Research Article

On the Characterization of Antineutrosophic Subgroup

Sudipta Gayen,1 S. A. Edalatpanah,2 Sripati Jha,3 and Ranjan Kumar4

1Centre for Data Science, Faculty of Engineering & Technology, Siksha ‘O’ Anusandhan (Deemed to be University), Odisha, India
2Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran
3Department of Mathematics, National Institute of Technology Jamshedpur, Jharkhand, India
4School of Advanced Sciences, VIT-AP University, Amaravati AP, India

Correspondence should be addressed to Ranjan Kumar; ranjank.nit52@gmail.com

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This article gives some essential scopes to study the characterizations of the antineutrosophic subgroup and antineutrosophic normal subgroup. Again, several theories and properties have been mentioned which are essential for analyzing their mathematical framework. Moreover, their homomorphic properties have been discussed.

1. Introduction

Fuzzy set (FS) [1] theory was introduced to handle uncertain situations more precisely than crisp sets. But there may exist some complex uncertain situations for which even FS theory is insufficient. As a result, intuitionistic fuzzy set (IFS) [2] and neutrosophic set (NS) [3] theories evolved, where the latter is more capable of dealing with uncertainties. Apart from these, there exist several byproducts of these set theories, like interval-valued versions [4–6]; type-I, type-II, and type-III versions; and soft [7–9] and hard versions. Presently, these theories have been adopted by several researchers in different applied fields. Also, in several pure mathematical fields, these notions are being utilized. In abstract algebra, Rosenfeld [10] was the pioneer to do so. He defined and studied the characteristics of a fuzzy subgroup (FSG). Thereafter, Das [11] presented the concept of the level subgroup of a FSG and showed several beautiful relationships between them. Afterward, Anthony and Sherwood [12, 13] redefined FSG by applying general T-norms and defined function generated FSG and subgroup generated FSG. In 1984, Mukherjee and Bhattacharya [14] introduced normal versions of FSG and cosets. Furthermore, Biswas [15] established the concept of intuitionistic fuzzy subgroup (IFSG). Similarly, Çetkin and Aygün [16] developed the neutrosophic subgroup (NSG) and studied its homomorphic properties. They have also established some connections between an NSG and its level subgroup.

The concept of the antifuzzy subgroup (AFSG) [17] is a kind of dual to FSG. It was defined and characterized by Biswas in 1990. He has mentioned some relationships between FSG and AFSG and studied several other properties. Similarly, there is notion of the intuitionistic antifuzzy subgroup (IAFSG) [18], which was developed by Li et al. in 2009. They have also studied its homomorphic properties and established some connections with its intuitionistic fuzzy counterpart. Table 1 contains some contributions of various researchers involving different antialgebraic notions under uncertainty.

Hence, it is obvious that antiversions of FSG, IFSG, etc. have been adopted by different researchers for the anticipation of unique and impactful results. In neutrosophic group theory, so far, authors have discussed NSGs and some of their algebraic structures. But still, the antineutrosophic subgroup (ANSG) is undefined. Also, the relationship between NSG and ANSG are still unexplored. Hence, this can be a fruitful area which can generate some scope of future research. Based on the aforementioned gaps, the objectives of this paper are as follows:

(i) to introduce ANSG and investigate its algebraic features

(ii) to define the antineutrosophic normal subgroup (ANNSG) and explore its algebraic characteristics
degrees.

relative to the membership and nonmembership

Definition 4 (see [11]). Let \( \psi \) be a FS of \( V \). Then, the set \( \psi_t = \{ r \in V : \psi(r) \geq t \} \forall t \in [0,1] \) is denoted as a level subset of \( \psi \).

Definition 5 (see [17]). Let \( \varphi \) be a FS of \( V \). Then, the set \( \varphi_t = \{ r \in V : \varphi(r) \leq t \} \forall t \in [0,1] \) is denoted as a lower level subset of \( \varphi \).

Next, the notions of FSG, IFSG, NSG, and a few of their essential properties are addressed.

Table 1: Desk research of different antialgebraic notions.

<table>
<thead>
<tr>
<th>Author &amp; references</th>
<th>Year</th>
<th>Contributions in various fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kim et al. [19]</td>
<td>2005</td>
<td>Introduced the concept of antifuzzy ideals of near-rings and investigated some of its properties.</td>
</tr>
<tr>
<td>Feng &amp; Yao [20]</td>
<td>2012</td>
<td>Introduced (( \lambda, \mu ))-antifuzzy subgroups and studied its properties.</td>
</tr>
<tr>
<td>Kausar [21]</td>
<td>2019</td>
<td>Introduced intuitionistic fuzzy normal subrings and intuitionistic anti fuzzy normal subrings over a nonassociative ring and studied their properties.</td>
</tr>
<tr>
<td>Ejegwa et al. [22]</td>
<td>2021</td>
<td>Studied antifuzzy multigroup and its characteristics.</td>
</tr>
<tr>
<td>Hoskova-Mayerova &amp; Al Tahan [23]</td>
<td>2021</td>
<td>Introduced different operations on fuzzy multi-ideals of near-rings and defined antifuzzy multisubnear-rings of near-rings and study their properties.</td>
</tr>
<tr>
<td>Ahmad et al. [24]</td>
<td>2021</td>
<td>Defined kernel subgroup of a FSG and AFSG and presented several results involving them.</td>
</tr>
<tr>
<td>Kalaiarasi et al. [25]</td>
<td>2022</td>
<td>Studied the properties of ( \gamma )-antifuzzy normal subgroup and ( \gamma )-fuzzy normal subgroup and presented their application in gene mutation.</td>
</tr>
<tr>
<td>Hemabala &amp; Kumar [26]</td>
<td>2022</td>
<td>Introduced and analyzed anti neutrosophic multifuzzy ideals of ( \gamma ) near-ring and studied their product.</td>
</tr>
</tbody>
</table>

(iii) to figure out the relationships between NSG and ANSG

(iv) to study several homomorphic attributes of ANSG and ANNSG

This article has been structured in the following manner. In Section 2, desk research of FSG, IFSG, and NSG and their normal versions are given. Also, antiversions of FSG and IFSG are discussed. In Section 3, the notions of ANSG and normal versions are given. Also, antiversions of FSG and IFSG are discussed which are required for our current study. Here, some elementary set theories under uncertainties are discussed which are required for our current study.

2. Preliminaries

Here, some elementary set theories under uncertainties are discussed which are required for our current study.

Definition 1 (see [1]). A FS \( \lambda \) of a crisp set \( V \) is defined as

\[ \lambda : V \rightarrow [0,1] \]

Definition 2 (see [2]). An IFS \( \gamma \) of a crisp set \( V \) is defined as

\[ \gamma = \{ (r, t_\gamma(r), f_\gamma(r)) : r \in V \} \]

where \( t_\gamma \) and \( f_\gamma \) are, respectively, known as the membership and nonmembership degrees.

Definition 3 (see [3]). A NS \( \eta \) of a crisp set \( V \) is defined as

\[ \eta = \{ (r, t_\eta(r), i_\eta(r), f_\eta(r)) : r \in V \} \]

where \( t_\eta, i_\eta \), and \( f_\eta \) are, respectively, known as the truth, indeterminacy, and falsity degrees.

Definition 4 (see [11]). Let \( \psi \) be a FS of \( V \). Then, the set \( \psi_t = \{ r \in V : \psi(r) \geq t \} \forall t \in [0,1] \) is denoted as a level subset of \( \psi \).

Definition 5 (see [17]). Let \( \varphi \) be a FS of \( V \). Then, the set \( \varphi_t = \{ r \in V : \varphi(r) \leq t \} \forall t \in [0,1] \) is denoted as a lower level subset of \( \varphi \).

2.1. Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Subgroup

Definition 6 (see [10]). For a classical group \( V \), a FS \( \psi \) is denoted as a FSG iff \( \forall m, r \in V \), the subsequent conditions are fulfilled:

(i) \( \psi(m \cdot r) \geq \min \{ \psi(m), \psi(r) \} \)

(ii) \( \psi(r^{-1}) \geq \psi(r) \)

Theorem 7 (see [10]). \( \psi \) is a FSG of \( V \) iff \( \forall m, r \in V \)

\[ \psi(mr^{-1}) \leq \min \{ \psi(m), \psi(r) \} \]

Proposition 8 (see [10]). Homomorphic image and preimage of a FSG is a FSG.

Theorem 9 (see [11]). Let \( V \) be a classical group and \( \psi \in \text{FSG}(V) \), then \( \forall t \in [0,1] \) with \( \psi(e) \geq t \), \( \psi_t \) are classical subgroups of \( V \).

Theorem 10 (see [11]). Let \( V \) be a classical group and \( \forall t \in [0,1] \) with \( \psi(e) \geq t \), \( \psi_t \) are classical subgroups of \( V \), then \( \psi \) is a FSG of \( V \).

Definition 11 (see [11]). Let \( \psi \) be a FSG of a classical group \( V \). Then, \( \forall t \in [0,1] \) and \( \psi(e) \geq t \) the subgroups \( \psi_t \) are termed as level subgroups of \( \psi \).

Definition 12 (see [15]). For a classical group \( V \), an IFS \( \gamma = \{ (r, t_\gamma(r), f_\gamma(r)) : r \in V \} \) is denoted an IFSG iff \( \forall m, r \in V \),

(i) \( t_\gamma(m \cdot r) \geq \min \{ t_\gamma(m), t_\gamma(r) \} \)

(ii) \( t_\gamma(r^{-1}) \geq t_\gamma(r) \)

(iii) \( f_\gamma(m \cdot r) \leq \max \{ f_\gamma(m), f_\gamma(r) \} \)

(iv) \( f_\gamma(r^{-1}) \leq f_\gamma(r) \)
**Proposition 13** (see [15]). For a classical group \( V \), an IFS \( \gamma = \{(m, t_\gamma(m), f_\gamma(m))\}: m \in V \) is an IFSG iff \( \forall m, r \in V \)

(i) \( t_\gamma(m \cdot r^{-1}) \geq \min \{ t_\gamma(m), t_\gamma(r) \} \)

(ii) \( f_\gamma(m \cdot r^{-1}) \leq \max \{ f_\gamma(m), f_\gamma(r) \} \)

**Theorem 14** (see [27]). Let \( V \) and \( R \) be two classical groups and \( l: V \rightarrow R \) be a homomorphism. Also, let \( \gamma \in \text{IFSG}(V) \) and \( \gamma' \in \text{IFSG}(R) \). Then,

(i) If \( \gamma \) has the supremum property, then \( l(\gamma) \in \text{IFSG}(R) \)

(ii) \( \Gamma^{-1}(\gamma') \in \text{IFSG}(V) \)

**Definition 15** (see [27]). Let \( \gamma \) be an IFS of \( V \) and let \( s_1, s_2 \in [0, 1] \) with \( s_1 + s_2 \leq 1 \). Then, the set \( \gamma_{(s_1,s_2)} = \{ m \in V : t_\gamma(m) \geq s_1 \text{ and } f_\gamma(m) \leq s_2 \} \) is known as \( (s_1,s_2) \)-level set of \( \gamma \).

**Theorem 16** (see [27]). Let \( V \) be a classical group and \( \gamma \in \text{IFSG}(V) \). Then, \( \forall s_1, s_2 \in [0, 1] \) with \( t_\gamma(e) \geq s_1 \) and \( f_\gamma(e) \leq s_2 \), \( \gamma_{(s_1,s_2)} \) are classical subgroups of \( V \).

**Theorem 17** (see [27]). Let \( V \) be a classical group and \( \forall s_1, s_2 \in [0, 1] \) with \( t_\gamma(e) \geq s_1 \) and \( f_\gamma(e) \leq s_2 \), \( \gamma_{(s_1,s_2)} \) are classical subgroups of \( V \). Then, \( \gamma \in \text{IFSG}(V) \).

**Definition 18** (see [16]). For a classical group \( V \), a NS \( \delta \) is defined as an NSG of \( V \) iff the subsequent terms are fulfilled:

(i) \( \delta(m \cdot r) \geq \min \{ \delta(m), \delta(r) \} \), i.e., \( t_\delta(m \cdot r) \geq \min \{ t_\delta(m), t_\delta(r) \} \), \( i_\delta(m \cdot r) \geq \min \{ i_\delta(m), i_\delta(r) \} \) and \( f_\delta(m \cdot r) \leq \max \{ f_\delta(m), f_\delta(r) \} \)

(ii) \( \delta(m^{-1}) \geq \delta(m) \), i.e., \( t_\delta(m^{-1}) \geq t_\delta(r) \), \( i_\delta(m^{-1}) \geq i_\delta(r) \), and \( f_\delta(m^{-1}) \leq f_\delta(r) \)

A set of all the NSGs will be signified as NSG(\( V \)). Here, note that \( t_\delta \) and \( i_\delta \) are following Definition 6, i.e., they are FSGs of \( V \) whereas, \( f_\delta \) is following Definition 24, i.e., it is an AFS of \( V \).

**Theorem 19** (see [16]). For a classical group \( V \delta \in \text{NSG}(V) \) iff \( \forall m, r \in V \)

\[ \delta(m \cdot r^{-1}) \geq \min \{ \delta(m), \delta(r) \}, \]

\[ i.e., t_\delta(m \cdot r^{-1}) \geq \min \{ t_\delta(m), t_\delta(r) \}, i_\delta(m \cdot r^{-1}) \geq \min \{ i_\delta(m), i_\delta(r) \}, \text{ and } f_\delta(m \cdot r^{-1}) \leq \max \{ f_\delta(m), f_\delta(r) \}. \]

**Theorem 20** (see [16]). \( \delta \in \text{NSG}(V) \) iff the p-level sets \( (t_\delta)_p, (i_\delta)_p \) and p-lower level set \( (f_\delta)_p \) are classical subgroups of \( V \forall p \in [0, 1] \).

**Theorem 21** (see [16]). Homomorphic image and preimage of any NSG is a NSG.

**Definition 22** (see [16]). For a classical group \( V \), a neutrosophic \( \delta \) is called an NNSG of \( V \) iff \( \forall m, r \in V \)

\[ \delta(m \cdot r \cdot m^{-1}) \leq \delta(r), \]

i.e., \( t_\delta(m \cdot r \cdot m^{-1}) \leq t_\delta(r), i_\delta(m \cdot r \cdot m^{-1}) \leq i_\delta(r) \), and \( f_\delta(m \cdot r \cdot m^{-1}) \geq f_\delta(r) \).

The set of all NNSG of \( V \) will be signified as NNSG(\( V \)). Also, notice that \( \eta \in \text{NNSG}(V) \) implies that \( t_\delta \) and \( i_\delta \) are fuzzy normal subgroups (FNSG) of \( V \) and \( f_\delta \) is the antifuzzy normal subgroup (AFNSG) of \( V \).

**Theorem 23** (see [16]). Homomorphic image and preimage of any NSG is a NSG.

In the next segment, the notions of AFSG and IAIFSG are discussed.

2.2 Antifuzzy Subgroup and Intuitionistic Antifuzzy Subgroup

**Definition 24** (see [17]). For a classical group \( V \), a FS \( \varphi \) is denoted as an AFSG of \( V \) if \( \forall m, r \in V \), the subsequent terms are fulfilled:

(i) \( \varphi(m \cdot r) \leq \max \{ \varphi(m), \varphi(r) \} \)

(ii) \( \varphi(r^{-1}) \leq \varphi(r) \)

**Theorem 25** (see [17]). \( \varphi \) is an AFSG of \( V \) if \( \forall m, r \in V \)

\[ \varphi(m \cdot r^{-1}) \leq \max \{ \varphi(m), \varphi(r) \}. \]

**Proposition 26** (see [17]). \( \varphi \) is a FSG of the group \( V \) iff its complement \( \varphi^c \) is an AFSG of \( V \).

**Definition 27** (see [17]). Let \( \varphi \) be an AFSG of a group \( V \). Then, \( \forall t \in [0, 1] \) such that \( t \geq \mu(\varphi) \), the \( \varphi \)-subgroups \( \varphi_t \) are called lower-level subgroups of \( \varphi \).

**Proposition 28** (see [17]). Let \( \varphi \) be an AFSG of \( V \). Then, \( \forall t \in [0, 1] \) such that \( t \geq \mu(\varphi) \), the \( \varphi \)-subgroups \( \varphi_t \) are classical subgroups of \( V \).

**Proposition 29** (see [17]). Let \( \varphi \) be a FS of a classical group \( V \) such that \( \varphi_t \) is a classical subgroup of \( V \forall t \in [0, 1] \) with \( t \geq \mu(\varphi) \). Then, \( \mu \) is an AFSG of \( V \).

**Definition 30** (see [28]). For a classical group \( V \), an IFS \( \gamma = \{(m, t_\gamma(m), f_\gamma(m)): m \in V \} \) is called an IAIFSG of \( V \) iff \( \forall m, r \in V \)

(i) \( t_\gamma(m \cdot r^{-1}) \leq \max \{ t_\gamma(m), t_\gamma(r) \} \)

(ii) \( f_\gamma(m \cdot r^{-1}) \geq \max \{ f_\gamma(m), f_\gamma(r) \} \)

**Proposition 31** (see [28]). \( \gamma \) is an IAIFSG of the group \( V \) iff its complement \( \gamma^c \) is an IAIFSG of \( V \).
Theorem 32 (see [28]). \( \gamma \in \text{IFSG}(V) \) iff \( \forall s_1, s_2 \in [0, I] \) with \( t_\gamma(e) \geq s_1 \) and \( f_\gamma(e) \leq s_2, (s_1, s_2) \)-level set of \( \gamma \), i.e., \( \gamma(s_1,s_2) \) are classical subgroups of \( V \).

Theorem 33 (see [18]). Homomorphic image and preimage of any IAIFSG is an IAIFSG.

In the following section, the notions of ANSG and ANNSG have been introduced and some of their fundamental properties are discussed.

3. Antineutrosophic Subgroup

Definition 34. For a classical group \( V \), a neutrosophic set \( \eta \) is called an ANSG of \( V \) iff the following terms are fulfilled:

(i) \( \eta(m \cdot r) \leq \max \{ \eta(m), \eta(r) \} \), i.e., \( t_\eta(m \cdot r) \leq \max \{ t_\eta(m), \eta(r) \} \), \( i_\eta(m \cdot r) \leq \max \{ i_\eta(m), i_\eta(r) \} \), and \( f_\eta(m \cdot r) \geq \min \{ f_\eta(m), f_\eta(r) \} \)

(ii) \( \eta(r^{-1}) \leq \eta(r) \), i.e., \( t_\eta(r^{-1}) \leq t_\eta(r) \), \( i_\eta(r^{-1}) \leq i_\eta(r) \), and \( f_\eta(r^{-1}) \geq f_\eta(r) \)

The set of all ANSGs will be signified as ANSG(\( V \))

Proposition 35. \( \eta \in \text{ANSG}(V) \) iff \( t_\eta \) and \( i_\eta \) are AFSGs of \( V \) and \( f_\eta \) is FSG of \( V \).

Proof. Let \( \eta \in \text{ANSG}(V) \) then from Definition 34, it is evident that \( t_\eta \) and \( i_\eta \) are following Definition 24, i.e., they are AFSGs of \( V \). Whereas \( f_\eta \) is following Definition 6, i.e., it is a FSG of \( V \). Again, if \( t_\eta \) and \( i_\eta \) are AFSGs of \( V \) and \( f_\eta \) is a FSG of \( V \) then \( \eta \in \text{ANSG}(V) \). \( \square \)

Example 36. Let \( V = \{ 1, i, -i, -1 \} \) be a classical group of order 4 and \( \eta \) be a neutrosophic set of \( V \), where the memberships of truth \( t_\eta \), indeterminacy \( i_\eta \), and falsity \( f_\eta \) of elements in \( \eta \) are given in Figure 1.

Notice that \( t_\eta \) and \( i_\eta \) are following Definition 24, i.e., are AFSGs of \( V \). Again, \( f_\eta \) is following Definition 6, i.e., is a FSG of \( V \). Hence, \( \eta \) is an ANSG of \( V \).

Example 37. Let \( V = \{ a, e \} \) be a classical group of order 2 and \( \eta \) be a NS of \( V \), where considering \( \theta \in [\pi/4, \pi/2] \), let \( \eta = \{ (a, \sin \theta/2, \sin \theta/4, \sin \theta + \cos \theta/2), (e, \cos \theta/2, \cos \theta/4, \sin \theta + \cos \theta/2) \} \). In Figures 2 and 3, memberships of \( a \) and \( e \) have been described graphically.

Here, \( \eta \) is following Definition 34 and hence it is an ANSG.

Theorem 38. Let \( \eta \in \text{ANSG}(V) \) where \( V \) is a classical group.

Then, \( \forall r \in V \)

(i) \( \eta(r^{-1}) = \eta(r) \)

(ii) \( \eta(e) \leq \eta(r) \), where \( e \) is the neutral element of \( V \)

Proof. Here, \( f_\eta \) is a FSG and both \( t_\eta \) and \( i_\eta \) are AFSGs of \( V \), by Definition 6. So, \( f_\eta(r) = f_\eta(r^{-1}) \leq f_\eta(r^{-1}) \) and hence \( f_\eta(r^{-1}) = f_\eta(r) \). Again, from Definition 24, \( t_\eta(r^{-1}) \leq t_\eta(r) \). So, \( t_\eta(r^{-1}) = t_\eta(r^{-1}) \leq t_\eta(r^{-1}) \) and hence \( t_\eta(r^{-1}) = t_\eta(r) \). Similarly, using Definition 24, we can prove \( i_\eta(r^{-1}) = i_\eta(r) \). So, \( \eta(r^{-1}) = \eta(r) \)

(ii) Using Definition 6, we have \( f_\eta(r) = f_\eta(r \cdot r^{-1}) \geq \max \{ f_\eta(r), f_\eta(r^{-1}) \} \).

Similarly, using Definition 24, we have

\[ i_\eta(r) = i_\eta(r \cdot r^{-1}) \leq \max \{ i_\eta(r), i_\eta(r^{-1}) \} \]

Hence, \( \eta(e) \leq \eta(r) \)

\( \square \)

Theorem 39. \( \eta \in \text{ANSG}(V) \) iff \( \forall m, r \in V \)

\( \eta(m \cdot r^{-1}) \leq \max \{ \eta(m), \eta(r) \} \).

Proof. Let \( \eta \in \text{ANSG}(V) \). Then, by Definition 34, we have \( \eta(m \cdot r^{-1}) \leq \max \{ \eta(m), \eta(r^{-1}) \} \).

Again, by Definition 34

\( \eta(r^{-1}) = \eta(r) \)

Conversely, let \( \eta(m \cdot r^{-1}) \leq \max \{ \eta(m), \eta(r^{-1}) \} \). So,

\[ t_\eta(m \cdot r^{-1}) \leq \max \{ t_\eta(m), t_\eta(r) \}, \]

\[ i_\eta(m \cdot r^{-1}) \leq \max \{ i_\eta(m), i_\eta(r) \}, \]

\[ f_\eta(m \cdot r^{-1}) \geq \max \{ f_\eta(m), f_\eta(r) \}. \]

Notice that,

\[ t_\eta(r^{-1}) = t_\eta(e \cdot r^{-1}) \leq \max \{ t_\eta(e), t_\eta(r) \} = \max \{ t_\eta(r^{-1}), t_\eta(r) \} \leq \max \{ t_\eta(r), t_\eta(r), t_\eta(r) \} = t_\eta(r). \]

Similarly, \( i_\eta(r^{-1}) \leq i_\eta(r) \) and \( f_\eta(r^{-1}) \geq f_\eta(r) \).

Again,

\[ t_\eta(m \cdot r) = t_\eta(m \cdot (r^{-1})^{-1}) \leq \max \{ i_\eta(m), i_\eta(r^{-1}) \} \]

\[ \leq \max \{ i_\eta(m), i_\eta(r) \}. \]
min \{f_\eta(m), f_\eta(r)\} can be proved. Hence, \(\eta\) satisfies Definition 34, i.e., \(\eta \in \text{ANSG}(V)\).

**Theorem 40.** \(\eta \in \text{ANSG}(V)\) iff \(\eta^c \in \text{NSG}(V)\).

**Proof.** If we take the complement of \(\eta\), i.e., \(\eta^c\) then corresponding degree of truth and degree of falsity will interchange their positions in \(\eta^c\). Also, the degree of indeterminacy will have its complement, i.e., \(\tilde{\eta}^c = 1 - \tilde{\eta}\). In other words, if

\[
\eta = \left\{ (r, t_\eta(r), i_\eta(r), f_\eta(r)) : r \in V \right\} \quad \text{then} \quad \eta^c = \left\{ (r, f_\eta(r), \tilde{\eta}^c(r), t_\eta(r)) : r \in V \right\}.
\]

(7)

Let \(\eta \in \text{ANSG}(V)\) then by Proposition 35 \(t_\eta\) and \(i_\eta\) are AFSGs of \(V\) and \(f_\eta\) is FSG of \(V\). So, in case of \(\eta, f_\eta\) and \(\tilde{\eta}\)
Proof. Let $\eta \in \text{ANSG}(V)$, $p \in [0, 1]$ and $m, r \in (i_p)_\eta$. Then, $t_\eta(m) \leq p$ and $t_\eta(r) \leq p$. Since $\eta \in \text{ANSG}(V)$, we have $t_\eta(m \cdot r^{-1}) \leq \max \{t_\eta(m), t_\eta(r)\} \leq p$ and hence $m \cdot r^{-1} \in (i_p)_\eta$. Similarly, it can be shown that $m \cdot r^{-1} \in (i_p)_\eta$ and $m \cdot r^{-1} \in (f_\eta)_p$. So, $(i_p)_\eta$, $(f_\eta)_p$, and $(f_\eta)_p$ are classical subgroups of $V$.

Conversely, let $\forall p \in [0, 1](i_p)_\eta$ is a classical subgroup of $V$. Let $m, r \in V$ such that $t_\eta(m) = p_1$ and $t_\eta(r) = p_2$ for some $p_1, p_2 \in [0, 1]$. Then, $m \in (i_p)_{p_1}$ and $r \in (i_p)_{p_2}$.

**Example 41.** Let $(\mathbb{Z}_4, +)$ be the group of integers modulo 4 with usual addition and $\eta = \{(r, t_\eta(r), i_\eta(r), f_\eta(r)) : r \in \mathbb{Z}_4\}$ is a NS of $\mathbb{Z}_4$, where $t_\eta, i_\eta$ and $f_\eta$ are mentioned in Table 2.

According to Definition 34, $\eta$ is an ANSG of $\mathbb{Z}_4$.

Now $\eta' = \{(r, t_{\eta'}(r), i_{\eta'}(r), f_{\eta'}(r)) : r \in \mathbb{Z}_4\}$, where $t_{\eta'}, i_{\eta'},$ and $f_{\eta'}$ are mentioned in Table 3.

Here, according to Definition 18, $\eta'$ is a NSG of $\mathbb{Z}_4$.

**Theorem 42.** $\eta \in \text{ANSG}(V)$ iff the $p$-lower level sets $(i_p)_\eta$, $(f_\eta)_p$, and $p$-level set $(f_\eta)_p$ are classical subgroups of $V$ $\forall p \in [0, 1]$.

**Theorem 43.** Intersection of any two ANSG of any group is an ANSG.
Proof. Let $\eta_1, \eta_2 \in \text{ANSG}(V)$. To prove this, using Theorem 39, we can show that

\[
(\eta_1 \cap \eta_2) \cdot (m_1 \cdot r_1) \leq \max \{ (\eta_1 \cap \eta_2)(m), (\eta_1 \cap \eta_2)(r) \}, \quad \text{i.e.,}
\]

\[
t_{\eta_1 \cap \eta_2}(m \cdot r_1) \leq \max \left\{ t_{\eta_1}(m), t_{\eta_2}(m) \right\},
\]

\[
i_{\eta_1 \cap \eta_2}(m \cdot r_1) \leq \max \left\{ i_{\eta_1}(m), i_{\eta_2}(m) \right\},
\]

\[
f_{\eta_1 \cap \eta_2}(m \cdot r_1) \geq \min \left\{ f_{\eta_1}(m), f_{\eta_2}(m) \right\}.
\]

(8)

Here,

\[
t_{\eta_1 \cap \eta_2}(m \cdot r_1) \leq \max \left\{ t_{\eta_1}(m \cdot r_1), t_{\eta_2}(m \cdot r_1) \right\}
\]

\[
\leq \max \left\{ \max \left\{ t_{\eta_1}(m), t_{\eta_2}(r) \right\}, \max \left\{ t_{\eta_1}(m), t_{\eta_2}(r) \right\} \right\}
\]

\[
= \max \left\{ \max \left\{ t_{\eta_1}(m), t_{\eta_2}(m) \right\}, \max \left\{ t_{\eta_1}(r), t_{\eta_2}(r) \right\} \right\}
\]

\[
= \max \left\{ t_{\eta_1}(m), t_{\eta_2}(r) \right\}.
\]

(9)

Similarly, we can show that

\[
i_{\eta_1 \cap \eta_2}(m \cdot r_1) \leq \max \left\{ i_{\eta_1}(m), i_{\eta_2}(r) \right\}.
\]

(10)

Again,

\[
f_{\eta_1 \cap \eta_2}(m \cdot r_1) \geq \min \left\{ f_{\eta_1}(m \cdot r_1), f_{\eta_2}(m \cdot r_1) \right\}
\]

\[
\geq \min \left\{ \min \left\{ f_{\eta_1}(m), f_{\eta_2}(r) \right\}, \min \left\{ f_{\eta_1}(m), f_{\eta_2}(r) \right\} \right\}
\]

\[
= \min \left\{ \min \left\{ f_{\eta_1}(m), f_{\eta_2}(m) \right\}, \min \left\{ f_{\eta_1}(r), f_{\eta_2}(r) \right\} \right\}
\]

\[
= \min \left\{ f_{\eta_1}(m), f_{\eta_2}(r) \right\}.
\]

(11)

Hence, $\eta_1 \cap \eta_2 \in \text{ANSG}(V)$. □

Theorem 44. Homomorphic image of any ANSG is an ANSG.

Proof. Let $U_1$ and $U_2$ be two classical groups and $s : U_1 \rightarrow U_2$ be a homomorphism. Let $\eta \in \text{ANSG}(U_1)$. Then, $\forall m_1, m_2 \in U_1$, we have

\[
t_s(m_1 \cdot m_2^{-1}) = \min \left\{ t_s(m_1), t_s(m_2) \right\},
\]

\[
i_s(m_1 \cdot m_2^{-1}) = \min \left\{ i_s(m_1), i_s(m_2) \right\},
\]

\[
f_s(m_1 \cdot m_2^{-1}) \geq \min \left\{ f_s(m_1), f_s(m_2) \right\}.
\]

(12)

Here, we have to show that $s(\eta)$ is an ANSG of $U_2$. Let $\exists n_1, n_2 \in U_2$ such that $n_1 = s(m_1)$ and $n_2 = s(m_2)$.

Now, as $s$ is a group homomorphism, we have

\[
s(t_s)(n_1 \cdot n_2^{-1}) = \min_{m \in \text{ANSG}(n_1)} t_s(m) \leq t_s(m_1 \cdot m_2^{-1})
\]

\[
\leq \max \{ t_s(m_1), t_s(m_2) \}.
\]

(13)

Again, $s(t_s)(n_1) = \min \{ t_s(m_1), t_s(m_2) \}$.

Where-from max $s(t_s)(n_1) = t_s(m_1)$ and hence,

\[
s(t_s)(n_1 \cdot n_2^{-1}) \leq \min \{ t_s(m_1), t_s(m_2) \}
\]

\[
= \max \{ \max s(t_s)(n_1), \max s(t_s)(n_2) \}
\]

\[
= \max \{ s(t_s)(n_1), s(t_s)(n_2) \}.
\]

(14)

Similarly, it can be shown that $s(i_s)(n_1 \cdot n_2^{-1}) \leq \max \{ s(i_s)(n_1), s(i_s)(n_2) \}$.

Also,

\[
s(f_s)(n_1 \cdot n_2^{-1}) = \max_{m \in \text{ANSG}(n_1)} f_s(m) \geq f_s(m_1 \cdot m_2^{-1})
\]

\[
\geq \min \{ f_s(m_1), f_s(m_2) \}.
\]

(15)

Again $s(f_s)(n_1) = \max f_s(m) \geq t_s(m_1)$. Where-from min $s(f_s)(n_1) = f_s(m_1)$ and hence

\[
s(f_s)(n_1 \cdot n_2^{-1}) \geq \min \{ f_s(m_1), f_s(m_2) \}
\]

\[
= \min \{ \min s(f_s)(n_1), \min s(f_s)(n_2) \}
\]

\[
= \min \{ s(f_s)(n_1), s(f_s)(n_2) \}.
\]

(16)

So, $s(\eta)$ is an ANSG of $U_2$. □

Theorem 45. Homomorphic preimage of any ANSG is an ANSG.

Proof. Let $U_1$ and $U_2$ be two classical groups and $s : U_1 \rightarrow U_2$ be a homomorphism. Let $\delta \in \text{ANSG}(U_2)$. Then, $\forall n_1, n_2 \in U_2$, we have

\[
t_s(n_1 \cdot n_2^{-1}) \leq \max \{ t_s(n_1), t_s(n_2) \},
\]

\[
i_s(n_1 \cdot n_2^{-1}) \leq \max \{ i_s(n_1), i_s(n_2) \},
\]

\[
f_s(n_1 \cdot n_2^{-1}) \geq \min \{ f_s(n_1), f_s(n_2) \}.
\]

(17)

Here, we have to show that $s^{-1}(\delta)$ is an ANSG of $U_1$. □
Let \( m_1, m_2 \in U_1 \). Since \( s \) is a group homomorphism,
\[
s^{-1}(t_{\delta})(m_1 \cdot m_2) = t_{\delta}(s(m_1) \cdot s(m_2)) = t_{\delta}(s(m_1), t_{\delta}(s(m_2))) = \max \{ t_{\delta}(s(m_1)), s^{-1}(t_{\delta}(m_2)) \}.
\]
(18)

Similarly, we can show that
\[
s^{-1}(i_{\delta})(m_1 \cdot m_2) \leq \max \{ s^{-1}(i_{\delta}(m_1)), s^{-1}(i_{\delta}(m_2)) \},
\]
\[
s^{-1}(f_{\delta})(m_1 \cdot m_2) \geq \min \{ s^{-1}(f_{\delta}(m_1)), s^{-1}(f_{\delta}(m_2)) \}.
\]
(19)

Hence, \( s^{-1}(\delta) \) is an ANSG of \( U_1 \).

**Theorem 46.** Let \( \eta \in \text{ANSG}(V) \) and \( l \) be a homomorphism on \( V \). Let \( \eta^{-1} : V \rightarrow [0, 1] \times [0, 1] \) be defined as \( \eta^{-1}(r) = \eta^{-1}(r^{-1}) \) for any \( r \in V \) then \( \eta^{-1} \in \text{ANSG}(V) \) and \( (l(\eta)^{-1}) = l(\eta^{-1}). \)

**Proof.**

\[
\eta^{-1}(m \cdot r^{-1}) = \eta(m \cdot r^{-1}) = \eta(r^{-1} \cdot m^{-1}) = \eta(r \cdot m^{-1}) \leq \max \{ \eta(r), \eta(m^{-1}) \} = \max \{ \eta^{-1}(r), \eta^{-1}(m) \} [\text{as } \eta \text{ is an ANSG}]
\]
\[
= \max \{ \eta^{-1}(m), \eta^{-1}(r) \}.
\]
(20)

Hence, by Theorem 39, \( \eta^{-1} \in \text{ANSG}(V) \).

**Example 49.** Let \( V = \{ e, m, r, mr \} \) be the Klein's 4-group and \( \eta = \{ (r, r_\eta(r), i_\eta(r), f_{\eta}(r)) : r \in V \} \) is a NS of \( V \), where \( t_{\eta}, i_\eta \), and \( f_\eta \) are mentioned in Table 4.

Here, \( \eta \) follows Definition 48, i.e., it is an ANNSG.

**Proposition 50.** \( \eta \in \text{ANSG}(V) \) iff \( t_\eta \) and \( i_\eta \) are AFNSs of \( V \) and \( f_\eta \) is FNS of \( V \).

**Proof.** Using Definition 48, this can be observed.

**Theorem 51.** Intersection of any two ANSG of any group is an ANSG.

**Proof.** Using Theorem 43, this can be proved.

**Theorem 52.** Let \( \eta \in \text{ANSG}(V) \). Then, the subsequent conditions are equivalent:

(i) \( \eta \in \text{ANNSG}(U) \)

(ii) \( \eta(m \cdot r \cdot m^{-1}) = \eta(r), \forall m, r \in V \)

(iii) \( \eta(m \cdot r) = \eta(V \cdot m), \forall m, r \in V \)

**Proof.** Let (i) be true. Then, by Definition 48, we have \( \eta(m \cdot r \cdot m^{-1}) \leq \eta(r) \), i.e., \( t_\eta(m \cdot r \cdot m^{-1}) \leq t_\eta(r), i_\eta(m \cdot r \cdot m^{-1}) \leq i_\eta(r) \), and \( f_\eta(m \cdot r \cdot m^{-1}) \geq f_\eta(r) \).

To prove (ii), we need to show
\[
t_\eta(m \cdot r \cdot m^{-1}) \geq t_\eta(r),
\]
\[
i_\eta(m \cdot r \cdot m^{-1}) \geq i_\eta(r),
\]
(23)

and
\[
f_\eta(m \cdot r \cdot m^{-1}) \leq f_\eta(r).
\]

In other words, we need to prove
\[
t_\eta(m \cdot r \cdot m^{-1}) = t_\eta(r),
\]
\[
i_\eta(m \cdot r \cdot m^{-1}) = i_\eta(r),
\]
(24)

and
\[
f_\eta(m \cdot r \cdot m^{-1}) = f_\eta(r).
\]
Table 4: Membership values of elements belonging to $\eta$.

<table>
<thead>
<tr>
<th></th>
<th>$t_\eta$</th>
<th>$i_\eta$</th>
<th>$f_\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>0.1</td>
<td>0.5</td>
<td>0.9</td>
</tr>
<tr>
<td>$a$</td>
<td>0.3</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$b$</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>$ab$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Notice that

$$t_\eta(m^{-1} \cdot r \cdot m) = t_\eta(m^{-1} \cdot r \cdot (m^{-1})^{-1}) \leq t_\eta(r). \quad (25)$$

Again,

$$t_\eta(r) = t_\eta(m^{-1} \cdot (m \cdot r \cdot m^{-1}) \cdot m) \leq t_\eta(m \cdot r \cdot m^{-1}). \quad (26)$$

Hence, $t_\eta(m \cdot r \cdot m^{-1}) = t_\eta(r)$.

Similarly, it can be shown that $i_\eta(m \cdot r \cdot m^{-1}) = i_\eta(r)$ and $f_\eta(m \cdot r \cdot m^{-1}) = f_\eta(r)$. Hence (i) $\Rightarrow$ (ii).

Let condition (ii) be true. In (ii), substituting $r$ in place of $r \cdot m^{-1}$ (iii) can easily be proved. So, (ii) $\Rightarrow$ (iii).

Let condition (iii) be true. Applying $\eta(m \cdot r) = \eta(r \cdot m)$ in $t_\eta(m \cdot r \cdot m^{-1})$, we have

$$t_\eta(m \cdot r \cdot m^{-1}) = t_\eta(r \cdot m^{-1} \cdot m) = t_\eta(r) \leq t_\eta(r). \quad (27)$$

So, (iii) $\Rightarrow$ (i).

Theorem 53. $\eta \in \text{ANNSG}(V)$ iff the $p$-lower level sets $(t_\eta)_p$, $(i_\eta)_p$, and $p$-level set $(f_\eta)_p$ are classical normal subgroups of $V \forall p \in [0, 1]$.

Proof. Using Theorem 42, this can be proved.

Theorem 54. Let $\eta \in \text{ANNSG}(V)$. The set $U_\eta = \{ m \in V : \eta(m) = \eta(e) \}$ is a classical normal subgroup of $V$, where $e$ is the identity element of $V$.

Proof. Since $\eta \in \text{ANNSG}(V)$, we have $\eta \in \text{ANSG}(V)$. Let $m, r \in U_\eta$, then by Theorem 39

$$\eta(m \cdot r^{-1}) \leq \max \{ \eta(m), \eta(r) \} = \max \{ \eta(e), \eta(e) \} = \eta(e). \quad (28)$$

Again, by Theorem 38, we have $\eta(m \cdot r^{-1}) \geq \eta(e)$ and hence $\eta(m \cdot r^{-1}) = \eta(e)$, i.e., $m \cdot r^{-1} \in U_\eta$. Since $\eta \in \text{ANSG}(V)$, we have

$$\eta(m \cdot r \cdot m^{-1}) = \eta(r \cdot m \cdot m^{-1}) = \eta(r) = \eta(e), \quad (29)$$

i.e., $m \cdot r \cdot m^{-1} \in U_\eta$ or $U_\eta$ is a normal subgroup of $V$.

Theorem 55. Let $\eta \in \text{ANNSG}(V)$ and $l$ be a homomorphism on $V$. Then, the homomorphic pre-image of $\eta$, i.e., $l^{-1}(\eta) \in \text{ANNSG}(V)$.

Proof. Using Theorem 44, we have $l^{-1}(\eta) \in \text{ANNSG}(V)$. Then, by Proposition 50, we can easily prove normality of $l^{-1}(\eta)$. Hence, $l^{-1}(\eta) \in \text{ANNSG}(V)$.

Theorem 56. Let $\eta \in \text{ANNSG}(V)$ and $l$ be a surjective homomorphism on $V$. Then the homomorphic image of $\eta$, i.e., $l(\eta) \in \text{ANNSG}(V)$.

Proof. Using Theorem 44, we have $l(\eta) \in \text{ANNSG}(V)$. Again, by Proposition 50, the normality condition can easily be proved. So, $l(\eta) \in \text{ANNSG}(V)$.

4. Conclusion

The studies of ANSG and its normal version might open some new directions of research. Here, homomorphism has been introduced in ANSG and ANNSG to understand their algebraic characteristics. Moreover, connections with their nonantiversions are provided. For these, numerous examples, theories, and propositions are given. In the future, these studies can be further extended by introducing various notions like the antineutrosophic ideal, antineutrosophic ring, antineutrosophic field, and antineutrosophic topological space. Furthermore, their interval-valued versions can be introduced and studied.

Data Availability

This work is a contribution towards the theoretical development of fuzzy algebra and its generalizations. The data that support the findings of this study are not publicly available due to the fact that they were created specifically for this study. We have not used any additional data set for drafting this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


