= \xi(x)\int_{|y|\le\rho}\hat{y}^{\perp}\theta(x+y)\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{4-\beta}}\frac{(y_1,y_2)}{|y|}dy\\
&\quad +\int_{|y|\le\rho}\left[(\xi(x+y)-\xi(x))\cdot\hat{y}^{\perp}\right]\theta(x+y)\\
&\quad \cdot\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{4-\beta}}\frac{(y_1,y_2)}{|y|}dy = I_{31}+I_{32}.
\end{aligned} \tag{62}$$

Since

$$P.V.\int_{\mathbb{R}^2}\frac{\tilde{y}^{\perp}\theta(x+y)}{|y|^{4-\beta}}dy = -u(x), \tag{63}$$

then, we have

$$I_{31} \le C|u(x)|. \tag{64}$$

For the second term, we have

$$\begin{aligned}
I_{32} &= \left|P.V.\int_{|y|\le\rho/2}\left|\nabla\xi(x^*)y\hat{y}^{\perp}\theta(x+y)\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{4-\beta}}\frac{(y_1,y_2)}{|y|}dy\right|\right.\\
&\quad +\left|P.V.\int_{\rho/2\le|y|\le\rho}\right|\left[(\xi(x+y)-\xi(x))\cdot\hat{y}^{\perp}\right]\\
&\quad \left.\cdot\theta(x+y)\chi\left(\frac{|y|}{\rho}\right)\frac{1}{|y|^{4-\beta}}\frac{(y_1,y_2)}{|y|}dy\right|\\
&\le C(G\|\theta\|_{L^\infty})\int_0^{2\pi}\int_0^{\rho/2}\frac{1}{|y|^{3-\beta}}ydrd\theta'\\
&\quad +GP.V.\int_{\rho/2\le|y|\le\rho}\tilde{y}^{\perp}\theta(x+y)|y|^{-(3-\beta)}dy\\
&\le CG\|\theta\|_{L^\infty}\rho^{\beta-1}+G\|\theta\|_{L^2}\left(\int_{\rho/2\le|y|\le\rho}\left(\tilde{y}^{\perp}\right)^2|y|^{-2(3-\beta)}dy\right)^{1/2}\\
&= CG\|\theta\|_{L^\infty}\rho^{\beta-1}+CG\rho^{2(2-\beta)}\|\theta\|_{L^2},
\end{aligned} \tag{65}$$

where we have used the Hölder inequality and $x^*\in(x,x+y)$, and θ' represents the angle in the polar coordinate through the transformation. Thus, we have

$$|\alpha_{\text{in}}| \le CG\left[|u(x)|+\left(\rho^{\beta-1}G+\rho^{-\beta}\right)\|\theta\|_{L^\infty}+\rho^{2(2-\beta)}\|\theta\|_{L^2}\right]. \tag{66}$$

Combining (57) and (66) in (56), we have the following.

Lemma 10. *Assume that x is such that*

$$G=\sup_{|y|\le\rho}|\nabla\xi(x+y)|, \tag{67}$$

Then, $|\alpha(x)|$ *is bounded by*

$$|\alpha(x)| \leq C\Big[G|u(x)| + \left(G\rho^{\beta-1} + \rho^{2(2-\beta)}\right) \cdot \left(G\|\theta\|_{L^\infty} + \rho^{-(\beta+2)}\|\theta\|_{L^2}\right)\Big]. \tag{68}$$

Under the assumptions of Corollary 4, we know $G(x)$ is bounded by a constant $|x| > c$. Therefore, we can estimate α on $V = U \setminus \{|x| < C\}$ by

$$|\alpha| \leq \left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1} C. \tag{69}$$

In particular, the function $F = |\nabla^\perp\theta|/|\nabla\rho|$ is independent of σ. We write the material derivative of F as follows:

$$D_t\left(\frac{|\nabla^\perp\theta|}{|\nabla\rho|}\right) = \frac{1}{|\nabla\rho|}D_t\left(|\nabla^\perp\theta|\right) + |\nabla^\perp\theta|\left(D_t\left(\frac{1}{|\nabla\rho|}\right)\right) = \left(\alpha + |\nabla\rho|D_t\left(\frac{1}{|\nabla\rho|}\right)\right)\frac{|\nabla^\perp\theta|}{|\nabla\rho|}. \tag{70}$$

We estimate $|\nabla\rho|D_t(1/|\nabla\rho|)$ on V by

$$|\nabla\rho|\left|D_t\left(\frac{1}{|\nabla\rho|}\right)\right| = \frac{1}{|\nabla\rho|}|D_t|\nabla\rho|| = \frac{1}{|\nabla\rho|}\left|\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)|\nabla\rho|\right| \leq C_1\left|\frac{d\delta}{dt}\right| + C_2\left|\frac{d\beta}{dt}\right| + C_3\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1} C. \tag{71}$$

The Definition 1 shows that $|d\delta/dt| \leq C$. The same estimate $|d\beta/dt| \leq C$ can be obtained. Combing (69) and (71), we get

$$|\alpha + |\nabla\rho|D_t 1/|\nabla\rho|| \leq C\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1} + C \tag{72}$$

on V when $\|\nabla^\perp\theta\|_{L^\infty} > C$.

Then, by (70), we have

$$|D_t F| \leq C\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1} \cdot F \tag{73}$$

on V when $\|\nabla^\perp\theta\|_{L^\infty} \geq C$. However, because F only depends on ρ and t and by the proposition, it follows that inequality (73) only depends on ρ and t.

We try to estimate the material derivative of $|\nabla^\perp\theta|$ on U. If $|x| < C$, then, we find $\tilde{x} \in V$ such that $\rho(x) = \rho(\tilde{x})$. The inequality (73) holds for $\tilde{x}$, and therefore, it holds for x. Therefore, the inequality (73) holds both on V and U. By (70), we have the following identity

$$D_t\left(|\nabla^\perp\theta|\right) = |\nabla\rho| \cdot D_t F + F \cdot D_t(|\nabla\rho|). \tag{74}$$

For the second term in (74), by the fact that the boundedness of $|\nabla\rho|$ on V, we have

$$F \leq \left\|\nabla^\perp\theta\right\|_{L^\infty}. \tag{75}$$

Because F is independent of σ, the inequality (75) holds on U. We compute the material derivative of $|\nabla\rho|$ and estimate it on U as follows:

$$\begin{aligned}|D_t(|\nabla\rho|)| &= |d/dt|\nabla\rho| + u\cdot\nabla(|\nabla\rho|)| \\ &\leq C_1|d\delta/dt| + C_2|d\beta/dt| + |u\cdot\nabla(|\nabla\rho|)| \\ &\leq C_1|d\beta| + C_2|d\beta/dt| + C_3\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1}.\end{aligned} \tag{76}$$

Combing (73), (74), and (76), we have

$$D_t\left(|\nabla^\perp\theta|\right) \leq C\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1}\left\|\nabla^\perp\theta\right\|_{L^\infty}. \tag{77}$$

If $x \in \mathbb{R}^2 \setminus U$, then, $|\nabla\xi| \leq \phi(t)$. By using the upper bound of α, we obtain

$$D_t\left(|\nabla^\perp\theta|\right) \leq C\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1}(C + \phi(t)), \tag{78}$$

on $x \in \mathbb{R}^2 \setminus U$. Therefore

$$\left|\frac{d}{dt}\left\|\nabla^\perp\theta\right\|_{L^\infty}\right| \leq C\left\|\nabla^\perp\theta\right\|_{L^\infty}^{-1}(C + \phi(t)). \tag{79}$$

Let $B = \|\nabla^\perp\theta\|_{L^\infty}$, integrating (79) on time t, we have

$$\begin{aligned}\left(B^2 - B^2(0)\right) &\leq 2\int_0^T C(C + \phi(t))dt, \\ B^2 &\leq 2\int_0^T C(C + \phi(t))dt + B^2(0).\end{aligned} \tag{80}$$

We have

$$\left\|\nabla^\perp\theta\right\|_{L^\infty} \leq \left[2\int_0^T C(C + \phi(t))dt + \left\|\nabla^\perp\theta(0)\right\|_{L^\infty}^2\right]^{1/2}. \tag{81}$$

At time $t = 0$, $\|\nabla^\perp\theta\|_{L^\infty} < \infty$. It shows that B is bounded, and $\|\nabla^\perp\theta\|_{L^\infty}$ is bounded, and it proves the non-blow-up for solution, and this proves the Corollaries 4 and 6.

3.3. Proof of Theorem 11. Let $F = F(x, t)$ is a solution to (1), and a level curve of F can be parameterized by

$$x_2 = f_\pm(x, t), \quad x_1 \in [a, b], \tag{82}$$

with

$$\begin{aligned}&f_\pm(x, t) \in C^1([a, b] \cap [0, T^*)), \\ &f_-(x_1, t) < f_+(x_1, t) \quad \forall x_1 \in [a, b], t \in [0, T),\end{aligned} \tag{83}$$

in the sense that

$$F(x, f_\pm(x, t), t) = F_\pm(t), \quad x_1 \in [a, b]. \tag{84}$$

From (82) and (84), we have

$$\frac{\partial F}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial f_\pm}{\partial x_1}=0, \tag{85}$$

$$\frac{\partial F}{\partial t}+\frac{\partial F}{\partial x_2}\frac{\partial f_\pm}{\partial t}=0. \tag{86}$$

Combining (1), (85), and (86) and the relationship $\nabla^\perp \psi = u$, we have

$$\begin{aligned}\frac{\partial f_\pm}{\partial t}&=\frac{-\partial F/\partial t}{\partial F/\partial x_2}=\frac{(-\partial\psi/\partial x_2,(\partial\psi/\partial x_1))\cdot(\partial F/\partial x_1,(\partial F/\partial x_2))}{\partial F/\partial x_2}\\&=\left(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\right)\cdot\left(\frac{\partial F/\partial x_1}{\partial F/\partial x_2},1\right)\\&=\left(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\right)\cdot\left(\frac{\partial f_\pm}{\partial x_1},1\right),\\\frac{\psi(x_1,f_\pm(x,t),t)}{\partial x_1}&=\frac{\partial\psi}{\partial x_1}+\frac{\partial\psi}{\partial x_2}\frac{\partial f_\pm}{\partial x_1}\\&=\left(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\right)\cdot\left(\frac{\partial f_\pm}{\partial x_1},1\right),\end{aligned} \tag{87}$$

where $(,)\cdot(,)$ denotes the inner product or dot product, i.e., $(a,b)\cdot(c,d)=ac+bd$, (a,b,c,d are scalar functions). Therefore

$$\frac{\partial f_\pm}{\partial t}=\frac{\psi(x_1,f_\pm(x,t),t)}{\partial x_1}. \tag{88}$$

Utilizing this formula, we can obtain an explicit equation for the change of time of the area between two fixed points a, b and two level curves (f_-, f_+),

$$\begin{aligned}&\frac{d}{dt}\left[\int_a^b (f_+(x_1,t)-f_-(x_1,t))dx_1\right]\\&\quad=\psi(b,f_+(b,t),t)-\psi(a,f_+(a,t),t)\\&\qquad+\psi(a,f_-(a,t),t)-\psi(b,f_-(b,t),t).\end{aligned} \tag{89}$$

Suppose that two-level curves f_+ and f_- collapse when t tends to T^* uniformly in $a\le x_1\le b$, i.e.

$$M(t)=f_+(x,t)-f_-(x,t)\sim\frac{1}{b-a}\int_a^b (f_+(x_1,t)-f_-(x_1,t))dx_1, \tag{90}$$

That is to say, the distance between two level sets are comparable for $a\le x_1\le b$. Denote

$$\delta(x_1,t)=\max_{x_1\in[a,b]}|f_+(x_1,t)-f_-(x_1,t)|, \tag{91}$$

be the thickness of the front and call the length $b-a$ of the interval $[a,b]$, the length of the front.

One assumption is that

$$\min_{x_1\in[a,b]}(f_+(x_1,t)-f_-(x_1,t))>c_1\cdot\delta(x_1,t), \tag{92}$$

for all $a\le x_1\le b$ and all $t\in[0,T^*)$. By the previous results of [11], we know that if (92) holds, then, we say $F_\pm(t)$ form a semiuniform front.

Proof of Theorem 11. By (89) and (90), we have

$$\begin{aligned}&\left|\frac{d}{dt}\frac{1}{b-a}\left[\int_a^b (f_+(x_1,t)-f_-(x_1,t))dx_1\right]\right.\\&\quad=\frac{1}{b-a}|\psi(b,f_+(b,t),t)-\psi(a,f_+(a,t),t)\\&\qquad+\psi(a,f_-(a,t),t)-\psi(b,f_-(b,t),t)|\\&\quad\le\frac{2}{b-a}\sup_{a\le x_1\le b}[\psi(x_1,f_+(x_1,t),t)-\psi(x_1,f_-(x_1,t),t)].\end{aligned} \tag{93}$$

□

By virtue of (20) in Lemma 8 (where p and q are different point in $[a,b]$), we take $p=f_+(x_1,t)$ and $q=f_-(x_1,t)$, and we have

$$\begin{aligned}&\left|\frac{d}{dt}\frac{1}{b-a}\left[\int_a^b (f_+(x_1,t)-f_-(x_1,t))dx_1\right]\right|\\&\quad\le\frac{C_1}{b-a}\sup_{a\le x_1\le b}|f_+(x_1,t)-f_-(x_1,t)|.\end{aligned} \tag{94}$$

That is to say

$$\left|\frac{d}{dt}M(t)\right|\le\frac{C_1}{b-a}|M(t)|. \tag{95}$$

Therefore, by Gronwall's inequality, (90) and (91), we have $\delta(t)>e^{-(C_1t+C_2)}$, and this allows us to rule out the formation of sharp fronts, and this yields (8) and concludes the proof.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest with others.

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