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Research Article

An Approximation Method for Variational Inequality with Uncertain Variables

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In this paper, a Stieltjes integral approximation method for uncertain variational inequality problem (UVIP) is studied. Firstly, uncertain variables are introduced on the basis of variational inequality. Since the uncertain variables are based on nonadditive measures, there is usually no density function. Secondly, the expected value model of UVIP is established after the expected value is discretized by the Stieltjes integral. Furthermore, a gap function is constructed to transform UVIP into an uncertain constraint optimization problem, and the optimal value of the constraint problem is proved to be the solution of UVIP. Finally, the convergence of solutions of the Stieltjes integral discretization approximation problem is proved.

1. Introduction

VIP is a significant branch of inequality and a classical problem in mathematics, which has attracted many scholars. Through the unremitting efforts of many mathematicians, VIP has developed into an important subject with rich content and broad prospects in mathematical programming. These achievements involve rich mathematical theories, optimization theory, economics and engineering (see [1–7]), and so on. For the classical VIP, $\forall v \in S$, there is a point $u \in S \in R^n$ such that

$$(u-v)^T f(u) \geq 0, \tag{1}$$

where $S \neq \varnothing$ is closed convex and $f: S \longrightarrow R^n$ is a vector-valued function. Chen and Fukushima [8] presented the regularized gap function as follows:

$$\mathscr{R}(u) := \max_{v \in S}\left\{(u-v)^T f(u) - \frac{\gamma}{2}\|u-v\|_G^2\right\}, \tag{2}$$

where matrix G is symmetric and positive definite square and parameter $\gamma > 0$. $\|\cdot\|_G$ indicates the G-norm, which is given by $\|u\|_G = \sqrt{u^T G u}$, $u \in R^n$. It means $\mathscr{R}(u) \geq 0$, $\forall u \in S$, and $\mathscr{R}(u) = 0$ iff u is a solution of VIP (f, S). On the basis of these theories, we convert the VIP (1) into an optimization problem as follows:

$$\min_{u \in S} \mathscr{R}(u). \tag{3}$$

Generally, the minimization problem (3) does not involve uncertainties. However, it is just an ideal situation. All of these characteristics may lead to the uncertainty. Therefore, many researchers have systematically studied variational inequalities with random variables. That is,

$$\left(u-u^{\dagger}\right)^T f\left(u^{\dagger}, \omega\right) \geq 0, \quad \forall u \in S, \omega \in \Omega, \tag{4}$$

where Ω is a stochastic sample space and the mapping $f: R^n \times \Omega \longrightarrow R^n$. Due to the randomness of the function f, there is generally no solution to problem (4). By calculating expected value $\mathbb{E}[f(u^\dagger,\omega)]$ over ω, problem (4) is transformed into as follows:

$$\left(u-u^\dagger\right)^T\mathbb{E}\left[f\left(u^\dagger,\omega\right)\right]\geq 0. \quad (5)$$

This problem is widely used in economics, management, and operations research. It was investigated in references such as [9–11]. Based on probability theory, the SVIP in literature [8] is studied. It is well known that probability is based on repeated tests, so it must have a large number of historical sample data to estimate probability. But in most conditions, it is hard to model a probability distribution due to the nonrepeatability of events, such as unprecedented sudden natural disasters, crisis management and emergency of acute infectious diseases, and so on. Liu [12] created uncertainty theory, which is based on nonadditive measure, to deal with these uncertain phenomena.

In the past few years, uncertainty theory has become a very fruitful subject. At the same time, many successful applications have been made at home and abroad (see [12–22]). Chen and Zhu [23] introduced the uncertain variable into the VIP and established the uncertain variational inequality problem (UVIP). They constructed the expected value model to solve the UVIP as follows:

$$\left(u-u^\dagger\right)^T\mathbb{E}\left[f\left(u^\dagger,\zeta\right)\right]\geq 0,\quad \forall u\in S,\zeta\in\Xi, \quad (6)$$

where Ξ is the set of uncertain variables and the mapping $f: R^n\times\Xi\longrightarrow R^n$.

Based on uncertainty theory, an approximation problem on UVIP is studied in this paper. It is clear that SVIP and UVIP are both natural generalizations of deterministic variational inequalities. Other contents of this paper are as follows. The second section reviews the basic concepts and properties of some uncertainty theories, including uncertain variables and uncertain expectations. In Section 3, research on the convergence of the approximation problem generated by the Stieltjes integral discrete approximation method (SDA for short) will be finished. Finally, a conclusion summarizes and prospects the future research work.

2. Preliminaries

In this section, we will give some definitions and lemmas. Firstly, we collect the concepts and properties in uncertainty space. Supposed that Γ is a nonempty set and $\mathscr{L}$ is a σ-algebra over Γ. Then, $(\Gamma,\mathscr{L})$ is called a measurable space; each element Λ in Γ is called an event. So $\mathscr{M}\{\Lambda\}\in R$ presents the belief degree that Λ occurs. So $(\Gamma,\mathscr{L},\mathscr{M})$ is an uncertainty space, which is defined by Ξ. To deal with belief degrees rightly, Liu [12] presented three axioms as follows:

(1) $\mathscr{M}\{\Gamma\}=1$

(2) $\mathscr{M}\{\Lambda\}+\mathscr{M}\{\Lambda^c\}=1$

(3) $\mathscr{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\leq\sum_{i=1}^{\infty}\mathscr{M}\{\Lambda_i\}$, where $\Lambda_1,\Lambda_2,\cdots$ are sequence of events

Definition 1 (see [12]). Let $\zeta\in\Xi$. If the following exists,

$$\mathbb{E}[\zeta]=\int_0^{+\infty}\mathscr{M}\{\zeta\geq u\}\mathrm{d}u-\int_{-\infty}^{0}\mathscr{M}\{\zeta\leq u\}\mathrm{d}u, \quad (7)$$

then $\mathbb{E}[\zeta]$ is the expected value of uncertain variable ζ.

Theorem 2 (see [12]). *Let $\zeta\in\Xi$ and Φ be the uncertainty distribution of ζ. If $\mathbb{E}[\zeta]$ exists, then*

$$\mathbb{E}[\zeta]=\int_{-\infty}^{+\infty}td\Phi. \quad (8)$$

Theorem 3 (see [12]). *Let $\zeta\in\Xi$ and Φ be the uncertainty distribution of ζ. If $\mathbb{E}[\zeta]$ exists,*

$$\mathbb{E}[F(\zeta)]=\int F(t)d\Phi(t). \quad (9)$$

3. SDA Method and Its Convergence

In this section, we will provide the convergence of SDA method and regularized gap functions $\mathscr{R}(u,\zeta)$ on the set S on the basis of uncertainty theory. It turns out that for $\gamma>0$, there is an optimal solution for problem (1). Therefore, we can find a fixed point $u^\dagger\in S\subset R^n$ such that

$$\left(u-u^\dagger\right)^T\mathbb{E}\left[f\left(u^\dagger,\zeta\right)\right]\geq 0,\quad \forall u\in S,\zeta\in\Xi, \quad (10)$$

where $f: R^n\times\Xi\longrightarrow R^n$ and Ξ is an uncertain space. Furthermore, we present the regularized gap function

$$\mathscr{R}(u,\zeta)\coloneqq\max_{v\in S}\left\{(u-v)^T\mathbb{E}[f(u,\zeta)]-\frac{\gamma}{2}\|u-v\|_G^2\right\}, \quad (11)$$

where parameter $\gamma>0$ and matrix G is positive definite and symmetric. Now, we can convert (10) into an optimization problem as follows:

$$\min_{u\in S}\mathscr{R}(u,\zeta). \quad (12)$$

In this section, in order to solve problem (12), we will propose a Stieltjes integral discrete approximation method (abbreviated as SDA), and the convergence of the method is studied. In most cases, there is no density function in the uncertain distribution. Then, it is difficult to calculate the uncertain expectation directly, so we use the Stieltjes integral to calculate. The distribution function is discretized before that, and we introduced the following definitions.

Definition 4 (division of interval by the Stieltjes integral [24]). Let $f(x)$ be a bounded function on the interval $[a,b]$ and $\kappa(x)$ be a bounded variation function on $[a,b]$, and make a division of interval $T: a=x_0<x_1<\cdots<x_n=b$ and

a group of "intermediate points," $x_{i-1} \le \xi \le x_i (i = 1, 2, \cdots, n)$, and make a sum:

$$\sum_{i=1}^{n} f(\xi_i)[\kappa(x_i) - \kappa(x_{i-1})]. \quad (13)$$

Set $\delta(T) = \max_{0 \le i \le n} |x_i - x_{i-1}|$. When $\delta(T) \longrightarrow 0$, the sum tends to a certain finite limit; then, $f(x)$ is said to be $R-S$ integrable about $\kappa(x)$ on the interval $[a, b]$. This limit is recorded as $\int_a^b f(x)dx$.

From the division of interval by the Stieltjes integral (10), we have $\forall \Delta\Phi_i = \Phi_i - \Phi_{i-1}, \exists \zeta_i, \text{s.t.} \Phi(\zeta_i) \in \Delta\Phi_i, i = 1, 2, \cdots$; the expectation of $f(u, \zeta)$ is

$$\begin{aligned} \mathbb{E}[f(u,\zeta)] &= \int f(u,t)d\Phi(t) \\ &= \sum_{i=1}^{\infty} f(u,\zeta_i)\Delta\Phi_i(\zeta_i), \quad u \in S, \zeta \in \Xi. \end{aligned} \quad (14)$$

According to the arbitrariness of $\Delta\Phi$, let $\Delta\Phi_i = \Phi_i(t_i) - \Phi_{i-1}(t_{i-1}) = 1/N$ and $\zeta_i \in (t_{i-1}, t_i)$; then,

$$\begin{aligned} \mathbb{E}[f(u,\zeta)] &= \int f(u,t)d\Phi(t) = \sum_{i=1}^{\infty} f(u,\zeta_i)\Delta\Phi_i(\zeta_i) \\ &= \lim_{N\longrightarrow\infty} \frac{1}{N}\sum_{i=1}^{N} f(u,\zeta_i) = \lim_{N\longrightarrow\infty} \mathbb{E}[f_N(u,\zeta)]. \end{aligned} \quad (15)$$

Therefore, we have the discrete approximation of (12) as follows:

$$\begin{aligned} &\min \mathscr{R}_N(u,\zeta) \\ &= \max_{v\in S}\left\{(u-v)^T\left[\frac{1}{N}\sum_{i=1}^{N} f(u,\zeta_i)\right] - \frac{\gamma}{2}\|u-v\|_G^2\right\} \quad \text{s.t.} \quad u \in S. \end{aligned} \quad (16)$$

Definition 5 (see [1]). Let $G \in R^{n\times n}$ be a symmetric positive definitive matrix and S be a convex subset of R^n. $\Theta_{S,G}(u)$ is a solution set of the following optimization model:

$$\min_y \|u-v\|_G^2 = \min_y (u-v)^T G(u-v), \quad \text{s.t.} \quad v \in S, \quad (17)$$

where the operator $\Theta_{S,G} : R^n \longrightarrow S$ is a skewed projection mapping for fixed $u \in R^n$.

Definition 6. In addition, we made the following assumptions in this section:

(1) S is a nonempty and compact set of R^n

(2) There exists a function $\varphi(\zeta)$ which is integrable and

$$\sup_{u\in S} \|f(u,\zeta)\| \le \phi(\zeta), \quad \zeta \in \Xi. \quad (18)$$

Suppose that (1) and (2) hold, we call $f(u, \zeta)$ as ϕ-bounded function.

The following theorem will provide the uniform convergence of the approximate problem (12).

Theorem 7. *Suppose that $f(u, \zeta)$ is ϕ-bounded function on S, $\forall \zeta \in \Xi$, it is continuous with respect to u. Then, we have*

(a) $\mathbb{E}[f(u, \zeta)]$ is finite and continuous

(b) $\mathbb{E}[f_N(u, \zeta)]$ uniformly converges to $\mathbb{E}[f(u, \zeta)]$ and

$$\lim_{N\longrightarrow\infty} \max_{u\in S} \|\mathbb{E}[f_N(u,\zeta)] - \mathbb{E}[f(u,\zeta)]\| = 0. \quad (19)$$

(c) $\{\mathscr{R}_N(u, \zeta)\}$ uniformly converges to $\mathscr{R}(u, \zeta)$ and

$$\lim_{N\longrightarrow\infty} \max_{u\in S} \|\mathscr{R}_N(u,\zeta) - \mathscr{R}(u,\zeta)\| = 0. \quad (20)$$

Proof.

(a) Since $f(u, \zeta)$ is continuous on S, $\forall \varepsilon > 0$, and $\exists \delta > 0$, when $|u - u_0| < \delta$, it holds

$$|f(u,\zeta) - f(u_0,\zeta)| < \varepsilon. \quad (21)$$

Then, we have

$$\begin{aligned} |\mathbb{E}[f(u,\zeta)] - \mathbb{E}[f(u_0,\zeta)]| &= \left|\int f(u,t)d\Phi(t) - \int f(u_0,t)d\Phi(t)\right| \\ &= \int |f(u,t) - f(u_0,t)|d\Phi(t) \\ &< \varepsilon \int 1 d\Phi(t). \end{aligned} \quad (22)$$

$\phi(\zeta)$ is an integrable function, so it is monotonous, and the range of the function is between zero and one. Therefore, it is bounded; it means that $\mathbb{E}[f(u, \zeta)]$ is continuous. From Definition 6, $f(u, \zeta)$ is ϕ-bounded function; we have

$$\mathbb{E}[f(u,\zeta)] = \int f(u,t)d\Phi(t) \le \int \phi(t)d\Phi(t). \quad (23)$$

Since $\phi(\zeta)$ is integrable, we have $\int \phi(t)d\Phi(t)$ which is finite. Therefore, (a) is hold.

(b) From equation (15), it can be seen that

$$\begin{aligned} \mathbb{E}[f(u,\zeta)] &= \int f(u,t)d\Phi(t) = \sum_{i=1}^{\infty} f(u,\zeta_i)\Delta\Phi_i(t) \\ &= \lim_{N\longrightarrow\infty} \frac{1}{N}\sum_{i=1}^{N} f(u,\zeta_i) = \lim_{N\longrightarrow\infty} \mathbb{E}[f_N(u,\zeta)], \end{aligned} \quad (24)$$

and it means that $\forall \varepsilon > 0$, $\exists N_0 > 0$, when $N > N_0$; it holds

$$|\mathbb{E}[f_N(u,\zeta)] - \mathbb{E}[f(u,\zeta)]| < \varepsilon. \tag{25}$$

From the fact that u is arbitrary, so

$$\max_{u\in S}|\mathbb{E}[f_N(u,\zeta)] - \mathbb{E}[f(u,\zeta)]| < \varepsilon. \tag{26}$$

From the fact that ε is arbitrary, $\mathbb{E}[f_N(u,\zeta)]$ uniformly converges to $\mathbb{E}[f(u,\zeta)]$, that is,

$$\lim_{N\longrightarrow\infty} \max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)] - \mathbb{E}[f(u,\zeta)]\| = 0. \tag{27}$$

(c) It follows from Li et al. [25] that the problem $\max_{v\in S}\{(u-v)^T\mathbb{E}[f(u,\zeta)] - (\gamma/2)\|u-v\|_G^2\}$ is essentially equal to the problem $\min_{v\in S}\|v-(u-\gamma^{-1}G^{-1}\mathbb{E}[f(u,\zeta)])\|_G^2$. So it is easy to have that $\forall u \in R^n$, $\forall \zeta \in \Xi$, and

$$\begin{aligned}\mathscr{R}(u,\zeta) &= (u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] \\ &\quad - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2, \\ \mathscr{R}_N(u,\zeta) &= (u-\mathscr{P}_N(u,\zeta))^T\mathbb{E}[f_N(u,\zeta)] \\ &\quad - \frac{\gamma}{2}\|u-\mathscr{P}_N(u,\zeta)\|_G^2,\end{aligned} \tag{28}$$

where

$$\mathscr{P}(u,\zeta) = \Theta_{S,G}\left(u-\gamma^{-1}G^{-1}\mathbb{E}[f(u,\zeta)]\right), \tag{29}$$

$$\mathscr{P}_N(u,\zeta) = \Theta_{S,G}\left(u-\gamma^{-1}G^{-1}\mathbb{E}[f_N(u,\zeta)]\right), \tag{30}$$

$$\mathbb{E}[f_N(u,\zeta)] = \frac{1}{N}\Sigma_{i=1}^N f(u,\zeta_i). \tag{31}$$

Let $\mathscr{R}(u,\zeta)$: $R^n \times \Xi \longrightarrow R^n > 0$ be a function defined by (11). $\forall u \in S$, $\zeta \in \Xi$, and $\mathscr{R}(u,\zeta) = 0$ iff u is a solution of FVIP (f,S). Therefore, u is a solution of (16) iff it solves (10), so

$$\begin{aligned}&|\mathscr{R}(u,\zeta) - \mathscr{R}_N(u,\zeta)| \\ &= \left|(u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2 \right.\\ &\quad \left. - \left\{(u-\mathscr{P}_N(u,\zeta))^T\mathbb{E}[f_N(u,\zeta)] - \frac{\gamma}{2}\|u-\mathscr{P}_N(u,\zeta)\|_G^2\right\}\right| \\ &= \left|(u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2 \right.\\ &\quad \left. - (u-\mathscr{P}_N(u,\zeta))^T\mathbb{E}[f_N(u,\zeta)] + \frac{\gamma}{2}\|u-\mathscr{P}_N(u,\zeta)\|_G^2\right| \\ &\le \left|(u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] - (u-\mathscr{P}_N(u,\zeta))^T\mathbb{E}[f_N(u,\zeta)]\right| \\ &\quad + \left|\frac{\gamma}{2}\|u-\mathscr{P}_N(u,\zeta)\|_G^2 - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2\right| \\ &= \left|(u-\mathscr{P}(u,\zeta))^T(\mathbb{E}[f(u,\zeta)] - \mathbb{E}[f_N(u,\zeta)]) \right.\\ &\quad \left. + (\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta))^T\mathbb{E}[f_N(u,\zeta)]\right| \\ &\quad + \left|\frac{\gamma}{2}\|u-\mathscr{P}_N(u,\zeta)\|_G^2 - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2\right| \\ &\le \|u-\mathscr{P}(u,\zeta\| \cdot \|\mathbb{E}[f(u,\zeta)] - \mathbb{E}[f_N(u,\zeta)]\| \\ &\quad + \|\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta)\| \cdot \|\mathbb{E}[f_N(u,\zeta)]\| \\ &\quad + \left|\frac{\gamma}{2}\|u-\mathscr{P}_N(u,\zeta)\|_G^2 - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2\right|.\end{aligned} \tag{32}$$

Since $\mathscr{R}(u,\zeta) \ge 0$, we have

$$\begin{aligned}&(u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] \\ &\quad - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2 \ge 0, \\ &\frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2 \le (u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] \\ &\qquad \le \|u-\mathscr{P}(u,\zeta\| \cdot \|\mathbb{E}[f(u,\zeta)]\|.\end{aligned} \tag{33}$$

Denote the smallest eigenvalue of G by $\lambda_{\min}$. Note that

$$\begin{aligned}&\sqrt{\lambda_{\min}}\|u\| \le \|u\|_G, \\ &\|u-\mathscr{P}(u,\zeta)\| \cdot \|\mathbb{E}[f(u,\zeta)]\| \le \frac{1}{\sqrt{\lambda_{\min}}}\|u-\mathscr{P}(u,\zeta)\|_G \\ &\qquad \cdot \|\mathbb{E}[f(u,\zeta)]\|.\end{aligned} \tag{34}$$

Further, we can conclude that

$$\|u-\mathscr{P}(u,\zeta)\| \le \frac{\|u-\mathscr{P}(u,\zeta)\|_G}{\sqrt{\lambda_{\min}}} \le \frac{2\|\mathbb{E}[f(u,\zeta)]\|}{\gamma\sqrt{\lambda_{\min}}}. \tag{35}$$

On account that S is nonempty and compact, so $\exists K > 0$, it holds

$$\begin{aligned}\|\mathbb{E}[f(u,\zeta)]\| &< K, \\ \|\mathbb{E}[f_N(u,\zeta)]\| &< K.\end{aligned} \tag{36}$$

Furthermore, we can conclude

$$\begin{aligned}\|u-\mathscr{P}(u,\zeta)\| &< \frac{2K}{\gamma\sqrt{\lambda_{\min}}}, \\ \|u-\mathscr{P}_N(u,\zeta)\| &< \frac{2K}{\gamma\sqrt{\lambda_{\min}}}.\end{aligned} \tag{37}$$

Moreover, from the nonexpansive property of the projection operator, it holds

$$\begin{aligned}\|\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta)\| &= \left\|\Theta_{S,G}\left(u-\gamma^{-1}G^{-1}\mathbb{E}[f_N(u,\zeta)]\right) \right.\\ &\quad \left. - \Theta_{S,G}\left(u-\gamma^{-1}G^{-1}\mathbb{E}[f(u,\zeta)]\right)\right\|_G \\ &\le \left\|\gamma^{-1}G^{-1}\mathbb{E}[f_N(u,\zeta)] - \gamma^{-1}G^{-1}\mathbb{E}[f(u,\zeta)]\right\|_G \\ &\le \gamma^{-1}\left\|G^{-1}\right\| \cdot \|\mathbb{E}[f_N(u,\zeta)] - \mathbb{E}[f(u,\zeta)]\|.\end{aligned} \tag{38}$$

Then, we can get

$$\begin{aligned}
&\left|\|u-\mathscr{P}_N(u,\zeta)\|_G^2-\|u-\mathscr{P}(u,\zeta)\|_G^2\right| \\
&\quad=\left|(u-\mathscr{P}_N(u,\zeta))^T G(u-\mathscr{P}_N(u,\zeta))\right. \\
&\qquad\left.-(u-\mathscr{P}(u,\zeta))^T G(u-\mathscr{P}(u,\zeta))\right| \\
&\quad=\left|(\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta))^T G(\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta))\right| \\
&\quad\le\|(\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta))\|\cdot\|G\|\cdot\|(\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta))\| \\
&\quad=\|(\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta))\|^2\cdot\|G\| \\
&\quad\le\gamma^{-2}\cdot\left\|G^{-1}\right\|\cdot\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2.
\end{aligned} \tag{39}$$

From (a) and (b), $\mathbb{E}[f_N(u,\zeta)]$ uniformly converges to $\mathbb{E}[f(u,\zeta)]$. So $\forall\delta>0$, when $N>N_0$, $\exists N_0$ such that

$$\begin{aligned}
&\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|\le\frac{\delta}{2\gamma^{-1}K\left(\left(2/\sqrt{\lambda_{\min}}\right)+\left\|G^{-1}\right\|\right)}, \\
&\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2\le\frac{\delta}{\gamma^{-1}\left\|G^{-1}\right\|}.
\end{aligned} \tag{40}$$

From $\|u-\mathscr{P}(u,\zeta)\|<(2/\gamma\sqrt{\lambda_{\min}})K$ and $\|u-\mathscr{P}_N(u,\zeta)\|<(2/\gamma\sqrt{\lambda_{\min}})K$, $\|\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta)\|\le\gamma^{-2}\cdot\|G^{-1}\|\cdot\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2$, $|\|u-\mathscr{P}_N(u,\zeta)\|_G^2-\|u-\mathscr{P}(u,\zeta)\|_G^2|\le\gamma^{-2}\cdot\|G^{-1}\|\cdot\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2$, and $\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2\le(\delta/(\gamma^{-1}\|G^{-1}\|))$, we can get

$$\begin{aligned}
&\max_{u\in S}|\mathscr{R}(u,\zeta)-\mathscr{R}_N(u,\zeta)| \\
&\quad\le\max_{u\in S}\|u-\mathscr{P}(u,\zeta)\|\cdot\max_{u\in S}\|(\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)])\| \\
&\qquad+\max_{u\in S}\|\mathscr{P}_N(u,\zeta)-\mathscr{P}(u,\zeta)\|\cdot\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]\| \\
&\qquad+\frac{\gamma}{2}\max_{u\in S}\left|\|u-\mathscr{P}_N(u,\zeta)\|_G^2-\|u-\mathscr{P}(u,\zeta)\|_G^2\right| \\
&\quad\le\frac{2}{\gamma\sqrt{\lambda_{\min}}}K\cdot\max_{u\in S}\|(\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)])\| \\
&\qquad+\gamma^{-1}\left\|G^{-1}\right\|\cdot\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|\cdot K \\
&\qquad+\frac{\gamma}{2}\cdot\gamma^{-2}\cdot\left\|G^{-1}\right\|\cdot\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2 \\
&\quad=\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\| \\
&\qquad\cdot\left\{\frac{2}{\gamma\sqrt{\lambda_{\min}}}K+\gamma^{-1}\left\|G^{-1}\right\|\cdot K\right\} \\
&\qquad+\frac{1}{2\gamma}\cdot\left\|G^{-1}\right\|\max_{u\in S}\|\mathbb{E}[f_N(u,\zeta)]-\mathbb{E}[f(u,\zeta)]\|^2 \\
&\quad\le\frac{\delta}{2\gamma^{-1}K\left(\left(2/\sqrt{\lambda_{\min}}\right)+\left\|G^{-1}\right\|\right)} \\
&\qquad\cdot\left\{\frac{2}{\gamma\sqrt{\lambda_{\min}}}K+\gamma^{-1}\left\|G^{-1}\right\|\cdot K\right\} \\
&\qquad+\frac{1}{2\gamma}\cdot\left\|G^{-1}\right\|\cdot\frac{\delta}{\gamma^{-1}\left\|G^{-1}\right\|}=\delta.
\end{aligned} \tag{41}$$

Then,

$$\lim_{N\longrightarrow\infty}\max_{u\in S}\|\mathscr{R}_N(u,\zeta)-\mathscr{R}(u,\zeta)\|=0. \tag{42}$$

That is, $\mathscr{R}_N(u,\zeta)$ uniformly converges to $\mathscr{R}(u,\zeta)$. □

Since the condition of uniform convergence is strong, there will be inevitable mistakes in the calculation process. Here, we weaken the condition of the function and then prove it.

Definition 8 (see [26]). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence and the function f be lower semicontinuous. $\{f_n\}$ epiconverges to f:

(i) $\forall\{u_n\}$ s.t. $\lim_{n\longrightarrow\infty}u_n=u$, there holds $\liminf_{n\longrightarrow\infty}f_n(u_n)\ge f(u)$, $\forall u$

(ii) $\exists\{v_n\}$ s.t. $\lim_{n\longrightarrow\infty}v_n=v$, there holds $\limsup_{n\longrightarrow\infty}f_n(v_n)\le f(v)$,$\forall v$

Lemma 9. *Assume that $f(u,\zeta)$ is ϕ-bounded function and function of the sequence $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverges to the function $\mathbb{E}[f(u,\zeta)]$. Then, $\mathscr{P}_N(u_N,\zeta)$ approaches to $\mathscr{P}(u,\zeta)$.*

Proof. To prove $\mathscr{P}_N(u_N,\zeta)$ approaches to $\mathscr{P}(u,\zeta)$, we will prove the following:

(a) $\forall\{u_N\}$ s.t. $\lim_{N\longrightarrow\infty}u_N=u$, $\forall\delta>0,\exists N^*>0,N>N^*$, then

$$\mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta)\ge-\delta. \tag{43}$$

(b) $\exists\{v_N\}$ s.t. $\lim_{N\longrightarrow\infty}v_N=v$, $\forall\delta>0,\exists N^*>0,N>N^*$, then

$$\mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta)\le\delta. \tag{44}$$

Firstly, we prove (a). Recall that

$$\begin{aligned}
\mathscr{R}_N(u_N,\zeta)&=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-\frac{\gamma}{2}\|u_N-v\|_G^2, \\
\mathscr{R}(u,\zeta)&=(u-v)^T\mathbb{E}[f(u,\zeta)]-\frac{\gamma}{2}\|u-v\|_G^2.
\end{aligned} \tag{45}$$

By [27] and (29), we can get that $\mathscr{P}_N(u_N,\zeta)$ is the unique optimal solution of $\min_{u\in S}\mathscr{R}_N(u_N,\zeta)$; and $\mathscr{P}(u,\zeta)$ is the only optimal solution of $\min_{u\in S}\mathscr{R}(u,\zeta)$. So we have

$$\begin{aligned}
&\mathscr{R}_N(u_N,\zeta)-\mathscr{R}(u,\zeta)\\
&\quad=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-\frac{\gamma}{2}\|u_N-v\|_G^2\\
&\qquad-\left\{(u-v)^T\mathbb{E}[f(u,\zeta)]-\frac{\gamma}{2}\|u-v\|_G^2\right\}\\
&\quad=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-(u-v)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{\|u_N-v\|_G^2-\|u-v\|_G^2\right\}\\
&\quad=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-(u-v)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-v)^TG(u_N-v)-(u-v)^TG(u-v)\right\}\\
&\quad=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-(u-v)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-u-v+y)^TG(u_N-u-v+y)\right.\\
&\quad=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-(u-v)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-u)^TG(u_N-u)\right\}.
\end{aligned}\tag{46}$$

Because $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverges to the function $\mathbb{E}[f(u,\zeta)]$ if for any u, $\forall\{u_N\}$ s.t. $\lim_{n\longrightarrow\infty}u_N=u$. So for any $\varepsilon>0,\exists N_0>0,N>N_0$, s.t. $\liminf_{N\longrightarrow\infty}\mathbb{E}[f_N(u_N,\zeta)]\geq\mathbb{E}[f(u,\zeta)]$, that is,

$$\begin{aligned}
\mathbb{E}[f_N(u_N,\zeta)]&\geq\mathbb{E}[f(u,\zeta)]-\varepsilon,\\
\mathbb{E}[f_N(u_N,\zeta)]\mathbb{E}[f(u,\zeta)]&\geq-\varepsilon.
\end{aligned}\tag{47}$$

Then,

$$\begin{aligned}
&\mathscr{R}_N(u_N,\zeta)-\mathscr{R}(u,\zeta)\\
&\quad=(u_N-v)^T\mathbb{E}[f_N(u_N,\zeta)]-(u_N-v)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad+(u_N-v)^T\mathbb{E}[f(u,\zeta)]-(u-v)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-u)^TG(u_N-u)\right\}\\
&\quad=(u_N-v)^T\{\mathbb{E}[f_N(u_N,\zeta)]-\mathbb{E}[f(u,\zeta)]\}\\
&\qquad+(u_N-u-v+y)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-u)^TG(u_N-u)\right\}\\
&\quad\geq\left|(u_N-v)^T\right|(-\varepsilon)+(u_N-u)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-u)^TG(u_N-u)\right\}.
\end{aligned}\tag{48}$$

From $\forall\{u_N\}$ s.t. $\lim_{n\longrightarrow\infty}u_N=u$, so for any $\varepsilon>0,\exists N_1>0$, $N>N_1$, we have

$$u_N-u>-\varepsilon.\tag{49}$$

Then, for any $\varepsilon>0,\exists N_2\in\max\{N_0,N_1\},N>N_2$, we have

$$\begin{aligned}
&\mathscr{R}_N(u_N,\zeta)-\mathscr{R}(u,\zeta)\\
&\quad\geq\left|(u_N-v)^T\right|(-\varepsilon)+(u_N-u)^T\mathbb{E}[f(u,\zeta)]\\
&\qquad-\frac{\gamma}{2}\left\{(u_N-u)^TG(u_N-u)\right\}\\
&\quad\geq-\varepsilon\left|(u_N-v)^T\right|-\varepsilon|\mathbb{E}[f(u,\zeta)]|-\varepsilon^2\frac{\gamma}{2}\|G\|.
\end{aligned}\tag{50}$$

By $u_N\in S,v\in S,\gamma>0$, so $(u_N-v)^T,(\gamma/2)\|G|$ are finite, and $\mathbb{E}[f(u,\zeta)]$ is finite; then, $\mathscr{R}_N(u_N,\zeta)$ approaches $\mathscr{R}(u,\zeta)$. And $\mathscr{P}_N(u_N,\zeta)$ is the only optimal solution to $\min_{u\in S}\mathscr{R}_N(u_N,\zeta)$; and $\mathscr{P}(u,\zeta)$ is the only optimal solution to $\min_{u\in S}\mathscr{R}(u,\zeta)$. So $\mathscr{P}_N(u_N,\zeta)$ approaches to $\mathscr{P}(u,\zeta)$; that is, for any $\delta>0$, $\exists N_3\in\max\{N_0,N_1\},N>N_3$, we have

$$\mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta)\geq-\delta.\tag{51}$$

Next, we prove (b). Because $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverges to the function $\mathbb{E}[f(u,\zeta)]$ if $\exists\{v_N\}$ s.t. $\lim_{n\longrightarrow\infty}v_N=v$, such that $\limsup_{N\longrightarrow\infty}\mathbb{E}[f_N(v_N,\zeta)]\leq\mathbb{E}[f(v,\zeta)]$, that is,

$$\begin{aligned}
\mathbb{E}[f_N(v_N,\zeta)]&\leq\mathbb{E}[f(v,\zeta)]+\varepsilon,\\
\mathbb{E}[f_N(v_N,\zeta)]-\mathbb{E}[f(v,\zeta)]&\leq\varepsilon.
\end{aligned}\tag{52}$$

That means that there exists a sequence $\{v_N\}$ converging to v, and it holds

$$\begin{aligned}
&\|\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta)\|\\
&\quad=\left\|\mathrm{Proj}_{S,G}\left(v_N-\gamma^{-1}G^{-1}\mathbb{E}[f_N(v_N,\zeta)]\right)\right.\\
&\qquad\left.-\mathrm{Proj}_{S,G}\left(v-\gamma^{-1}G^{-1}\mathbb{E}[f(v,\zeta)]\right)\right\|_G\\
&\quad\leq\left\|(v_N-v)-\gamma^{-1}G^{-1}\mathbb{E}[f_N(v_N,\zeta)]-\gamma^{-1}G^{-1}\mathbb{E}[f(v,\zeta)]\right\|_G\\
&\quad\leq\|v_N-v\|+\gamma^{-1}\left\|G^{-1}\right\|\cdot\|\mathbb{E}[f_N(v_N,\zeta)]-\mathbb{E}[f(v,\zeta)]\|\\
&\quad\leq\delta+\gamma^{-1}\left\|G^{-1}\right\|\varepsilon.
\end{aligned}\tag{53}$$

From (a) and (b), we can get that $\mathscr{P}_N(u_N,\zeta)$ approaches to $\mathscr{P}(u,\zeta)$, so the proof is completed. □

Theorem 10. *Assume that $f(u,\zeta)$ is ϕ-bounded function and every function of the sequence $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverges to the function $\mathbb{E}[f(u,\zeta)]$. Then, $\mathscr{R}_N(u_N,\zeta)$ epiconverge to $\mathscr{R}(u,\zeta)$.*

Proof. To prove $\{\mathscr{R}_N(u_N,\zeta)\}$ epiconverge to $\{\mathscr{R}(u,\zeta)\}$, we will prove the following:

(I) If for any u, $\forall\{u_N\}$ s.t. $\lim_{N\longrightarrow\infty}u_N=u$, then $\liminf_{N\longrightarrow\infty}\mathscr{R}_N(u_N)\geq\mathscr{R}(u)$

(II) $\exists\{v_N\}$ s.t. $\lim_{N\longrightarrow\infty}v_N=v$, then $\limsup_{N\longrightarrow\infty}\mathscr{R}_N(v_N)\leq\mathscr{R}(u)$

First of all, we prove (I). Because for any u, $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverges to the function $\mathbb{E}[f(u,\zeta)]$, $\forall\{u_N\}$ s.t. $\lim_{N\longrightarrow\infty} u_N = u$, there holds $\liminf_{N\longrightarrow\infty}\mathbb{E}[f_N(u_N,\zeta)] \geq \mathbb{E}[f(u,\zeta)]$, that is,

$$\begin{aligned}\mathbb{E}[f_N(u_N,\zeta)] &\geq \mathbb{E}[f(u,\zeta)] - \varepsilon_1,\\ \mathbb{E}[f_N(u_N,\zeta)] - \mathbb{E}[f(u,\zeta)] &\geq -\varepsilon_1.\end{aligned} \tag{54}$$

From

$$\begin{aligned}\mathscr{R}(u,\zeta) &= (u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2,\\ \mathscr{R}_N(u_N,\zeta) &= (u_N-\mathscr{P}_N(u_N,\zeta))^T\mathbb{E}[f_N(u_N,\zeta)]\\ &\quad - \frac{\gamma}{2}\|u_N-\mathscr{P}_N(u_N,\zeta)\|_G^2.\end{aligned} \tag{55}$$

We then obtain

$$\begin{aligned}&\mathscr{R}_N(u_N,\zeta) - \mathscr{R}(u,\zeta)\\ &\quad= (u_N-\mathscr{P}_N(u_N,\zeta))^T\mathbb{E}[f_N(u_N,\zeta)]\\ &\qquad - \frac{\gamma}{2}\|u_N-\mathscr{P}_N(u_N,\zeta)\|_G^2 - (u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)]\\ &\qquad - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2\\ &\quad= (u_N-\mathscr{P}_N(u_N,\zeta))^T\{\mathbb{E}[f_N(u_N,\zeta)] - \mathbb{E}[f(u,\zeta)]\}\\ &\qquad + (u_N-\mathscr{P}_N(u_N,\zeta)-u+\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)]\\ &\qquad - \frac{\gamma}{2}\|u_N-\mathscr{P}_N(u_N,\zeta)\|_G^2 + \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2\\ &\quad\geq \left|(u_N-\mathscr{P}_N(u_N,\zeta))^T\right|(-\varepsilon_1)\\ &\qquad + (u_N-\mathscr{P}_N(u_N,\zeta)-u+\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)]\\ &\qquad - \frac{\gamma}{2}\|u_N-\mathscr{P}_N(u_N,\zeta)\|_G^2 + \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2\\ &\quad= \left|(u_N-\mathscr{P}_N(u_N,\zeta))^T\right|(-\varepsilon_1)\\ &\qquad + (u_N-u+\mathscr{P}(u,\zeta)-\mathscr{P}_N(u_N,\zeta))^T\mathbb{E}[f(u,\zeta)]\\ &\qquad - \frac{\gamma}{2}\Big\{(u-u_N+\mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta))^T G(u-u_N\\ &\qquad + \mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta))\Big\}.\end{aligned} \tag{56}$$

From $\{u_N\}$ converging to u, so for any $\varepsilon_2 > 0, \exists N_1 > 0$, $N > N_1$, we have

$$u_N - u \geq -\varepsilon_2. \tag{57}$$

By (51) in Lemma 9 and (56), that is, for any $\varepsilon_3 > 0$, $\exists N_3 \in \max\{N_0, N_1\}, N > N_3$, we have $\mathscr{P}_N(u_N,\zeta) - \mathscr{P}(u,\zeta) \geq -\varepsilon_3$. We can get

$$\begin{aligned}&\mathscr{R}_N(u_N,\zeta) - \mathscr{R}(u,\zeta)\\ &\quad\geq \left|(u_N-\mathscr{P}_N(u_N,\zeta))^T\right|(-\varepsilon_1)\\ &\qquad + (u_N-u+\mathscr{P}(u,\zeta)-\mathscr{P}_N(u_N,\zeta))^T\mathbb{E}[f(u,\zeta)]\\ &\qquad - \frac{\gamma}{2}\Big\{(u-u_N+\mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta))^T G(u-u_N\\ &\qquad + \mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta))\Big\}\\ &\quad\geq \left|(u_N-\mathscr{P}_N(u_N,\zeta))^T\right|(-\varepsilon_1) - (\varepsilon_2+\varepsilon_3)\mathbb{E}[f(u,\zeta)]\\ &\qquad + \frac{\gamma}{2}\Big\{(-\varepsilon_2-\varepsilon_3)^T G(-\varepsilon_2-\varepsilon_3\Big\}.\end{aligned} \tag{58}$$

Obviously, $u_N \in S$, $\mathscr{P}_N(u_N,\zeta)$ is the the unique optimal solution of problem $\min_{u\in S}\mathscr{R}_N(u_N,\zeta)$; and $\mathscr{P}(u,\zeta)$ is the the unique optimal solution of problem $\min_{u\in S}\mathscr{R}(u,\zeta)$, so $|u_N-\mathscr{P}_N(u_N,\zeta)|^T$ is finite. That is, for any $\varepsilon > 0, \exists N_2 > 0$, $N > N_2 > N_1$, we have

$$\mathscr{R}_N(u_N,\zeta) - \mathscr{R}(u,\zeta) \geq -\varepsilon. \tag{59}$$

It means that

$$\liminf_{N\longrightarrow\infty}\mathscr{R}_N(u_N,\zeta) \geq \mathscr{R}(u,\zeta). \tag{60}$$

Furthermore, we prove (II). From

$$\begin{aligned}\mathscr{R}(u,\zeta) &= (u-\mathscr{P}(u,\zeta))^T\mathbb{E}[f(u,\zeta)] - \frac{\gamma}{2}\|u-\mathscr{P}(u,\zeta)\|_G^2,\\ \mathscr{R}_N(u_N,\zeta) &= (u_N-\mathscr{P}_N(u_N,\zeta))^T\mathbb{E}[f_N(u_N,\zeta)]\\ &\quad - \frac{\gamma}{2}\|u_N-\mathscr{P}_N(u_N,\zeta)\|_G^2.\end{aligned} \tag{61}$$

Because $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverges to the function $\mathbb{E}[f(u,\zeta)]$, $\exists\{v_N\}$ s.t. $\lim_{N\longrightarrow\infty} v_N = v$, such that $\limsup_{N\longrightarrow\infty}\mathbb{E}[f_N(v_N,\zeta)] \leq \mathbb{E}[f(v,\zeta)]$, that is,

$$\begin{aligned}\mathbb{E}[f_N(v_N,\zeta)] &\leq \mathbb{E}[f(v,\zeta)] + \varepsilon_1,\\ \mathbb{E}[f_N(v_N,\zeta)] - \mathbb{E}[f(v,\zeta)] &\leq \varepsilon_1.\end{aligned} \tag{62}$$

We then obtain that there exists a sequence $\{v_N\}$ converging to v; it holds

$$\begin{aligned}
&\mathscr{R}_N(v_N,\zeta)-\mathscr{R}(v,\zeta)\\
&\quad=(v_N-\mathscr{P}_N(v_N,\zeta))^T\mathbb{E}[f_N(v_N,\zeta)]-\frac{\gamma}{2}\|v_N-\mathscr{P}_N(v_N,\zeta)\|_G^2\\
&\qquad-(v-\mathscr{P}(v,\zeta))^T\mathbb{E}[f(v,\zeta)]+\frac{\gamma}{2}\|v-\mathscr{P}(v,\zeta)\|_G^2\\
&\quad\le\Big|(v_N-\mathscr{P}_N(v_N,\zeta))^T\mathbb{E}[f_N(v_N,\zeta)]\\
&\qquad-(v_N-\mathscr{P}(v_N,\zeta))^T\mathbb{E}[f(v,\zeta)]\\
&\qquad+(v_N-\mathscr{P}(v_N,\zeta))^T\mathbb{E}[f(v,\zeta)]-\frac{\gamma}{2}\|v_N-\mathscr{P}_N(v_N,\zeta)\|_G^2\\
&\qquad-(v-\mathscr{P}(v,\zeta))^T\mathbb{E}[f(v,\zeta)]+\frac{\gamma}{2}\|v-\mathscr{P}(v,\zeta)\|_G^2\Big|\\
&\quad=\Big|(v_N-\mathscr{P}_N(v_N,\zeta))^T(\mathbb{E}[f_N(v_N,\zeta)]-\mathbb{E}[f(v,\zeta)])\\
&\qquad+(v_N-\mathscr{P}(v_N,\zeta)-v+\mathscr{P}(v,\zeta))^T\mathbb{E}[f(v,\zeta)]\Big|\\
&\qquad+\Big|\frac{\gamma}{2}\|v-\mathscr{P}(v,\zeta)\|_G^2-\|v_N-\mathscr{P}_N(v_N,\zeta)\|_G^2\Big|\\
&\quad=\Big|(v_N-\mathscr{P}_N(v_N,\zeta))^T\varepsilon_1\\
&\qquad+(v_N-v+\mathscr{P}(v,\zeta)-\mathscr{P}(v_N,\zeta))^T\mathbb{E}[f(v,\zeta)]\Big|\\
&\qquad+\Big|\frac{\gamma}{2}\Big\{(v-\mathscr{P}(v,\zeta))^TG(v-\mathscr{P}(v,\zeta))\\
&\qquad-(v_N-\mathscr{P}_N(v_N,\zeta))^TG(v_N-\mathscr{P}_N(v_N,\zeta))\Big\}\Big|\\
&\quad\le\|(v_N-\mathscr{P}_N(v_N,\zeta))\|\varepsilon_1\\
&\qquad+\|v_N-v+\mathscr{P}(v,\zeta)-\mathscr{P}_N(v_N,\zeta)\|\|\mathbb{E}[f(v,\zeta)]\|\\
&\qquad+\frac{\gamma}{2}\|(v-v_N+\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta))\|\|G\|\\
&\qquad\cdot\|(v-v_N+\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta))\|\\
&\quad\le\|v_N-\mathscr{P}_N(v_N,\zeta)\|\varepsilon_1+\{\|v_N-v\|\\
&\qquad+\|\mathscr{P}(v,\zeta)-\mathscr{P}_N(v_N,\zeta))\|\}\|\mathbb{E}[f(v,\zeta)]\|\\
&\qquad+\frac{\gamma}{2}\{(\|(v-v_N\|+\|\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta))\|)\cdot\|G\|\\
&\qquad\cdot(\|(v-v_N\|+\|\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta))\|)\}.
\end{aligned}\tag{63}$$

By $\{v_N\}$ converging to v, so for any $\varepsilon>0,\exists N_1>0,N>N_1$, we have

$$v_N-v<\varepsilon.\tag{64}$$

And since $\mathscr{R}(v,\zeta)\ge 0$, we have

$$\begin{aligned}
&(v-\mathscr{P}(v,\zeta))^T\mathbb{E}[f(v,\zeta)]\\
&\quad-\frac{\gamma}{2}\|v-\mathscr{P}(v,\zeta)\|_G^2\ge 0,\\
&\quad\frac{\gamma}{2}\|v-\mathscr{P}(v,\zeta)\|_G^2\le(v-\mathscr{P}(v,\zeta))^T\mathbb{E}[f(v,\zeta)]\\
&\qquad\qquad\le\|v-\mathscr{P}(v,\zeta)\|\cdot\|\mathbb{E}[f(v,\zeta)]\|.
\end{aligned}\tag{65}$$

Note that

$$\begin{aligned}
&\sqrt{\lambda_{\min}}\|v\|\le\|v\|_G,\\
&\|v-\mathscr{P}(v,\zeta)\|\cdot\|\mathbb{E}[f(v,\zeta)]\|\le\frac{1}{\sqrt{\lambda_{\min}}}\|v-\mathscr{P}(v,\zeta)\|_G\\
&\qquad\cdot\|\mathbb{E}[f(v,\zeta)]\|,
\end{aligned}\tag{66}$$

where $\lambda_{\min}$ indicate the smallest eigenvalue of G. Further, we can conclude that

$$\begin{aligned}
\|v-\mathscr{P}(v,\zeta)\|&\le\frac{1}{\sqrt{\lambda_{\min}}}\|v-\mathscr{P}(v,\zeta)\|_G\\
&\le\frac{2}{\gamma\sqrt{\lambda_{\min}}}\|\mathbb{E}[f(v,\zeta)]\|.
\end{aligned}\tag{67}$$

Because S is a compact and nonempty set on R^n, then $\exists M>0$; it holds

$$\begin{aligned}
&\|\mathbb{E}[f(v,\zeta)]\|<M,\\
&\|\mathbb{E}[f_N(v,\zeta)]\|<M.
\end{aligned}\tag{68}$$

Furthermore, it is not difficult to show that

$$\begin{aligned}
&\|v-\mathscr{P}(v,\zeta)\|<\frac{2}{\gamma\sqrt{\lambda_{\min}}}M,\\
&\|v_N-\mathscr{P}_N(v_N,\zeta)\|<\frac{2}{\gamma\sqrt{\lambda_{\min}}}M.
\end{aligned}\tag{69}$$

From (63), $v_N-v\le\varepsilon_2$, by (53) in Lemma 9; that is, for any $\varepsilon_3>0,\exists N_3\in\max\{N_0,N_1\},N>N_3$, we have $\mathscr{P}_N(u_N,\zeta)-\mathscr{P}(u,\zeta)\le\varepsilon_3$. $\|\mathbb{E}[f(v,\zeta)]\|<M$, and $\|v-\mathscr{P}(v,\zeta)\|<(2/\gamma\sqrt{\lambda_{\min}})M$ and $\|v_N-\mathscr{P}_N(v_N,\zeta)\|<(2/\gamma\sqrt{\lambda_{\min}})M$; we have

$$\begin{aligned}
&\mathscr{R}_N(v_N,\zeta)-\mathscr{R}(v,\zeta)\\
&\quad\le\|(v_N-v\|+\|\mathscr{P}(v,\zeta)-\mathscr{P}_N(v_N,\zeta))\|\|\mathbb{E}[f(v,\zeta)]\|\\
&\qquad+\varepsilon_1\|v_N-\mathscr{P}_N(v_N,\zeta)\|+\frac{\gamma}{2}\{(\|(v-v_N\|\\
&\qquad+\|\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta))\|)\cdot\|G\|\\
&\qquad\cdot(\|v-v_N\|+\|\mathscr{P}_N(v_N,\zeta)-\mathscr{P}(v,\zeta))\|)\}\\
&\quad\le\left(\varepsilon_2+\varepsilon_3+\gamma^{-1}\|G^{-1}\|\varepsilon_1\right)M+\frac{2\varepsilon_1}{\gamma\sqrt{\lambda_{\min}}}M\\
&\qquad+\frac{\gamma}{2}\left(\varepsilon_2+\varepsilon_3+\gamma^{-1}\|G^{-1}\|\varepsilon_1\right)\cdot\|G\|\\
&\qquad\cdot\left(\varepsilon_2+\varepsilon_3+\gamma^{-1}\|G^{-1}\|\varepsilon_1\right)\\
&\quad=\left(\varepsilon_2+\varepsilon_3+\gamma^{-1}\|G^{-1}\|\varepsilon_1\right)M+\frac{2\varepsilon_1}{\gamma\sqrt{\lambda_{\min}}}M\\
&\qquad+\frac{\gamma}{2}\left(\varepsilon_2+\varepsilon_3+\gamma^{-1}\|G^{-1}\|\varepsilon_1\right)^2\cdot\|G\|.
\end{aligned}\tag{70}$$

That is, for any $\varepsilon > 0, \exists N_2 > 0, N > N_2 > N_1$, we have

$$\mathscr{R}_N(v_N,\zeta) - \mathscr{R}(v,\zeta) \le \varepsilon. \quad (71)$$

It means that there exists a sequence $\{v_N\}$ that converges to v, so that

$$\limsup_{N\longrightarrow\infty} \mathscr{R}_N(v_N,\zeta) \le \mathscr{R}(v,\zeta). \quad (72)$$

From (60) and (72), we can get that $\{\mathscr{R}_N(u_N,\zeta)\}$ epi-converge to $\{\mathscr{R}(u,\zeta)\}$. □

Theorem 11. *Suppose that $f(u,\zeta)$ is ϕ-bounded function, and $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverge to $\mathbb{E}[f(u,\zeta)]$. Then, we have*

$$\lim_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) = \min_{u\in S} \mathscr{R}(u,\zeta). \quad (73)$$

Proof. Note that, by $f(u,\zeta)$ is ϕ-bounded function, for every N, $\min_{u\in S} \mathscr{R}_N(u,\zeta)$ and $\min_{u\in S} \mathscr{R}(u,\zeta)$ are both finite. So, in order to prove $\lim_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) = \min_{u\in S} \mathscr{R}(u,\zeta)$, we can prove the following:

(a) $\limsup_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) \le \min_{u\in S} \mathscr{R}(u,\zeta)$

(b) $\liminf_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) \ge \min_{u\in S} \mathscr{R}(u,\zeta)$

We first prove (a). Let $\forall\varepsilon > 0$. $\exists u_\varepsilon \in S$ s.t.

$$\mathscr{R}(u_\varepsilon,\zeta) \le \min_{u\in S} \mathscr{R}(u,\zeta) + \varepsilon. \quad (74)$$

From Theorem 10, we have $\mathscr{R}_N(u_N,\zeta) \longrightarrow \mathscr{R}(u,\zeta)$; it means that $\exists u_N$, s.t. $\lim_{N\longrightarrow\infty} u_N = u_\varepsilon$; there holds $\limsup_{N\longrightarrow\infty} \mathscr{R}_N (u_N,\zeta) \le \mathscr{R}(u_\varepsilon,\zeta)$. Therefore, we have

$$\begin{aligned} \limsup_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) &\le \limsup_{N\longrightarrow\infty} \mathscr{R}_N(u_N,\zeta), \\ \limsup_{N\longrightarrow\infty} \mathscr{R}_N(u_N,\zeta) &\le \mathscr{R}(u_\varepsilon,\zeta), \\ \mathscr{R}(u_\varepsilon,\zeta) &\le \min_{u\in S} \mathscr{R}(u,\zeta) + \varepsilon. \end{aligned} \quad (75)$$

By the arbitrariness of ε, we have that

$$\limsup_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) \le \min_{u\in S} \mathscr{R}(u,\zeta). \quad (76)$$

Next, we prove (b). $\forall\varepsilon > 0, \exists\{u_N\} \subset S$ s.t.

$$0 \le \mathscr{R}_N(u_N,\zeta) \le \min_{u\in S} \mathscr{R}_N(u,\zeta) + \varepsilon. \quad (77)$$

Then, $\exists\{u_{N_k}\}$ s.t. $\lim_{N\longrightarrow\infty} u_{N_k} = u_\varepsilon \in S$ such that

$$\lim_{N\longrightarrow\infty} \mathscr{R}N_k(u_{N_k},\zeta) = \liminf_{N\longrightarrow\infty} \mathscr{R}_N(u_N,\zeta). \quad (78)$$

Therefore, we have that

$$\liminf_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) \ge \liminf_{N\longrightarrow\infty} \mathscr{R}_{N_k}(u_{N_k},\zeta). \quad (79)$$

Since, by Theorem 10, the sequence $\{\mathscr{R}_N(u_N,\zeta)\}$ epi-converges to $\mathscr{R}(u,\zeta)$, that is, for every sequence $\{u_N\}$ converging to u, we have

$$\liminf_{N\longrightarrow\infty} \mathscr{R}_N(u_N,\zeta) \ge \mathscr{R}(u,\zeta). \quad (80)$$

Then,

$$\begin{aligned} &\liminf_{N\longrightarrow\infty} \mathscr{R}_{N_k}(u_{N_k},\zeta) \ge \mathscr{R}(u_\varepsilon,\zeta), \\ &\mathscr{R}(u_\varepsilon,\zeta) \ge \min_{u\in S} \mathscr{R}(u,\zeta), \\ &\liminf_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) \ge \liminf_{N\longrightarrow\infty} \mathscr{R}_{N_k}(u_{N_k},\zeta) \ge \mathscr{R}(u_\varepsilon,\zeta) \\ &\qquad \ge \min_{u\in S} \mathscr{R}(u,\zeta). \end{aligned} \quad (81)$$

Because ε is arbitrary, we have

$$\liminf_{N\longrightarrow\infty} \min_{u\in S} \mathscr{R}_N(u,\zeta) \ge \min_{u\in S} \mathscr{R}(u,\zeta). \quad (82)$$

The conclusion follows from (76) and (82) immediately. □

Theorem 12. *Suppose that $\mathbb{E}[f_N(u_N,\zeta)]$ epiconverge to $\mathbb{E}[f(u,\zeta)]$. Suppose that function $f(u,\zeta)$ is uniformly monotone with respect to u, there exists a function $\Psi(u)$ which is nonnegative integrable, $\forall u, v \in R^n$, $\forall N > 0$,*

$$(u - v)^T [f(u,\zeta) - f(v,\zeta)] \ge \Psi(\zeta) \|u - v\|^2. \quad (83)$$

Here, u_N is an optimal solution of (16), and $E[\Psi(\zeta)] > 0$. Then, the sequence $\{u_N\}$ converges to the unique solution of (10).

Proof. From

$$\begin{aligned} \mathscr{R}(u,\zeta) &= (u - \mathscr{P}(u,\zeta))^T \mathbb{E}[f(u,\zeta)] - \frac{\gamma}{2} \|u - \mathscr{P}(u,\zeta)\|_G^2, \\ \mathscr{R}_N(u_N,\zeta) &= (u_N - \mathscr{P}_N(u_N,\zeta))^T \mathbb{E}[f_N(u_N,\zeta)] \\ &\quad - \frac{\gamma}{2} \|u_N - \mathscr{P}_N(u_N,\zeta)\|_G^2. \end{aligned} \quad (84)$$

$\arg\inf \mathscr{R}$ and $\arg\inf \mathscr{R}_N$ are the optimal solution sets of (11) and (16). Let $u_{N_k} \in \arg\inf \mathscr{R}_{N_k}$, $u \in \arg\inf \mathscr{R}$. By Theorem 11, we have $\lim_{k\longrightarrow\infty} u_{N_k} = u \in S$. And from the assumptions, it shows that $\mathbb{E}[f(\cdot,\zeta)]$ is uniformly monotone and $\mathbb{E}[\Psi(\zeta)]>0$. So we have

$$(u-v)^T[f(u,\zeta)-f(v,\zeta)] \geq \Psi(\zeta)\|u-v\|^2,$$
$$\mathbb{E}\left[(u-v)^T[f(u,\zeta)-f(v,\zeta)]\right] \geq \mathbb{E}\left[\Psi(\zeta)\|u-v\|^2\right],$$
$$(u-v)^T\{\mathbb{E}[f(u,\zeta)]-\mathbb{E}[f(v,\zeta)]\} \geq \mathbb{E}[\Psi(\zeta)]\|u-v\|^2. \tag{85}$$

$\mathbb{E}[\Psi(\zeta)]>0$, so $(u-v)^T\{\mathbb{E}[f(u,\zeta)]-\mathbb{E}[f(v,\zeta)]\}\geq 0$. By the arbitrariness of u, v, we have

$$(u-v)^T\{\mathbb{E}[f(u,\zeta)]-\mathbb{E}[f(v,\zeta)]\} \geq 0. \tag{86}$$

So, $u=v$ means the uniqueness of the solution to problem (10), denoted by $u^\dagger$. It is not difficult to show that $u^\dagger$ is also the unique solution of (12). Therefore, $u^\dagger$ is a unique cluster point of the bounded sequence $\{u_N\}$. □

4. Conclusions

In this paper, we studied the SDA method for solving the UVIP. By constructing the gap function (11), the uncertain variational inequality problem is transformed into an optimization problem (12). Then, we propose the SDA method to solve it. Also, we research the convergence of the optimization problem. Finally, the correctness of the SDA method is proved; that is, the solution of the approximation problem (16) obtained by the SDA method converges to the solution of the original uncertain variational inequality (10).

In this paper, we have done some work on the Stieltjes integral discrete approximation of uncertain variational inequalities and obtained the related theoretical results, which have good theoretical and practical significance. Future studies are as follows: we can consider the displacement gap function to establish the correlation model; and we can consider to apply this method to the solution of uncertain complementary functions.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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