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Research Article

Application of Müntz Orthogonal Functions on the Solution of the Fractional Bagley–Torvik Equation Using Collocation Method with Error Stimate

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This paper uses Müntz orthogonal functions to numerically solve the fractional Bagley–Torvik equation with initial and boundary conditions. Müntz orthogonal functions are defined on the interval $[0, 1]$ and have simple and distinct real roots on this interval. For the function $f \in L^2(0, 1)$, we obtain the best unique approximation using Müntz orthogonal functions. We obtain the Riemann–Liouville fractional integral operator for Müntz orthogonal functions so that we can reduce the complexity of calculations and increase the speed of solving the problem, which can be seen in the process of running the Maple program. To solve the fractional Bagley–Torvik equation with initial and boundary conditions, we use Müntz orthogonal functions and consider simple and distinct real roots of Müntz orthogonal functions as collocation points. By using the Riemann–Liouville fractional integral operator that we define for the Müntz orthogonal functions, the process of numerically solving the fractional Bagley–Torvik equation that is solved using Müntz orthogonal functions is reduced, and finally, we reach a system of algebraic equations. By solving algebraic equations and obtaining the vector of unknowns, the fractional Bagley–Torvik equation is solved using Müntz orthogonal functions, and the error value of the method can be calculated. The low error value of this numerical solution method shows the high accuracy of this method. With the help of the Müntz functions, we obtain the error bound for the approximation of the function. We have obtained the error bounds for the numerical method using which we solved the fractional Bagley–Torvik equation with initial and boundary conditions. Finally, we have given a numerical example to show the accuracy of the solution of the method presented in this paper. The results of solving this example using Müntz orthogonal functions and comparing the results with other methods that have been used the solve this example show the higher accuracy of the method proposed in this paper.

1. Introduction

To approximate the fractional Bagley–Torvik equation via Müntz orthogonal functions, we have the following [1]:

$$AD^2 f(t) + BD^{\frac{3}{2}} f(t) + Cf(t) = g(t), \tag{1}$$

with the initial condition

$$f(0) = f_0, f'(0) = f'_0, \tag{2}$$

or the boundary condition

$$f(0) = f_0, f(1) = f_1. \tag{3}$$

To solve the fractional Bagley–Torvik equation, several numerical solutions and analytical solutions have been used. Hybrid functions approximation [1] fractional-order Legendre collocation method [2], Haar wavelet [3], Laplacetransform [4], Laguerre polynomials [5], shifted Chebyshev operational matrix [6], Legendre artificial neural network method [7], Chebyshev collocation method [8], the fractional Taylor method [9], exponential integrators [10], Gegenbauer wavelet method [11], Müntz–Legendre polynomials [12], discrete spline methods [13], Hermit solution [14], local discontinuous Galerkin approximations [15], numerical inverse Laplace transform [16], generalized Fibonacci operational tau algorithm [17], Jacobi collocation methods [18], polynomial least squares method [19], and fast multiscale Galerkin algorithm [20] are methods by which Bagley–Torvik equation solved numerically. In the study of Alshammari et al. [21], residual power series are used to obtain the numerical solution of a class of Bagley–Torvik problems in Newtonian fluid, and in the study of Karaaslan et al. [22], using the discontinuous Galerkin method that can be combined in the equation of motion of a plate immersed in a Newtonian fluid, the numerical solution of Bagley–Torvik equation has been discussed. Analytical solutions of the generalized Bagley–Torvik equation [23], Sumudu transformation method [24], generalized differential transform [25], Sine–Gordon expansion method, and Bernoulli equation method [26] are analytical solutions for solving the Bagley–Torvik equation in this work.

Several numerical techniques have been proposed for solving fractional integrodifferential equations, such as Legendre wavelet [27], Euler function [28], Chebyshev series [29], and also methods such as stable least residue [30], discrete Galerkin [31], homotopy perturbation [32], variational interaction [33], Runge–Kutta convolution quadrature [34], and Hermite spectral collocation [35].

Fractional derivatives can be used to solve classical problems in viscous fluid motion [36]. A fluid half-space is considered, and the plate at the boundary is allowed to initiate a general transverse motion. It will be shown that the shear pressure at any point of the fluid can be expressed in terms of the time derivative of the fractional order for the fluid velocity characteristic. The equation of motion [36]

$$\rho \frac{\partial \nu}{\partial t} = \mu \frac{\partial^2 \nu}{\partial z^2}, \tag{4}$$

is a diffusion equation where ρ is the fluid concentration, μ is the viscosity, ν is the transverse velocity of the fluid, t is the time and z is the distance from the wet plate. By taking the Laplace transform, the following ordinary differential equation is obtained

$$\rho[s\overline{\nu}(s,z) - \nu(0,x)] = \mu \frac{d^2\overline{\nu}(s,z)}{dz^2}, \tag{5}$$

where in

$$\overline{\nu}(s,z) = \int_0^\infty e^{-st}\nu(t,z)dt = L[\nu(t,z)], \tag{6}$$

and $\nu(0,z)$ is the characteristic of the initial velocity of the fluid. If the initial velocity in the fluid is assumed to be zero and the boundary conditions are applied, then the fluid velocity at the wetted surface must match the plane velocity, and the fluid velocity in the half-space must be bounded. $\overline{\nu}_p(s)$ is the speed conversion of the plate, and

$$\overline{\nu}(s,z) = \overline{\nu}_p(s)e^{\left(\frac{\rho s}{\mu}\right)^{\frac{1}{2}}z}. \tag{7}$$

After obtaining the transformation of the fluid velocity characteristic, $\overline{\sigma}(s,z)$, by writing the transformation of the shear pressure relation of for a Newtonian fluid

$$\sigma(t,z) = \mu \frac{\partial \nu(t,z)}{\partial z}, \tag{8}$$

and

$$\overline{\sigma}(s,z) = \mu \frac{\partial \overline{\nu}(s,z)}{\partial z}, \tag{9}$$

the result for the pressure conversion in the speed conversion condition of Equation (7) as

$$\overline{\sigma}(s,z) = \sqrt{\mu\rho}\sqrt{s}\overline{\nu}(s,z), \tag{10}$$

it is expressed. This relation can be written again as a time relation. So

$$\overline{\sigma}(s,z) = \sqrt{\mu\rho}\frac{1}{\sqrt{s}}s\overline{\nu}(s,z), \tag{11}$$

and

$$\overline{\sigma}(s,z) = \sqrt{\mu\rho}L\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)t^{\frac{1}{2}}}\right]L\left[\frac{\partial \nu}{\partial t}\right]. \tag{12}$$

Therefore, the pressure is equal to the convolution of two functions of time, that is

$$\sigma(t,z) = \frac{\sqrt{\mu\rho}}{\Gamma\left(\frac{1}{2}\right)}\int_0^t \frac{\frac{\partial}{\partial \tau}\nu(\tau,z)}{(t-\tau)^{\frac{1}{2}}}d\tau. \tag{13}$$

Since $\nu(0,z)$ reached zero, it can be shown that Equation (13) is equivalent to [36]

$$\sigma(t,z) = \sqrt{\mu\rho}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\partial}{\partial t}\int_0^t \frac{\nu(\tau,z)}{(t-\tau)^{\frac{1}{2}}}d\tau. \tag{14}$$

Equation (14) can be written as follows:

$$\sigma(t,z) = \sqrt{\mu\tau}D_{(t)}^{\frac{1}{2}}[\nu(t,z)]. \tag{15}$$

The index (t) indicates the fractional derivative with respect to time. Equation (15) seems to be an unusual relationship between pressure and velocity, but using this method provides significant satisfaction in pressure–velocity relationships for Newtonian fluid. Equation (8) is a combination equation for a Newtonian fluid. Equation (15) describes the relationship between pressure and velocity in the fluid for any semi-infinite fluid domain and any allowed velocity at the boundary. The priority of this problem is to show the behavior of a real physical system using fractional derivatives [36].

This article is organized into six sections. In Section 1, the introduction is presented. Section 2 is devoted to the preliminary and practical definitions. In Section 3, Müntz orthogonal functions and the best unique approximation for the arbitrary function are introduced also the Riemann–Liouville fractional integral operator for the Müntz orthogonal functions is defined to shorten the process of solving the fractional Bagley–Torvik. Section 4 is dedicated to the method of numerical solution of the fractional Bagley–Torvik equation with initial-boundary conditions using orthogonal functions as well as the error bound of the presented method. In Section 5, a numerical example is given to demonstrate the efficiency and capability of the proposed method. Finally, Section 6 is allocated to the general conclusion.

2. Preliminaries and Notations

2.1. Fractional Integral and Derivative

Definition 1. The Riemann–Liouville fractional integral operator of the order α with the assumption $\alpha \in \mathbb{R}^+$ and I^α in $[a, b]$ is defined as follows [37]:

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{f(s)}{(t-s)^{1-\alpha}} ds = \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1} \times f(t) & \alpha > 0, \\ f(t) & \alpha = 0, \end{cases} \tag{16}$$

where $t^{\alpha-1} \times f(t)$ is the convolution product of $f(t)$ and $t^{\alpha-1}$.

Definition 2. The Caputo fractional derivative operator of order α with the assumption $\alpha \in \mathbb{R}^+$ and ${}^C D^\alpha$ is defined as follows [37]:

$$\begin{aligned} {}^C D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \\ n - 1 &< \alpha \leq n, \; n \in \mathbb{N}. \end{aligned} \tag{17}$$

As it is obvious for $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the Caputo derivative and Riemann–Liouville integral satisfy the following properties [37]:

1. $$I^\alpha({}^C D^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} f^k(0) \frac{t^k}{k!}, \tag{18}$$

2. $${}^C D^\alpha(f(t)) = I^{n-\alpha}\, {}^C D^n f(t), \; n - 1 < \alpha \leq n. \tag{19}$$

where λ_1, λ_2, and c are real constants.

Definition 3. The two-scale fractal derivatives with dimensions α and β with respect to t and x, respectively, are defined as follows [38]:

$$\frac{\partial f}{\partial t^\alpha}(t_0, x) = \Gamma(1+\alpha) \lim_{\substack{t - t_0 \longrightarrow \Delta t \\ \Delta t \neq 0}} \frac{f(t,x) - f(t_0, x)}{(t - t_0)^\alpha}, \tag{20}$$

$$\frac{\partial f}{\partial x^\beta}(t, x_0) = \Gamma(1+\beta) \lim_{\substack{x - x_0 \longrightarrow \Delta x \\ \Delta x \neq 0}} \frac{f(t,x) - f(t, x_0)}{(x - x_0)^\beta}. \tag{21}$$

Definition 4. The He's fractional derivative defined as $(0 < \alpha \leq 1)$ [39]

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (s-t)^{n-\alpha-1} (f_0(s) - f(s)) ds. \tag{22}$$

Definition 5. Suppose that $f \in H^r(0,1)$ with integers $r \geq 0$ where [40]

$$H^r(a,b) = \left\{ v \in \mathbb{C}^{r-1}([a,b]) : \frac{d}{dx} v^{r-1} \in L^2(a,b) \right\}, \tag{23}$$

is the Sobolev space. Let a function $f \in H^1(a,b)$ and $v \in (0,1)$. The Atangana–Baleanu–Caputo fractional derivative of order v of u with a based point a is defined as follows [41, 42]:

$${}^{ABC}_{\;\;a} D_t^v f(t) = \frac{B(v)}{1-v} \int_a^t f'(s) E_v \left[-\frac{v}{1-v} (t-s)^v \right] ds, \tag{24}$$

where $B(v)$ is defined as

$$B(v) = 1 - v + \frac{v}{\Gamma(v)'}, \tag{25}$$

one parameter and two-parameter Mittag–Leffler functions are defined as follows, respectively.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \; \alpha > 0, \; z \in \mathbb{C}. \tag{26}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; \alpha > 0, \; \beta > 0, \; z \in \mathbb{C}. \tag{27}$$

3. Müntz Orthogonal Functions and Their Properties

3.1. Müntz Orthogonal Functions

Definition 6. The family $\{P_n(t)\}_{n=0}^{\infty}$ is Müntz orthogonal functions and is defined as follows [43]:

Table 1: Roots of $P_n(t)$.

n	t_1	t_2	t_3	t_4	t_5
1	0.3678794412	–	–	–	–
2	0.06442096633	0.6374173264	–	–	–
3	0.01871588194	0.2651887508	0.7969679223	–	–
4	0.007047297639	0.1154772486	0.4569410332	0.8683835323	–
5	0.003221796109	0.05672067679	0.2565492462	0.5974812127	0.9100748739

$$P_n(t) = R_n(t) + S_n(t)\ \ln(t),\ n = 0, 1, 2, \ldots,\ t \in [0, 1]. \tag{28}$$

$R_n(t)$ and $S_n(t)$ are algebraic polynomials of degree $[n/2]$ and $[(n-1)/2]$, respectively, which are as follows:

$$R_n(t) = \sum_{\nu=0}^{\left[\frac{n}{2}\right]} a_\nu^{(n)} t^\nu,\ S_n(t) = \sum_{\nu=0}^{\left[\frac{n-1}{2}\right]} b_\nu^{(n)} t^\nu. \tag{29}$$

If $n = 2m$ is an even number, then for each $0 \le \nu \le m-1$, we have the following:

$$a_\nu^{(2m)} = -\binom{m+\nu}{m}^2 \binom{m}{\nu}^2 \left[\frac{2m+1}{2\nu+1} + 2(m-\nu) \sum_{j=0, j\neq\nu}^{m-1} \frac{2j+1}{(j-\nu)(j+\nu+1)}\right], \tag{30}$$

and

$$b_\nu^{(2m)} = -(m-\nu)\binom{m+\nu}{m}^2 \binom{m}{\nu}^2. \tag{31}$$

For $\nu = m$,

$$a_m^{(2m)} = \binom{2m}{m}^2,\ b_m^{(2m)} = 0. \tag{32}$$

If $n = 2m+1$ is an odd number, then for each $0 \le \nu \le m$, we have the following:

$$a_\nu^{(2m+1)} = \binom{m+\nu}{m}^2 \binom{m}{\nu}^2 \left[\frac{2m+1}{2\nu+1} + 2(m+\nu+1) \sum_{j=0, j\neq\nu}^{m} \frac{2j+1}{(j-\nu)(j+\nu+1)}\right], \tag{33}$$

and

$$b_\nu^{(2m+1)} = (m+\nu+1)\binom{m+\nu}{m}^2 \binom{m}{\nu}^2. \tag{34}$$

Some Müntz orthogonal functions are shown as follows:

$$\begin{aligned}
&P_0(t) = 1,\\
&P_1(t) = 1 + \ln(t),\\
&P_2(t) = -3 + 4t - \ln(t),\\
&P_3(t) = 9 - 8t + 2(1+6t)\ln(t),\\
&P_4(t) = -11 - 24t + 36t^2 - 2(1+18t)\ln(t),\\
&P_5(t) = 19 + 276t - 294t^2 + 3(1+48t+60t^2)\ln(t),\\
&P_6(t) = -21 - 768t + 390t^2 + 400t^3 - 3(1+96t+300t^2)\ln(t).
\end{aligned} \tag{35}$$

Table 1 shows the roots of $P_n(t)$ for $n = 1, 2, 3, 4, 5$.

Theorem 1. *The Müntz orthogonal function $P_n(t)$ for $n = 0, 1, 2, \ldots$, has exactly n simple and distinct real root in $[0, 1]$ [43].*

3.2. Function Approximation Using Orthogonal Müntz Basis. Let $\{P_0(t), P_1(t), \ldots, P_N(t)\}$ be a set of orthogonal Müntz functions and

$$Y = \operatorname{span}\{P_0(t), P_1(t), \ldots, P_N(t)\}. \tag{36}$$

If f is an arbitrary element of $L^2(0,1)$ then f is the best approximation out of Y such that $I_N f \in Y$, that is

$$\forall y \in Y : \|f - I_N f\| \le \|f - y\|, \tag{37}$$

where $I_N f \in Y$, there exist unique coefficients $c_0, c_1, \ldots, c_N$ such that

$$f \simeq I_N f = \sum_{n=0}^{N} c_n P_n(t) = C^T \varphi(t), \tag{38}$$

where

$$C^T = [c_0, c_1, \ldots, c_N], \tag{39}$$

and

$$\varphi^T = [P_0(t), \dots, P_N(t)]. \tag{40}$$

3.3. Fractional Riemann–Liouville Integral Operator for Müntz Functions. The Riemann–Liouville fractional integral I^α is defined in Equation (16) for orthogonal Müntz functions as follows:

$$I^\alpha \varphi(t) = \overline{\varphi}(t, \alpha), \tag{41}$$

where

$$\overline{\varphi}(t, \alpha) = [I^\alpha P_0(t), I^\alpha P_1(t), \dots, I^\alpha P_N(t)]. \tag{42}$$

To obtain $I^\alpha P_n(t)$ by taking the Laplace transform from Equation (28), we have the following:

$$\begin{aligned} L(P_n(t)) &= L\left(\sum_{\nu=0}^{\left[\frac{n}{2}\right]} a_\nu^{(n)} t^\nu + \sum_{\nu=0}^{\left[\frac{n-1}{2}\right]} b_\nu^{(n)} t^\nu \ln(t)\right) \\ &= \sum_{\nu=0}^{\left[\frac{n}{2}\right]} a_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\nu+1}} + \sum_{\nu=0}^{\left[\frac{n-1}{2}\right]} b_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\nu+1}} \left(\sum_{k=1}^{\nu} \frac{1}{k} - \ln(s)\right). \end{aligned} \tag{43}$$

By using Equation (16), we get the following:

$$\begin{aligned} L(I^\alpha P_n(t)) &= L\left(\frac{1}{\Gamma(\alpha)} t^{\alpha-1} \times P_n(t)\right) \\ &= \sum_{\nu=0}^{\left[\frac{n}{2}\right]} a_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\alpha+\nu+1}} + \sum_{\nu=0}^{\left[\frac{n-1}{2}\right]} b_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\alpha+\nu+1}} \left(\sum_{k=1}^{\nu} \frac{1}{k} - \ln(s)\right). \end{aligned} \tag{44}$$

Taking the inverse Laplace transform of Equation (44) yields $I^\alpha P_n(t)$

$$\begin{aligned} I^\alpha P_n(t) &= \sum_{\nu=0}^{\left[\frac{n}{2}\right]} a_\nu^{(n)} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha} \\ &\quad + \sum_{\nu=0}^{\left[\frac{n-1}{2}\right]} b_\nu^{(n)} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha} \left(\sum_{k=1}^{\nu} \frac{1}{k} - \sum_{l=1}^{\nu+\alpha} \frac{1}{l} + \ln(t)\right). \end{aligned} \tag{45}$$

4. Numerical Method and Its Error Estimation

4.1. Müntz Collocation Method. In this section, it is explained how the fractional Bagley–Torvik equation with initial conditions and boundary conditions is solved using Müntz orthogonal functions.

A) To solve Equation (1) with conditions given Equation (2), the Riemann–Liouville integral is taken from Equation (1)

$$I^2\left(A\, {}^C D^2 f(t) + B\, {}^C D^{\frac{3}{2}} f(t) + C f(t)\right) = I^2 g(t). \tag{46}$$

Using Equations (18) and (19), we have the following:

$$A(f(t) - f_0' t - f_0) + B\left(I^{\frac{1}{2}}(f(t) - f_0' t - f_0)\right) + C I^2 f(t) = I^2 g(t). \tag{47}$$

According to Equation (38), the approximation solution for $f(t)$ is as follows:

$$f(t) = C^T \varphi(t) = c_0 P_0(t) + c_1 P_1(t) + \cdots + c_N P_N(t). \tag{48}$$

According to Equation (41), we have the following:

$$I^2 f(t) = C^T \overline{\varphi}(t, 2),\ I^{\frac{1}{2}} f(t) = C^T \overline{\varphi}\left(t, \frac{1}{2}\right). \tag{49}$$

By substituting Equations (48) and (49) in Equation (47), the following equation is obtained

$$\begin{aligned} &A(C^T \varphi(t) - f_0' t - f_0) + B\left(C^T \overline{\varphi}\left(t, \frac{1}{2}\right)\right. \\ &\left. - I^{\frac{1}{2}}(f_0' t + f_0)\right) + C^T \overline{\varphi}(t, 2) = I^2 g(t). \end{aligned} \tag{50}$$

Interpolating Equation (50) in points t_i, $i = 0, 1, \dots, N$ which are the roots of Müntz orthogonal functions $P_{N+1}(t)$, we have the following:

$$\begin{aligned} &A(C^T \varphi(t_i) - f_0' t_i - f_0) + B\left(C^T \overline{\varphi}\left(t_i, \frac{1}{2}\right)\right. \\ &\left. - I^{\frac{1}{2}}(f_0' t_i + f_0)\right) + C^T \overline{\varphi}(t_i, 2) = I^2 g(t_i). \end{aligned} \tag{51}$$

These equations lead to getting $N+1$ algebraic equations, which can be solved for the unknown vector C^T. The results show that the value of $f(t)$ in Equation (48) can be approximated by using Müntz orthogonal functions.

B) To solve Equation (1) with conditions given Equation (3), we suppose

$${}^C D^2 f(t) = C^T \varphi(t) = c_0 P_0(t) + c_1 P_1(t) + \cdots + c_N P_N(t). \tag{52}$$

Using Equations (18) and (41), we have the following:

$$f(t) = C^T \overline{\varphi}(t, 2) + f_0' t + f_0. \tag{53}$$

Due to Equations (19) and (41), we also have the following:

$${}^C D^{\frac{3}{2}} f(t) = C^T \overline{\varphi}\left(t, \frac{1}{2}\right). \tag{54}$$

By substituting Equations (52)–(54) in Equation (1), the following equation is obtained

$$AC^T\varphi(t)+BC^T\overline{\varphi}\left(t,\frac{1}{2}\right)+C(C^T\overline{\varphi}(t,2)+f_0't+f_0)=g(t). \tag{55}$$

Let $t=1$ in Equation (53) and with conditions given Equation (3), we have the following:

$$f(1)=C^T\overline{\varphi}(1,2)+f_0'+f_0=f_1. \tag{56}$$

Now by interpolating Equation (55) in points t_i, $i=0,1,\ldots,N$ which are the roots of Müntz orthogonal functions $P_{N+1}(t)$ and considering Equation (56) that leads to generating $N+2$ equations and $N+2$ unknown coefficients which can be used to get the unknown vector C^T and f_0'.

Remark 1. One of the problems of this method is to find the roots of Müntz orthogonal functions for large n. With the help of maple software, We calculated the for $n\le 60$ up to 50 meaningful digits and there was numerical stability.

4.2. Error Estimation. First consider the following lemma that will be used in deriving our main convergence results.

Lemma 1. *Let $I_Nf=\sum_{n=0}^{N}c_nP_n(t)$ be the best approximation of f in Y. Then, if $r\le N+1$ we have the following:*

$$\|f-I_Nf\|_{L^2(0,1)}\le c(N+1)^{-r}\left\|f^{(r)}\right\|_{L^2(0,1)}, \tag{57}$$

and for $1\le\mu\le r$ we have the following:

$$\|f-I_Nf\|_{H^\mu(0,1)}\le c(N+1)^{2\mu-\frac{1}{2}-r}\left\|f^{(r)}\right\|_{L^2(0,1)}, \tag{58}$$

where c is a constant depending only on r.

Proof. Suppose that L_Nf be the truncated Legendre series of the function f, then according to Equation (5.4.11) in the study of Canuto et al. [44] for $r\le N+1$ we have the following:

$$\|f-L_Nf\|^2_{L^2(0,1)}\le c(N+1)^{-2r}\left\|f^{(r)}\right\|^2_{L^2(0,1)}. \tag{59}$$

AS I_Nf is the best approximation of f in L^2-norm, we can write

$$\|f-I_Nf\|^2_{L^2(0,1)}=\|f-L_Nf\|^2_{L^2(0,1)}\le c(N+1)^{-2r}\left\|f^{(r)}\right\|^2_{L^2(0,1)}, \tag{60}$$

and Equation (57) is proved. Equation (58) is proved using Equation (5.5.11) in the study of Canuto et al. [44] in a similar manner. □

Theorem 2. *If $f\in H^r(0,1)$ and $r\ge0$ then the error bound E_N for Equation (1) with the conditions given in Equation (2) is as following:*

$$\|E_N\|_{L^2(0,1)}\le\left(c(N+1)^{-r}\left\|f^{(r)}\right\|_{L^2(0,1)}\right)\left(A+\frac{B}{\Gamma\left(\frac{3}{2}\right)}+\frac{C}{\Gamma(3)}\right). \tag{61}$$

Proof. According to Equation (47), we have the following:

$$\begin{aligned}\|E_N\|_{L^2[0,1]}&=\left\|A\left(I_Nf(t)-f_0't-f_0\right)+B\left(I^{\frac{1}{2}}\left(I_Nf(t)-f_0't-f_0\right)\right)+CI^2I_Nf(t)-I^2g(t)\right\|_{L^2(0,1)}\\&=\|A\left(I_Nf(t)-f_0't-f_0\right)+B\left(I^{\frac{1}{2}}\left(I_Nf(t)-f_0't-f_0\right)\right)+CI^2I_Nf(t)\\&\quad-A\left(f(t)-f_0't-f_0\right)-B\left(I^{\frac{1}{2}}\left(f(t)-f_0't-f_0\right)\right)-CI^2f(t)_{L^2(0,1)}\\&\le A\|I_Nf(t)-f(t)\|_{L^2(0,1)}+B\left\|I^{\frac{1}{2}}I_Nf(t)-I^{\frac{1}{2}}f(t)\right\|_{L^2(0,1)}+C\|I^2I_Nf(t)-I^2f(t)\|_{L^2(0,1)}.\end{aligned} \tag{62}$$

From Equation (16) and Equation (57), we have the following:

$$\begin{aligned}\left\|I^{\frac{1}{2}}f(t)-I^{\frac{1}{2}}I_Nf(t)\right\|_{L^2(0,1)}&=\left\|\frac{1}{\Gamma\left(\frac{1}{2}\right)}t^{-\frac{1}{2}}\times(I_Nf(t)-f(t))\right\|_{L^2(0,1)}\\&\le\frac{1}{\Gamma\left(\frac{3}{2}\right)}\|I_Nf(t)-f(t)\|_{L^2(0,1)}\\&\le\frac{1}{\Gamma\left(\frac{3}{2}\right)}c(N+1)^{-r}\left\|f^{(r)}\right\|_{L^2(0,1)}.\end{aligned} \tag{63}$$

Similarly

$$\begin{aligned}\|I^2f(t)-I^2I_Nf(t)\|_{L^2(0,1)}&=\left\|\frac{1}{\Gamma(2)}t\times(I_Nf(t)-f(t))\right\|_{L^2(0,1)}\\&\le\frac{1}{\Gamma(3)}\|I_Nf(t)-f(t)\|_{L^2(0,1)}\\&\le\frac{1}{\Gamma(3)}c(N+1)^{-r}\left\|f^{(r)}\right\|_{L^2(0,1)}.\end{aligned} \tag{64}$$

Then Equation (61) is obtained based on Equations (57) and (62)–(64). □

Theorem 3. *If $f \in H^r(0,1)$ and $r \geq 0$, the error bound $\overline{E}_N$ for Equation (1) with the conditions in Equation (3) is as follows:*

$$\|\overline{E}_N\|_{L^2(0,1)} \leq \left(c(N+1)^{2\mu-\frac{1}{2}-r}\|f^{(r)}\|_{L^2(0,1)}\right)\left(A+\frac{B}{\Gamma\left(\frac{3}{2}\right)}\right) +Cc(N+1)^{-r}\|f^{(r)}\|_{L^2(0,1)}. \tag{65}$$

Proof. According to Equation (1), we have the following:

$$\begin{aligned}\|\overline{E}_N\|_{L^2(0,1)} &= \|A\,{}^CD^2I_Nf(t)+B\,{}^CD^{\frac{3}{2}}I_Nf(t)+CI_Nf(t)-g(t)\|_{L^2(0,1)}\\ &= \|A\,{}^CD^2I_Nf(t)+B\,{}^CD^{\frac{3}{2}}I_Nf(t)+CI_Nf(t)-A\,{}^CD^2f(t)-B\,{}^CD^{\frac{3}{2}}f(t)-Cf(t)\|_{L^2(0,1)}\\ &= \|A({}^CD^2I_Nf(t)-{}^CD^2f(t))+B\left({}^CD^{\frac{3}{2}}I_Nf(t)-{}^CD^{\frac{3}{2}}f(t)\right)+C(I_Nf(t)-f(t))\|_{L^2(0,1)}\\ &\leq A\|{}^CD^2I_Nf(t)-{}^CD^2f(t)\|_{L^2(0,1)}+B\|{}^CD^{\frac{3}{2}}I_Nf(t)-{}^CD^{\frac{3}{2}}f(t)\|_{L^2(0,1)}+C\|I_Nf(t)-f(t)\|_{L^2(0,1)}.\end{aligned} \tag{66}$$

Using Equations (16), (19), and (58), we have the following:

$$\begin{aligned}&\|{}^CD^{\frac{3}{2}}I_Nf(t)-{}^CD^{\frac{3}{2}}f(t)\|_{L^2(0,1)}\\ &=\|I^{\frac{1}{2}}\,{}^CD^2I_Nf(t)-I^{\frac{1}{2}}\,{}^CD^2f(t)\|_{L^2(0,1)}\\ &=\left\|\frac{1}{\Gamma\left(\frac{1}{2}\right)}t^{-\frac{1}{2}}\times({}^CD^2I_Nf(t)-{}^CD^2f(t))\right\|_{L^2(0,1)}\\ &\leq\frac{1}{\Gamma\left(\frac{3}{2}\right)}\|{}^CD^2I_Nf(t)-{}^CD^2f(t)\|_{L^2(0,1)}\\ &\leq\frac{1}{\Gamma\left(\frac{3}{2}\right)}\|I_Nf(t)-f(t)\|_{H^\mu(0,1)}\\ &\leq\frac{1}{\Gamma\left(\frac{3}{2}\right)}c(N+1)^{2\mu-\frac{1}{2}-r}\|f^{(r)}\|_{L^2(0,1)}.\end{aligned} \tag{67}$$

Also

$$\begin{aligned}\|{}^CD^2I_Nf(t)-{}^CD^2f(t)\|_{L^2(0,1)} &\leq\|I_Nf(t)-f(t)\|_{H^\mu(0,1)}\\ &\leq c(N+1)^{2\mu-\frac{1}{2}-r}\|f^{(r)}\|_{L^2(0,1)}.\end{aligned} \tag{68}$$

Then Equation (65) is obtained based on Equations (57) and (66)–(68). □

5. Numerical Example

In this section, a numerical example for solving the fractional Bagley–Torvik equation with initial-boundary conditions is given to show the efficiency and applicability of the numerical method in this paper.

Example 1. Consider following properties [1, 6]:

$$g(t)=\begin{cases}8 & 0\leqslant t<1,\\ 0 & t>1.\end{cases} \tag{69}$$

$$f(t)=8(y_U(t)-y_U(t-1)), \tag{70}$$

if

$$g(t)=8(U(t)-U(t-1)), \tag{71}$$

where

$$y_U(t)=U(t)\left(\frac{1}{A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{C}{A}\right)^r t^{2(r+1)}E^{(r)}_{\frac{1}{2},\frac{3r}{2}+3}\left(\frac{-B}{A}t^{\frac{1}{2}}\right)\right). \tag{72}$$

In the two-parameter Mittag-Leffler function with λ, $\mu>0$ indices, is equal to

$$E^{(r)}_{\lambda,\mu}(y)\equiv\frac{d^r}{dy^r}E_{\lambda,\mu}(y)=\sum_{j=0}^{\infty}\frac{(j+r)!y^j}{j!\Gamma(\lambda j+\lambda r+\mu)},\ r=0,1,2,\ldots. \tag{73}$$

$$f(0)=f_0=0,\ f'(0)=f'_0=0,\ f(1)=f_1=2.952583880. \tag{74}$$

By solving this example with the initial conditions and coefficients $A=1$, $B=1/2$, and $C=1/2$ are considered, then the approximate values of the function $f(t)$ are obtained. These approximate values in Table 2 are compared with the approximate values in the study of Mashayekhi and Razzaghi [1], which was obtained by using combined functions, and in the study of Ji et al. [6], which was obtained by using the shifted Chebyshev operational matrix. In Figure 1, the logarithmic values of the absolute errors for different values of N are depicted, which shows the exponential convergence.

Table 3 shows the results of the comparison of the method presented in this paper with the method in the study of Uddin and Ahmad [4], which is used to solve the fractional Bagley–Torvik equation using the Laplace transform.

Table 2: Approximate values of $f(t)$ for $A=1$, $B=1/2$ and $C=1/2$ and $N=16$.

t	Method [1] ($M=3, N=8$)	Method [6] ($N=16$)	Present method ($N=16$)	Exact
0.1	0.0364875	0.036487532	0.03648747992	0.03648747990
0.2	0.1406398	0.140639669	0.14063962121	0.14063962117
0.3	0.3074848	0.3074844733	0.30748462710	0.30748462713
0.4	0.5332842	0.533283636	0.53328410990	0.53328410988
0.5	0.8147568	0.814758247	0.81475694936	0.81475694938
0.6	1.1488372	1.148848315	1.14883742229	1.14883742227
0.7	1.5325655	1.532537770	1.53256542649	1.53256542650
0.8	1.9630293	1.963013767	1.96302925484	1.96302925484
0.9	2.4373338	2.437896842	2.43733397083	2.43733397084
1.0	2.9525839	–	2.95258387995	2.95258388004

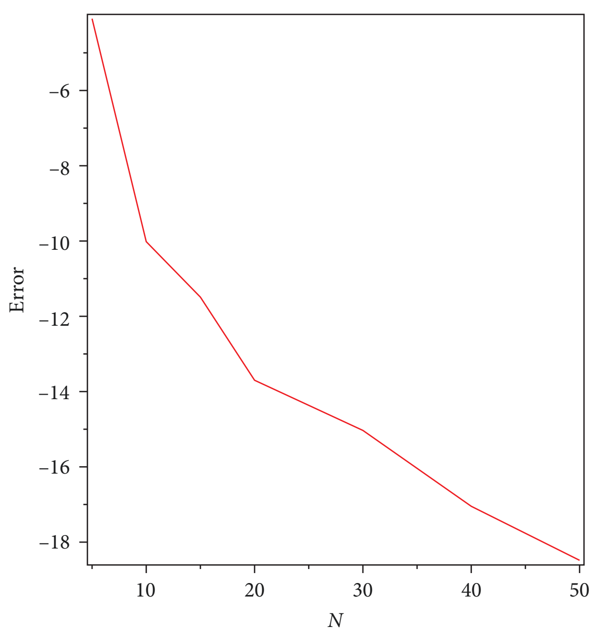

Figure 1: Logarithmic values of absolute errors for $A=1$, $B=1/2$ and $C=1/2$.

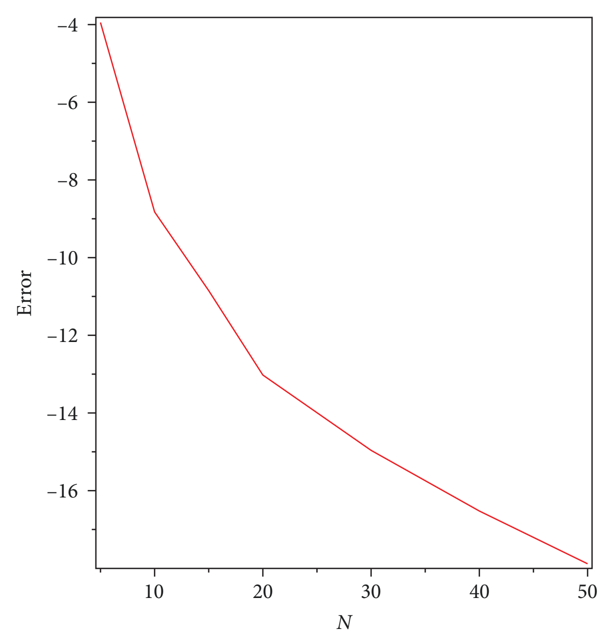

Figure 2: Logarithmic values of absolute errors for $A=B=C=1$.

Table 3: Absolute errors of $f(t)$ for $A=1$, $B=1/2$, and $C=1/2$.

N	Method [4] (C_1)	Method [4] (C_2)	Method [4] (C_3)	Present method
10	$7.7161e-3$	$4.9000e-8$	$1.0000e-9$	$9.6216e-11$
20	$1.0552e-5$	$1.0000e-9$	$1.0000e-9$	$2.0024e-14$
23	$9.8800e-7$	$1.0000e-9$	$1.0000e-9$	$3.1158e-14$
50	$1.0000e-9$	$1.0000e-9$	$1.0000e-9$	$3.2920e-19$
60	$1.0000e-9$	$1.0000e-9$	$2.9000e-8$	$2.9127e-19$
90	$1.0000e-9$	$2.0000e-9$	$1.1000e-1$	$2.2831e-24$
100	$1.0000e-9$	$1.3700e-7$	$9.5387e-0$	$1.3631e-24$

Table 4: Absolute errors of $f(t)$ for $A=B=C=1$.

t	Method [1] ($M=4, N=8$)	Present method ($N=14$)
0.5	$2.4e-9$	$1.2e-10$
1	$8.4e-9$	$1.0e-9$
2	$6.3e-8$	$2.3e-9$
5	$2.8e-4$	$1.2e-6$

C_1, C_2, and C_3 are three different paths for optimal parameters [4].

If the coefficients $A=B=C=1$ are considered, the values of logarithmic values of absolute errors are shown in Figure 2. In Table 4, the absolute errors of the function $f(t)$ for $A=B=C=1$ using the method presented in this paper are compared with the method in the study of Mashayekhi and Razzaghi [1].

Table 5 shows the results of the comparison of the method presented in this paper with the method in the study of Zolfaghari et al. [45], which is used to solve the fractional Bagley–Torvik equation using the homotopy perturbation method.

Moreover, by solving this example with the boundary conditions and coefficients $A=1$, $B=1/2$, and $C=1/2$ are

Table 5: Approximate values of $f(t)$ for $A = B = C = 1$ and $N = 10$.

t	Method [45]	Present method $N = 10$	Exact
0	0.0000	0.000000000	0.000000000
0.1	0.0335	0.033507355	0.033507310
0.2	0.1252	0.125221248	0.125221280
0.3	0.2677	0.267609447	0.267609420
0.4	0.4555	0.455435396	0.455435398
0.5	0.6843	0.684334730	0.684334753
0.6	0.9501	0.950392575	0.950392563
0.7	1.2494	1.249959149	1.249959133
0.8	1.5788	1.579557186	1.579557209
0.9	1.9351	1.935832303	1.965832293
1	2.3149	2.315525744	2.315525823

Table 6: Approximate values of $f(t)$ for $A = 1$, $B = 1/2$ and $C = 1/2$ and different values of N.

t	$N = 10$	$N = 30$	$N = 60$	Exact
0.1	0.036487404375	0.036487479864	0.0364874799011	0.0364874799009
0.2	0.140639724816	0.140639621157	0.1406396211738	0.1406396211740
0.3	0.307484496274	0.307484627118	0.3074846271334	0.3074846271337
0.4	0.533284101747	0.533284109806	0.5332841098751	0.5332841098756
0.5	0.814757087316	0.814756949317	0.8147569493835	0.8147569493833
0.6	1.148837366742	1.148837422295	1.1488374222698	1.1488374222703
0.7	1.532565319887	1.532565426547	1.5325654264985	1.5325654264982
0.8	1.963029329581	1.963029254846	1.9630292548366	1.9630292548369
0.9	2.437333969513	2.437333970854	2.4373339708438	2.4373339708440
1.0	2.952583880039	2.952583880039	2.9525838800390	2.9525838800390

Table 7: Approximate values of $f(t)$ for $A = B = C = 1$ and different values of N.

t	$N = 10$	$N = 30$	$N = 60$	Exact
0.1	0.033507162346	0.0335073099353	0.0335073100169	0.0335073100166
0.2	0.125221479995	0.1252212803110	0.1252212803466	0.1252212803471
0.3	0.267609140575	0.2676094195004	0.2676094195331	0.2676094195339
0.4	0.455435400934	0.4554353974487	0.4554353975927	0.4554353975939
0.5	0.684335033706	0.6843347530036	0.6843347531505	0.6843347531500
0.6	0.950392421075	0.9503925632445	0.9503925632016	0.9503925632028
0.7	1.249958912764	1.2499591326644	1.2499591325649	1.2499591325644
0.8	1.579557379566	1.5795572087353	1.5795572087127	1.5795572087133
0.9	1.935832277274	1.9358322930399	1.9358322930172	1.9358322930176
1.0	2.315525822793	2.3155258227926	2.3155258227926	2.3155258227926

considered, then the approximate values of the function $f(t)$ are obtained. In Table 6, approximate values are compared with exact values.

If the coefficients $A = B = C = 1$ are considered, in Table 7, the approximate values of function $f(t)$ for $N = 10$, $N = 30$, and $N = 60$, are compared with its exact values. As we can see, increasing the value N leads to a low absolute error and high accuracy of the method, which proves the stability of the proposed method.

6. Conclusions

In this paper, we utilize Müntz orthogonal functions to provide the approximate solution of the fractional Bagley–Torvik equation subjected to initial-boundary conditions. First, the functional equations are introduced, which help to solve the problem effectively. Then the fractional integral operator is defined as Müntz orthogonal functions that shorten the solution of the Bagley–Torvik fractional equation in a system of algebraic equations. The roots of Müntz orthogonal functions on the interval $[0, 1]$ are simple and distinct considered collocation points. Furthermore, solving these algebraic equations help us to approximate the desired function using Müntz orthogonal functions. In the end, several numerical examples with initial-boundary conditions are solved with this method. The results obtained by the present method are computationally more accurate compared with the existing methods for solving the fractional Bagley–Torvik equation.

Data Availability

No data were used for the research described in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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