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Research Article

Stability of Set Differential Equations in Fréchet Spaces

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In this paper, we investigate the stability of set differential equations in Fréchet space $\mathbb{F}$. Some comparison principles and stability criteria are established for set differential equations with the fact that every Fréchet space $\mathbb{F}$ is a projective limit of Banach spaces.

1. Introduction

In recent years, set differential equations (SDEs for short) have attracted extensive attention due to its intrinsic advantages. In 1970, Blasi and Iervolino [1] first began the study of SDEs in semilinear metric spaces. Nowadays, the theory of SDEs has been developed into an independent subject area. There are a few results on the existence, stability, and other properties of solutions for various equations, such as SDEs [2–11], set functional differential equations [12–16], set integrodifferential equations [17–20], SDEs on time scales [21, 22], SDEs with causal operators [23–27], and others [15, 28–31], and references are given therein. Systematic development of set differential equations has been provided by Lakshmikantham et al. [32] and Martynyuk [33].

However, with the development of SDEs, when we further try to extend the results to the case of infinite dimensional locally convex spaces, namely, Fréchet spaces which are not Banach spaces, most of the previous definitions are no longer available in the new framework. Apart from this, the increasing problems modeled in the framework of non-Banach infinite dimensional spaces appear in modern analysis, differential geometry, and theoretical physics, also inspiring us to seek alternative methods to investigate SDEs. Thus, we will study SDEs within the Fréchet framework in this paper. In fact, several basic results have been obtained in this field. For example, Galanis et al. [34] generalized the basic notions about SDEs to a Fréchet space using the fact that it is a projective limit of Banach spaces. In the new framework, they obtained the properties of the Hausdorff metric and constructed the continuity, and Hukuhara differentiability of set-valued mappings, which lay the foundation for the study of SDEs. In [35], Galanis et al. investigated the existence of solutions of SDE in Fréchet space. Wang et al. [36] obtained the criterion of rapid convergence of solutions for SDEs.

The major purpose of this paper is to establish some comparison results and further attempt to study the stability of solutions of SDEs in Fréchet space employing Lyapunov-like functions and comparison principles. We recall some necessary background materials on the study of SDEs in Banach spaces and Fréchet ones in Section 2. Subsequently, in Section 3, we establish some comparison results of SDEs in Fréchet space. Finally, some types of stability properties of solutions of SDEs in Fréchet space are derived in Section 4.

2. Preliminaries

We first give some concepts related to Banach space [32, 34, 35] that will be used in the later discussion.

Let $K_c(\mathbb{E})$ denote the collection of all nonempty, compact, and convex subsets of Banach space $\mathbb{E}$. For any two nonempty subsets X and Y on $\mathbb{E}$, the Hausdorff metric is determined by the formula

$$D[X, Y] = \max \left\{ \sup_{x\in X} \inf_{y\in Y} \|x - y\|, \sup_{y\in Y} \inf_{x\in X} \|x - y\| \right\}. \quad (1)$$

where $\|\cdot\|$ denotes the norm in $\mathbb{E}$. Then, $(K_c(\mathbb{E}), D)$ is a complete metric space.

For the space $K_c(\mathbb{E})$, if we define the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_c(\mathbb{E})$ is a semilinear metric space (see [32]).

For $A, B, \bar{A}, \bar{B}, C \in K_c(\mathbb{E})$ and $\lambda \in \mathbb{R}_+$, the Hausdorff metric (1) has the following properties:

$$\begin{gathered} D[A+C, B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \\ D[\lambda A, \lambda B] = \lambda D[A,B], \\ D[A,B] \le D[A,C] + D[C,B], \\ D\left[A+\bar{A}, B+\bar{B}\right] \le D[A,B] + D\left[\bar{A},\bar{B}\right]. \end{gathered} \tag{2}$$

For $I=[t_0, T]$, we call the mapping $F: I \longrightarrow K_c(\mathbb{E})$ is Hukuhara differentiable if there exists $\mathscr{D}_H F(t_0) \in K_c(\mathbb{E})$, such that

$$\begin{aligned} \lim_{h\longrightarrow 0^+} \frac{F(t_0+h)-F(t_0)}{h} &= \lim_{h\longrightarrow 0^+} \frac{F(t_0)-F(t_0-h)}{h} \\ &= \mathscr{D}_H F(t_0), \text{for } t_0 \in I, \end{aligned} \tag{3}$$

where the differences are in the sense of the Hausdorff difference.

Moreover, if there exists a mapping $\Psi: I \longrightarrow K_c(\mathbb{E})$, such that

$$F(t) = X_0 + \int_{t_0}^{t} \Psi(s)ds, X_0 \in K_c(\mathbb{E}), \tag{4}$$

where the integrable is in the sense of Bochner, then $D_H F(t) = \Phi(t)$ a.e. on I.

We denote the integral as follows

$$\begin{aligned} &(A)\int_{I_0} F(s)ds \\ &= \left[\int_{I_0} f(s)ds : f \text{ is a Bochner integrable selector of } F\right]. \end{aligned} \tag{5}$$

However, because the topology of $\mathbb{F}$ is derived from a set of seminorms rather than a single norm, the application of the classical definition of Hausdorff distance in $K_c(\mathbb{F})$ is impossible. Therefore, the above methodology is no longer available if the space we dealt with is not Banach space $\mathbb{E}$, but Fréchet space $\mathbb{F}$. In order to overcome the above shortcomings, Galanis et al. [34, 35] proposed a set of structures suitable for generalized locally convex topological vector spaces on Fréchet spaces. The main difference is that the classical Hausdorff metric D is replaced by a group of semimetrics in a Fréchet space.

Let $\mathbb{F}$ be a Fréchet space defined by a sequence $\{p_i\}_{i\in\mathbb{N}}$ of seminorms, and the sequence $\{p_i\}_{i\in\mathbb{N}}$ be increasing. Then, $\mathbb{F} \triangleq \varprojlim\{\mathbb{E}^i ; \rangle^{ji}\}_{i,j\in\mathbb{N}}$ can be seen as a projective limit of Banach spaces $\mathbb{E}$, where $\mathbb{E}^i$ denotes the completion of the quotient $\mathbb{F}/\text{Ker } p_i$, $i \in \mathbb{N}$, and ϱ^{ji} are the connecting morphisms

$$\varrho^{ji} : \mathbb{E}^j \longrightarrow \mathbb{E}^i : \left[x + \text{Ker } p_j\right]_j \mapsto [x + \text{Ker } p_i]_i, j \ge i, \tag{6}$$

the bracket $[\cdot]_i$ stands for the corresponding equivalence class, and ϱ^{ji} is a continuous mapping (see [37, 38]).

In this case, the space $K_c(\mathbb{F})$ can be viewed as a projective limit space with a corresponding structure to $\mathbb{E}^i$s:

$$K_c(\mathbb{F}) \equiv \varprojlim\left\{K_c\left(\mathbb{E}^i\right) ; \varrho^{ji}\right\}_{i,j\in\mathbb{N}}, \tag{7}$$

where the mapping $\varrho^{ji} : K_c(\mathbb{E}^j) \longrightarrow K_c(\mathbb{E}^i)$: $A \mapsto \varrho^{ji}(A)$. Each ρ^{ji} is continuous concerning the topologies induced by the Hausdorff metrics $D^{\mathbb{E}^j}, D^{\mathbb{E}^i}$ on $K_c(\mathbb{E}^j)$ and $K_c(\mathbb{E}^i)$. Therefore, every element A of $K_c(\mathbb{F})$ can be realized as

$$A \equiv \left(\varrho^i(A)\right)_{i\in\mathbb{N}} \equiv \varprojlim \varrho^i(A), \tag{8}$$

where $\varrho^i : \mathbb{F} \longrightarrow \mathbb{E}^i$ are the canonical projections of $\mathbb{F}$ to $\mathbb{E}^i$.

With the above definition, we can revise the notion of the Hausdorff metric in Fréchet space as follows:

Definition 1 (see [34]). Let $\mathbb{F}$ be a Fréchet space which is defined by a group of seminorms. For $X, Y \in K_c(\mathbb{F})$, we call i-Hausdorff metric between X and Y.

$$D^i[X,Y] = \max\left\{d_D^i(X,Y), d_D^i(Y,X)\right\}. \tag{9}$$

From the perspective of classical conceptions of Banach factors, the connection below exists: if $X^i, Y^i \in K_c(\mathbb{E}^i)$, $X = \varprojlim X^i$, $Y = \varprojlim Y^i$, then

$$D^i[X,Y] = D^{\mathbb{E}^i}\left[X^i, Y^i\right], \tag{10}$$

where the revised i-Hausdorff metric also satisfies the corresponding properties (2).

Let the set-valued mapping $F: I \longrightarrow K_c(\mathbb{F})$ can be seen as a projective limit of the corresponding mappings in the Banach factors $\mathbb{E}^i$; that is, $F = \varprojlim F^i$, in which $F^i : I \longrightarrow K_c(\mathbb{E}^i)$, denotes the canonical projections of the limit $K_c(\mathbb{F}) \equiv \varprojlim K_c(\mathbb{E}^i)$.

Proposition 2 (see [35]). *The mapping $F(=\varprojlim F^i)$ is Hukuhara differentiable at t_0 if and only if every F^i is Hukuhara differentiable at t_0 and*

$$\mathscr{D}_H F(t_0) = \varprojlim \mathscr{D}_H F^i(t_0). \tag{11}$$

3. Comparison Results of SDEs in Fréchet Space

In this section, we consider the IVP for SDEs in the Fréchet space with the framework of $K_c(\mathbb{F})$

$$\mathscr{D}_H X(t) = F(t, X(t)), X(t_0) = X_0, \quad t_0 \in I, \tag{12}$$

where $F \in C[I \times K_c(\mathbb{F}), K_c(\mathbb{F})]$, $X_0 \in K_c(\mathbb{F})$.

The mapping $X \in C^1[I, K_c(\mathbb{F})]$ is called a solution of SDEs (12) on I; that is

$$X(t) = X_0 + \int_{t_0}^{t} \mathscr{D}_H X(s) ds, t \in I. \tag{13}$$

Then, we associate the IVP (12) with the integral equations

$$X(t) = X_0 + \int_{t_0}^{t} F(s, X(s)) ds, t \in I. \tag{14}$$

In our further discussion, we will use the comparison differential equation

$$w' = g(t, w), w(t_0) = w_0 \geq 0, \tag{15}$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$, $g(t, 0) \equiv 0$.

Theorem 3. *Suppose that the following conditions are satisfied:*

B1: the projective limit $F = \varprojlim F^i$, $F^i \in C[I \times K_c(\mathbb{E}^i), K_c(\mathbb{E}^i)]$

B2: $D^i[F(t, U), F(t, W)] \leq g(t, D^i[U, W])$, $(t, U), (t, W) \in I \times K_c(\mathbb{F})$, $i \in \mathbb{N}$

B3: the scalar differential equation (15) has a maximal solution $r(t) \equiv r(t, t_0, w_0)$ *existing on I*

Then, $D^i[U(t), W(t)] \leq r(t), t \geq t_0$ *provided* $D^i[U_0, W_0] \leq w_0$, *where* $U(t) \equiv U(t; t_0, U_0)$ *and* $W(t) \equiv W(t; t_0, W_0)$ *are solutions of SDEs (12) on I.*

Proof. By the fact of the mapping F as a projective limit, Equations (12) can be reduced to a system of SDEs on the Banach spaces $\mathbb{E}^i$:

$$\mathscr{D}_H^i X(t) = F^i\left(t, X^i(t)\right), X^i(t_0) = X_0^i \in K_c\left(\mathbb{E}^i\right), t \geq t_0, \tag{16}$$

where X_0^i denotes the projection of X_0.

From condition B2, (10), and the previously stated Property 2.1, we have

$$D^{\mathbb{E}^i}\left[F^i\left(t, U^i\right), F^i\left(t, W^i\right)\right] \leq g\left(t, D^{\mathbb{E}^i}\left[U^i, W^i\right]\right), t \geq t_0, i \in \mathbb{N}. \tag{17}$$

According to the comparison principle [32] for SDEs in Banach space with the framework of $K_c(\mathbb{E})$, if $U^i(t)$ and $W^i(t)$ are solutions of (16) through (t_0, U_0^i), (t_0, W_0^i) on I, then

$$D^{\mathbb{E}^i}\left[U^i(t), W^i(t)\right] \leq r(t), t \geq t_0, \tag{18}$$

providing that $D^{\mathbb{E}^i}[U_0^i, V_0^i] \leq u_0$.

In addition, by considering the projective limits of $U(t) = \varprojlim U^i(t): I \longrightarrow K_c(\mathbb{F})$, we can obtain

$$\begin{aligned} \mathscr{D}_H U(t) &= \varprojlim\left(D_H^i U^i(t)\right) = \varprojlim F^i\left(t, U^i(t)\right) = F(t, U(t)), \\ U(t_0) &= \left(U^i(t_0)\right)_{i \in \mathbb{N}} = \left(U_0^i\right)_{i \in \mathbb{N}} = U_0, \end{aligned} \tag{19}$$

that means $U(t)$ is a solution of (12) through (t_0, U_0) on I. In the same way, we can claim that $W(t)$ is a solution of (12) through (t_0, W_0). Thus

$$D^i[U(t), W(t)] = D^{\mathbb{E}^i}\left[U^i(t), W^i(t)\right] \leq r(t), t \geq t_0, \tag{20}$$

providing $D^i[U_0, W_0] = D^{\mathbb{E}^i}[U_0^i, W_0^i] \leq w_0$.

This proves the claimed estimation of Theorem 3.

Next, we give a comparison result under weaker assumptions. □

Theorem 4. *Suppose that the conditions B1 and B3 are satisfied in Theorem 3, and*

$$\begin{aligned} &\limsup_{h \longrightarrow 0^+} \frac{1}{h}\left\{D^i[U + hF(t, U), W + hF(t, V)] - D^i[U, W]\right\} \\ &\quad \leq g\left(t, D^i[U, W]\right), \end{aligned} \tag{21}$$

Then, the conclusion of Theorem 3 holds.

Proof. By condition B1, the inequality (21) can be transformed into the inequality on Banach spaces:

$$\begin{aligned} &\limsup_{h \longrightarrow 0^+} \frac{1}{h}\left\{D^{\mathbb{E}^i}\left[U^i + hF^i\left(t, U^i\right), W^i + hF^i\left(t, W^i\right)\right] - D^{\mathbb{E}^i}\left[U^i, W^i\right]\right\} \\ &\quad \leq g\left(t, D^{\mathbb{E}^i}\left[U^i, W^i\right]\right), \end{aligned} \tag{22}$$

where $U^i, W^i \in K_c(\mathbb{E}^i)$, $F^i \in C[I \times K_c(\mathbb{E}^i), K_c(\mathbb{E}^i)]$.

Using the properties (2) and the inequality (21), we can obtain the inequality (17) hold. The rest argument is similar to the proof of Theorem 3. We omit here.

After that, we give a comparison theorem via Lyapunov-like functions which can be used to the discussion of the stability theory of Lyapunov in Fréchet space. □

Theorem 5. *Suppose that* $V \in C[I \times K_c(\mathbb{F}), \mathbb{R}_+]$, $X, Y \in K_c(\mathbb{F})$, *and.*

B4: $|V(t, X) - V(t, Y)| \leq L^i D^i[X, Y]$, L^i *is a positive constant*

B5: $\mathscr{D}^+V(t,X)=\limsup_{h\longrightarrow 0^+}(1/h)[V(t+h,X+hF(t,X))-V(t,X)]\le g(t,V(t,X))$

Then, $V(t,X(t))\le r(t), t\ge t_0$ *provided* $V(t_0,X_0)\le w_0$.

Proof. By Theorem 3, equations (12) can be transformed into the following SDEs on the Banach spaces $\mathbb{E}^i$:

$$\mathscr{D}_H^i X(t)=F^i\left(t,X^i(t)\right), X^i(t_0)=X_0^i\in K_c\left(\mathbb{E}^i\right), t\ge t_0. \quad (23)$$

where $X^i\in K_c(\mathbb{E}^i)$ and $F^i\in C[I\times K_c(\mathbb{E}^i),K_c(\mathbb{E}^i)]$. Meanwhile, we have

$$\begin{aligned}|V(t,X)-V(t,Y)|&=\left|V\left(t,\underleftarrow{\lim}X^i\right)-V\left(t,\underleftarrow{\lim}Y^i\right)\right|\\&\le\max\left|V^{E^i}\left(t,X^i\right)-V^{E^i}\left(t,Y^i\right)\right|_{i\in\mathbb{N}}\\&\le L^iD^i[X,Y]=L^iD^{E^i}\left[X^i,Y^i\right],\end{aligned} \quad (24)$$

where $V^{\mathbb{E}^i}\in C[\mathbb{R}_+\times K_c(\mathbb{E}^i),\mathbb{R}_+]$.

Similarly, we have

$$\begin{aligned}&\mathscr{D}^+V(t,X)\\&\le\max\left\{\limsup_{h\longrightarrow 0^+}\frac{1}{h}\left[V^{E^i}\left(t+h,X^i+hF^i\left(t,X^i\right)\right)-V^{E^i}\left(t,X^i\right)\right]\right\}_{i\in\mathbb{N}}\\&\le g\left(t,V\left(t,\underleftarrow{\lim}X^i\right)\right)=g\left(t,V^{E^i}\left(t,X^i\right)\right),\end{aligned} \quad (25)$$

namely,

$$\begin{aligned}&\left|V^{\mathbb{E}^i}\left(t,X^i\right)-V^{\mathbb{E}^i}\left(t,Y^i\right)\right|_{i\in\mathbb{N}}\le L^iD^{\mathbb{E}^i}\left[X^i,Y^i\right],\\&\left(\mathscr{D}^+V^{\mathbb{E}^i}\left(t,X^i\right)\right)_{i\in\mathbb{N}}\le\left(g\left(t,V^{\mathbb{E}^i}\left(t,X^i\right)\right)\right)_{i\in\mathbb{N}}.\end{aligned} \quad (26)$$

If $X^i(t)\triangleq X(t,t_0,X_0^i)$ is any solution of SDEs (12) existing on I and satisfies $V^i(t_0,X_0^i)\le w_0$, then, we obtain

$$\left(V^{\mathbb{E}^i}\left(t,X^i(t)\right)\right)_{i\in\mathbb{N}}\le r(t), t\ge t_0. \quad (27)$$

Moreover, by considering the projective limits of $X(t)=\underleftarrow{\lim}X^i(t)$, we have

$$\begin{aligned}&\mathscr{D}_HX(t)=\underleftarrow{\lim}\left(\mathscr{D}_H^iX^i(t)\right)=\underleftarrow{\lim}F^i\left(t,X^i(t)\right)=F(t,X(t)),\\&X(t_0)=\left(X_0^i\right)_{i\in\mathbb{N}}=X_0.\end{aligned} \quad (28)$$

Then, $X(t)$ is a solution of (12) on I. Furthermore, we obtain

$$V(t,X)=V\left(t,\underleftarrow{\lim}X^i\right)=V^{\mathbb{E}^i}\left(t,X^i\right)\le r(t), t\ge t_0. \quad (29)$$

This proves the claimed estimation of Theorem 5. □

4. Stability Criteria

In this section, we give the stability criteria via Lyapunov functions in Fréchet spaces.

Firstly, we give the following sets for convenience.

$$\begin{aligned}&S(\rho)=\left\{X\in K_c(\mathbb{F})\colon D^i[X,\theta]<\rho,\rho>0\text{ is a cons tan t}\right\},\\&\mathscr{K}=\{a\in C[[0,\rho),\mathbb{R}_+],|a(t)\text{is strictly increasing and } a(0)=0\},;\\&\mathscr{CK}=\{b\in C[\mathbb{R}_+\times[0,\rho),\mathbb{R}_+],|b(t,\cdot)\in\mathscr{K},\text{for all }t\in\mathbb{R}_+\}.\end{aligned} \quad (30)$$

Definition 6. The trivial solution $X=\theta$ of (12) is said to be

C1: equistable, if for each $\varepsilon>0$ and $t_0\in\mathbb{R}_+$, there exists a $\delta=\delta(t_0,\varepsilon)>0$, such that

$$D^i[X_0,\theta]<\delta\text{ implies }D^i[X(t),\theta]<\varepsilon, t\ge t_0, \quad (31)$$

where $X(t)$ is the solution of SDEs (12)

C2: uniformly stable, if the δ in (C1) is independent of t_0

C3: equiattractive, if for each $\varepsilon>0$ and $t_0\in\mathbb{R}_+$, there exist $\delta=\delta(t_0)>0$ and $T=T(t_0,\varepsilon)$, such that

$$D^i[X_0,\theta]<\delta\text{ implies }D^i[X(t),\theta]<\varepsilon, t\ge t_0+T \quad (32)$$

C4: uniformly attractive, if the δ and T in C3 are independent of t_0

C5: equiasymptotically stable, if C1 and C3 hold simultaneously

C6: uniformly asymptotically stable, if C2 and C4 hold simultaneously

Theorem 7. *Suppose the following conditions are satisfied:*

S1: $V\in C[I\times S(\rho),\mathbb{R}_+]$ *and for* $(t,X),(t,Y)\in I\times S(\rho)$ *such that*

$$|V(t,X)-V(t,Y)|\le L^iD^i[X,Y], L^i>0\text{ is a constant}, \quad (33)$$

$$\mathscr{D}^+V(t,X)=\limsup_{h\longrightarrow 0^+}\frac{1}{h}[V(t+h,X+hF(t,X))-V(t,X)]\le 0, \quad (34)$$

S2: for $(t,X)\in I\times S(\rho)$*, there exist* $a\in\mathscr{K}$ *and* $b\in\mathscr{CK}$ *satisfying*

$$a\left(D^i[X,\theta]\right)\le V(t,X)\le b\left(t,D^i[X,\theta]\right), (t,X)\in\mathbb{R}_+\times S(\rho). \quad (35)$$

Then, the trivial solution of SDEs (12) is equistable.

Proof. From the previous section, we know that equation (12) can be reduced to a system of SDEs on the Banach spaces $\mathbb{E}^i$ by considering the mapping F as a projective limit.

$$\mathscr{D}_H^i X(t) = F^i\left(t, X^i(t)\right), X^i(t_0) = X_0^i \in K_c\left(\mathbb{E}^i\right), t \geq t_0. \quad (36)$$

Using the known conditions, we obtain

$$\begin{aligned} |V(t,X) - V(t,Y)| &\leq \max \left| V^{\mathbb{E}^i}\left(t, X^i\right) - V^{\mathbb{E}^i}\left(t, Y^i\right) \right|_{i\in\mathbb{N}} \\ &\leq L^i D^{\mathbb{E}^i}\left[X^i, Y^i\right], \end{aligned} \quad (37)$$

where $V^{\mathbb{E}^i} \in C[\mathbb{R}_+ \times K_c(\mathbb{E}^i), \mathbb{R}_+]$ and $X^i, Y^i \in K_c(\mathbb{E}^i)$.

$$\begin{aligned} \mathscr{D}^+ V(t,X) &= \limsup_{h\longrightarrow 0^+} \frac{1}{h}\left[V\left(t+h, \underleftarrow{\lim} X^i h \underleftarrow{\lim} F^i\left(t, X^i\right)\right) - V\left(t, \underleftarrow{\lim} X^i\right)\right] \\ &\leq \max\left\{\limsup_{h\longrightarrow 0^+} \frac{1}{h}\left[V^{E^i}\left(t+h, X^i + hF^i\left(t, X^i\right)\right) - V^{E^i}\left(t, X^i\right)\right]\right\}_{i\in\mathbb{N}} \\ &\leq 0. \end{aligned} \quad (38)$$

Noticing that $D^i[X(\cdot),\theta] = D^{\mathbb{E}^i}[X^i(\cdot),\theta]$, we have

$$\begin{aligned} a\left(D^i[X,\theta]\right) &= a\left(D^{\mathbb{E}^i}\left[X^i,\theta\right]_{i\in\mathbb{N}}\right) = \left(a^i\left(D^{\mathbb{E}^i}\left[X^i,\theta\right]\right)\right)_{i\in\mathbb{N}} \\ &\leq V^{\mathbb{E}^i}\left(t, X^i\right) \leq b\left(t, \left(D^{\mathbb{E}^i}\left[X^i,\theta\right]\right)_{i\in\mathbb{N}}\right) \\ &= \left(b^i\left(t, D^{\mathbb{E}^i}\left[X^i,\theta\right]\right)\right)_{i\in\mathbb{N}}. \end{aligned} \quad (39)$$

where $V^{\mathbb{E}^i} \in C[\mathbb{R}_+ \times K_c(\mathbb{E}^i), \mathbb{R}_+]$ and $a^i \in \mathscr{K}$, $b^i \in \mathscr{CK}$.

Furthermore, we obtain

$$\begin{gathered} \left| V^{\mathbb{E}^i}\left(t, X_1^i\right) - V^{\mathbb{E}^i}\left(t, X_2^i\right) \right|_{i\in\mathbb{N}} \leq L^i D^{\mathbb{E}^i}\left[X_1^i, X_2^i\right], \\ \left(\mathscr{D}^+ V^{\mathbb{E}^i}\left(t, X^i\right)\right)_{i\in\mathbb{N}} \leq 0, \\ \left(a^i\left(D^{\mathbb{E}^i}\left[X^i,\theta\right]\right)\right)_{i\in\mathbb{N}} \leq V^{\mathbb{E}^i}\left(t, X^i\right) \leq \left(b^i\left(t, D^{\mathbb{E}^i}\left[X^i,\theta\right]\right)\right)_{i\in\mathbb{N}}. \end{gathered} \quad (40)$$

Similar to the proof of stability criteria in Theorem 3.4.1 [32], we know that the trivial solution of (36) is equistable. Thus, for $i \in \mathbb{N}$, $\varepsilon = \varepsilon^i > 0$, there exists a $\delta = \delta^i_{i\in\mathbb{N}} = \delta^i(t_0, \varepsilon^i)_{i\in\mathbb{N}} > 0$, such that

$$\begin{aligned} D^i[X_0,\theta] &= \left(D^{\mathbb{E}^i}\left[X_0^i,\theta\right]\right)_{i\in\mathbb{N}} < \delta^i_{i\in\mathbb{N}} = \delta \text{ implies } D^i[X(t),\theta] \\ &= \left(D^{\mathbb{E}^i}\left[X^i(t),\theta\right]\right)_{i\in\mathbb{N}} < \varepsilon^i_{i\in\mathbb{N}} = \varepsilon, \end{aligned} \quad (41)$$

that is, for each $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$, such that $D^i[X(t),\theta] < \varepsilon, t \geq t_0$ provided $D^i[X_0,\theta] < \delta$. Therefore, the trivial solution of (33) is equistable.

The proof is complete. □

Similar to the proof process of Theorem 7, we can obtain the following theorems.

Theorem 8. *Suppose that the conditions of Theorem 7 hold, except that inequality (34) can be strengthened to*

$$\begin{aligned} \mathscr{D}^+ V(t,X) &= \limsup_{h\longrightarrow 0^+} \frac{1}{h}[V(t+h, X + hF(t,X)) - V(t,X)] \\ &\leq -\beta[V(t,X)], \beta \in \mathscr{K}. \end{aligned} \quad (42)$$

Then, the trivial solution of SDEs (12) is equiasymptotic stable.

Theorem 9. *Suppose that the following conditions are satisfied:*

S3: for any $0 < \eta < \rho$, $(t,X) \in I \times S(\rho) \cap S^c(\eta)$, and $V \in C[I \times S(\rho) \cap S^c(\eta), \mathbb{R}_+]$, such that (33) and (34) hold;

S4: there exist $a, b \in \mathscr{K}$ satisfying

$$a\left(D^i[X,\theta]\right) \leq V(t,X) \leq b\left(D^i[X,\theta]\right), (t,X) \in I \times S(\rho). \quad (43)$$

Then, the trivial solution of SDEs (12) is uniformly stable.

Theorem 10. *Suppose that the conditions of Theorem 9 are satisfied except that the condition (S3) of Theorem 9 is strengthened to*

$$\begin{aligned} \mathscr{D}^+ V(t,X) &= \limsup_{h\longrightarrow 0^+} \frac{1}{h}[V(t+h, X + hF(t,X)) - V(t,X)] \\ &\leq -\varphi\left[\left(D^i[X,\theta]\right)\right], \end{aligned} \quad (44)$$

where $\varphi \in \mathscr{K}$. Then, the trivial solution of SDEs (12) is uniformly asymptotic stable.

Finally, we shall present the stability of SDEs in Fréchet space via the comparison principle which will unite the stability relations between SDEs and the comparison differential equation (15).

Theorem 11. *Suppose that the conditions B3 in Theorem 3 and S4 in Theorem 9 are satisfied, and*

S5: for $V \in C[I \times S(\rho), \mathbb{R}_+]$, and $(t,X), (t,Y) \in I \times S(\rho)$, such that

$$\begin{aligned} &|V(t,X) - V(t,Y)| \leq L^i D^i[X,Y], L > 0 \text{ is a constant}, \\ &\mathscr{D}^+ V(t,X) = \limsup_{h\longrightarrow 0^+} \frac{1}{h}[V(t+h, X + hF(t,X)) - V(t,X)] \\ &\qquad \leq g(t, V(t,X)), \end{aligned} \quad (45)$$

Then, the stable (uniform) stability properties of the trivial solution of (15) imply the corresponding equistable (uniform) stability of the trivial solution of SDEs (12).

Proof.

(i) Firstly, we prove the equistable of SDEs (12). Assume that the scalar differential equation (15) is stable. Hence, for the given $0 < \varepsilon < \rho$ and $b(\varepsilon)$, there exists a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$, such that

$$|u_0| < \delta_1 \text{provided} |w(t, t_0, w_0)| < b(\varepsilon), t \geq t_0. \quad (46)$$

Choosing $w_0 = V(t_0, X_0)$ and $\delta = \delta(t_0, \varepsilon) > 0$ satisfying $a(\delta) < \delta_1$.

Letting $D^i[X_0, \theta] < \delta$. Then, for any solution $X(t)$ of SDEs (12), we claim that $D^i[X(t), \theta] < \varepsilon$, $t \geq t_0$. If it is not true, then there exists a solution $X(t)$ of (12) with $D^i[X_0, \theta] < \delta$ and a $t_1 > t_0$ such that

$$D^i[X(t_1), \theta] = \varepsilon \text{ and } D^i[X(t), \theta] \leq \varepsilon < \rho, t_0 \leq t \leq t_1. \quad (47)$$

By Theorem 5, we obtain

$$V(t, X(t)) \leq r(t), t_0 \leq t \leq t_1. \quad (48)$$

Then,

$$V(t_0, X_0) \leq a\left(D^i[X_0, \theta]\right) < a(\delta) < \delta_1. \quad (49)$$

From (47) and (48), we get

$$b(\varepsilon) = b\left(D^i[X_1, \theta]\right) \leq V(t_1, X(t_1)) \leq r(t_1) < b(\varepsilon), \quad (50)$$

which is a contradiction. This proves that the trivial solution of SDE (12) is equistable.

(ii) Secondly, we prove the equiasymptotic stability of (12). Suppose that the solution of the scalar differential equation (15) is asymptotic stable. Then, the trivial solution of SDEs (12) is equistable. From (I), the trivial solution of comparison equation (15) is equistable. Therefore, we only need to prove that SDE (12) is equiattractive

Considering the attractivity of the trivial solution of (15), we take $\varepsilon = \rho$ and choose $\delta = \delta(t_0, \rho) > 0$. By setting $0 < \eta < \rho$, then for the given $b(\eta)$, there exist $\delta_2 = \delta_2(t_0) > 0$ and $T = T(t_0, \eta)$, such that

$$|u_0| < \delta_2 \text{ implies } r(t) < b(\eta), t \geq t_0 + T. \quad (51)$$

Choosing $w_0 = V(t_0, X_0)$ and $\delta_3 = \delta_3(t_0) > 0$ such that (47) holds, that is $a(\delta_3) < \delta_2$.

Let $\delta = \min\{\delta_2, \delta_3\}$, then using the conditions (S4) in Theorem 9, (51) and the result of Theorem 9, we obtain

$$b\left(D^i[X(t), \theta]\right) \leq V(t, X)) \leq r(t) < b(\eta), t \geq t_0 + T, \quad (52)$$

that is, the trivial solution of (12) is equiattractive. Furthermore, it is equiasymptotic stable.

(iii) Finally, the proof of the uniform stability is analogous to the proof of equistability, and what difference is only to choose the δ independent of t_0. The proof of the uniformly asymptotic stability is considered in the same way. Here, we omit the details.

□

Remark 12. Another way to prove the above Theorem is that, firstly, we transform the known conditions in Fréchet space into Banach space by the realization of the projective limit; then, we prove the stability results in Banach spaces; then, by the relation $D^i[X(\cdot), \theta] = D^{\mathbb{E}^i}[X^i(\cdot), \theta]$, where $X(\cdot) \in K_c(\mathbb{F})$ and $X(\cdot) \in K_c(\mathbb{E}^i)$, we can obtain the corresponding results.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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