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Research Article

A New Fundamental Asymmetric Wave Equation and Its Application to Acoustic Wave Propagation

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The irreducible representations of the extended Galilean group are used to derive the symmetric and asymmetric wave equations. It is shown that among these equations only a new asymmetric wave equation is fundamental. By being fundamental the equation gives the most complete description of propagating waves as it accounts for the Doppler effect, forward and backward waves, and makes the wave speed to be the same in all inertial frames. To demonstrate these properties, the equation is applied to acoustic wave propagation in an isothermal atmosphere, and to determine Lamb's cutoff frequency.

1. Introduction

In modern physics, a dynamical equation is called fundamental if it is local, has its own Lagrangian, and remains invariant with respect to the spatiotemporal transformations that form a group of the metric and internal symmetries that form a gauge group [1]. The latter is related to interactions, which do not affect the description of free particles and waves. The law of inertia for classical particles [2–4] and the Schrödinger [5] and Lévy-Leblond [6] equations for quantum particles are examples of nonrelativistic fundamental equations. Moreover, all the basic equations of relativistic classical and quantum physics are fundamental [1].

In nonrelativistic physics, space and time are Galilean, and their metrics are $ds_1^2 = dx^2 + dy^2 + dz^2$ and $ds_2^2 = dt^2$, respectively, with x, y, and z being the spatial coordinates and t being time. All transformations that leave the Galilean metrics unchanged form the Galilean group of the metric [6], meaning that the metrics preserve their forms in all inertial frames, and observers associated with these frames are called *Galilean observers*.

For free particles in classical mechanics (CM), the law of inertia is fundamental because it is local, has its well-known Lagrangian [2–4], and is also invariant with respect to the Galilean group of the metric [6]. This means that for all Galilean observers, the form of the equation describing this law is the same. On the other hand, the second law of dynamics may or may not be Galilean invariant depending on the form of its force [2–4].

The wave equation of classical physics describes the propagation of waves in a given background medium [7–9]; however, the wave equation is not fundamental because it is not Galilean invariant [6, 7]. There are two main reasons: (i) the Galilean metrics require wave equations to be asymmetric in space and time derivatives and (ii) Galilean invariance requires the wave speed to be the same in all inertial frames, which is violated by any classical wave propagating slower than the speed of light. Both reasons apply to the wave equation; therefore, it is not fundamental.

The Galilean group of the metric can be extended to make its structure similar to that of the Poincaré group [10, 11]. Let $\mathscr{G}_e$ be the extended Galilean group [9–12] with its mathematical structure $\mathscr{G}_e = [O(3) \otimes_s B(3)] \otimes_s [T(3+1) \otimes U(1)]$, where $O(3)$ and $B(3)$ are subgroups of rotations and boosts, respectively. In addition, $T(3+1)$ is an invariant Abelian subgroup of combined translations in space and time, and $U(1)$ is a one-parameter unitary subgroup. The subgroup $T(3+1)$ plays an important role in $\mathscr{G}_e$ because its irreducible representations (irreps) are well-known [13, 14], and they provide labels for all the irreps of $\mathscr{G}_e$ [13–15].

The Schrödinger equation of quantum mechanics (QM) is Galilean invariant since its form remains the same when all transformations of $\mathcal{G}_e$ are applied to it; however, the invariance requires that the phase factor is introduced [5, 16–18]. By being Galilean invariant, local, and with its Lagrangian known, the Schrödinger equation is the fundamental equation of QM as its form remains the same for all Galilean observers. Moreover, the scalar wavefunction of the equation transforms as one of the irreps of $\mathcal{G}_e$, which guarantees that all Galilean observers identify the same physical object represented by the function.

The Lévy-Leblond equation [(6)] whose spinor wave function describes elementary particles with spin in nonrelativistic QM is linear, has a Lagrangian, and is also Galilean invariant, which means that it is fundamental. Thus, the Lévy-Leblond and Schrödinger equations are two fundamental wave equations of nonrelativistic QM, and they describe particles with and without spin, respectively.

In relativistic classical physics, the wave equation for electromagnetic waves is fundamental since the speed of light remains the same in all inertial frames. However, there is no similar equation in nonrelativistic physics for classical waves propagating with speeds lower than the speed of light. Previous attempts to use different forms of the Schrödinger equation to describe the propagation of classical waves were made [19–23], but the resulting equations were not fundamental. Therefore, the main aim of this paper is to derive such an equation by following the recent work [24] in which a new asymmetric wave equation was discovered, and used to formulate a theory of cold dark matter [25].

In this paper, the conditions for the new asymmetric wave equation to become a fundamental wave equation for classical waves are established and discussed. To compare the wave description given by the nonfundamental and fundamental wave equations, both formulations are used to describe the propagation of acoustic waves in an isothermal atmosphere and to determine Lamb's cutoff frequency.

The paper is organized as follows: in Section 2, the basic equations are derived and discussed; the wave equations and their Lagrangians are obtained in Section 3; Galilean invariance of the wave equations is investigated in Section 4; applications of the obtained results to acoustic wave propagation are presented in Section 5; and conclusions are given in Section 6.

2. Derivation of Symmetric and Asymmetric Equations

The invariant Abelian subgroup $T(3+1)$ of combined translations in space and time plays an important role in $\mathcal{G}_e$ because its irreps are well-known [11, 13–15] and they provide labels for all the irreps of $\mathcal{G}_e$ [13, 14]. The conditions that the scalar wave function $\phi(t,\mathbf{x})$ transforms as one of the irreps of $\mathcal{G}_e$ are given by the following eigenvalue equations [17, 18] (see the appendix for their derivation):

$$i\frac{\partial}{\partial t}\phi(t,\mathbf{x}) = \omega\phi(t,\mathbf{x}), \tag{1}$$

$$-i\nabla\phi(t,\mathbf{x}) = \mathbf{k}\phi(t,\mathbf{x}), \tag{2}$$

where $\phi(t,\mathbf{x})$ is an eigenfunction of the generators of $T(3+1)$, and the eigenvalues ω and $\mathbf{k}$ are real constants that label the irreps. The generator of translation in time is $\hat{E} = i\partial/\partial t$, and the generator of translations in space is $\hat{P} = -i\nabla$, with $[\hat{E},\hat{P}] = 0$ being the commuting operators. The group $\mathcal{G}_e$ also has the generator of boosts given by $\hat{V} = t\hat{P}$, which means that the eigenvalues for the operators $\hat{V}$ and $\hat{P}$ must be the same [17, 18]. The fact that $\phi(t,\mathbf{x})$ obeys Equations (1) and (2) and transforms as one of the irreps of $\mathcal{G}_e$ means that all Galilean observers identify the same object, which is a wave under consideration, and their description of this wave is identical.

The obtained eigenvalue equations can be used to derive all wave equations of physics for scalar wave functions that are allowed to exist in Galilean space and time. In general, the derived dynamical equations can be divided into two separate families, namely, the symmetric equations, with the same order of space and time derivatives, and the asymmetric equations, with different orders of space and time derivatives [17]. Moreover, the equations can be of any order [18], but in this paper, only the second-order equations are considered.

The only second-order symmetric equation that can be derived from the eigenvalue equations is

$$\left[\frac{\partial^2}{\partial t^2} - C_1\nabla^2\right]\phi(t,\mathbf{x}) = 0, \tag{3}$$

where $C_1 = \omega^2/k^2$, with $k^2 = (\mathbf{k}\cdot\mathbf{k})$. Since C_1 is a real constant coefficient of an arbitrary value, there is an infinite set of these second-order equations, and they are called *wave-like equations* [(24)].

Two different asymmetric second-order equations resulting from Equations (1) and (2) can also be obtained [24]

$$\left[i\frac{\partial}{\partial t} + C_2\nabla^2\right]\phi(t,\mathbf{x}) = 0, \tag{4}$$

$$\left[\frac{\partial^2}{\partial t^2} - iC_3\mathbf{k}\cdot\nabla\right]\phi(t,\mathbf{x}) = 0, \tag{5}$$

where $C_2 = \omega/k^2$ and $C_3 = \omega^2/k^2 = C_1$ are arbitrary constants. This means that there are two infinite sets of second-order asymmetric equations.

The form of Equation (4) is the same as that of the Schrödinger equation [5], except for the presence of the coefficient C_2. Therefore, all equations of the same form as Equation (4) are called *Schrödinger-like equations*. However, the equations are given by Equation (5) with different coefficients C_3 are called *new asymmetric equations*, as originally named when the equation was first introduced [24]. It must also be noted that the constants C_1, C_2, and C_3 are expressed in terms of the eigenvalues, which label the irreps of $\mathcal{G}_e$.

In the previous work [24], it was shown that by using the de Broglie relationship [5], the coefficient C_2 expressed in terms of the labels of the irreps ω and k can be evaluated, and it becomes the same as the coefficient in the Schrödinger equation of QM [5]. The obtained Schrödinger equation does not include any potentials, which means that it describes free quantum particles of ordinary matter. Moreover, the coefficient C_3 of the new asymmetric equation was also evaluated, and the resulting equation was used to describe a quantum structure of dark matter particles [25]. In the following section, the coefficients C_1, C_2, and C_3 are evaluated in such a way that the resulting equations describe classical waves.

3. Wave Equations for Classical Waves

There are infinite sets of symmetric (Equation (3)) and asymmetric (Equations (4) and (5)) equations. To select equations that describe classical waves, the constants C_1, C_2, and C_3 must be expressed in terms of the wave frequency and wave vector as well as the wave speed. This can be achieved by identifying the labels of the irreps ω and k as the wave frequency and wave number, respectively, and introducing the characteristic wave speed, $c_w = \omega/k$. Then, Equation (3) becomes

$$\left[\frac{\partial^2}{\partial t^2} - c_w^2 \nabla^2\right]\phi(t, \mathbf{x}) = 0, \tag{6}$$

which is the well-known standard wave equation SWE [7–9]. Moreover, the Schrödinger-like and new asymmetric wave equations for classical waves can be written as

$$\left[i\frac{\partial}{\partial t} + \frac{c_w^2}{\omega}\nabla^2\right]\phi(t, \mathbf{x}) = 0, \tag{7}$$

$$\left[\frac{\partial^2}{\partial t^2} - ic_w^2 \mathbf{k}\cdot\nabla\right]\phi(t, \mathbf{x}) = 0. \tag{8}$$

Observe that the obtained standard, Schrödinger-like, and new asymmetric wave equations are of different forms, and yet they can be used to describe the free propagation of classical waves, as it is now demonstrated.

If the wave speed c_w is constant in the above wave equations, then it is easy to verify that the solutions to the SWE given by Equation (6) are either

$$\phi(t, \mathbf{x}) = Ae^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} + Be^{-i(\omega t + \mathbf{k}\cdot\mathbf{r})}, \tag{9}$$

or

$$\phi(t, \mathbf{x}) = Ce^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} + De^{i(\omega t + \mathbf{k}\cdot\mathbf{r})}, \tag{10}$$

where A, B, C, and D are constants to be determined by specifying boundary conditions. The solutions given by Equations (9) and (10) are equivalent, and they reflect the fact that $\sqrt{-1} = \pm i$, which means that the choice of solutions is a matter of convention, and it has no physical effect [7–9]. Moreover, the first and second solutions in Equation (9) describe the forward and backward waves, respectively, and the same is true for Equation (10). Substitution of any solution presented above into the SWE results in the dispersion relation $\omega^2 = k^2 c_w^2$, which verifies the choice of $C_1 = c_w^2$ selected for Equation (3).

For the Schrödinger-like wave equation given by Equation (7), the only solutions that describe classical waves are given by Equation (9) as after substituting any of these two solutions into the equation, the dispersion relation $\omega^2 = k^2 c_w^2$ is obtained; this relation justifies the choice of $C_2 = c_w^2/\omega$ in Equation (4). However, the solutions given by Equation (10) lead to the dispersion relation $\omega^2 = -k^2 c_w^2$, which does not represent waves, instead $\omega = -ikc_w$ describes exponentially decaying oscillations in the background medium.

The new asymmetric wave equation given by Equation (7) allows only for the solutions that can be written in the following form

$$\phi(t, \mathbf{x}) = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + De^{i(\mathbf{k}\cdot\mathbf{r} + \omega t)}, \tag{11}$$

where the first and second solutions represent the forward and backward propagating waves, respectively. Substituting any of these two solutions into the new asymmetric wave equation gives the dispersion relation $\omega^2 = k^2 c_w^2$, which justifies the choice of $C_3 = c_w^2$ in Equation (5). The two other remaining solutions of Equations (9) and (10) give the dispersion relation $\omega^2 = -k^2 c_w^2$, which does not describe waves but instead exponentially decaying oscillations with $\omega = +ik c_w$ in a medium where the waves propagate.

The presented results demonstrate that all three considered wave equations account for both the forward and backward waves, whose dispersion relations are the same, namely, $\omega^2 = k^2 c_w^2$, which allows expressing the coefficients C_1, C_2, and C_3 in terms of the wave speed c_w. It is also shown that the standard wave equation allows for two solutions identified with either $+i$ or $-i$. However, the Schrödinger-like wave equation is limited to the solutions with $-i$, while the new asymmetric wave equation allows only for the solutions with $+i$, which means that these two wave equations are complementary.

The derived three-wave equations are second order, thus, they are local, which is one of the requirements for them to be fundamental. Since the wave equations describe freely propagating waves, the requirement of gauge invariance [1] does not have to be considered. However, it remains to be determined whether these equations have Lagrangians, and whether they are Galilean invariant or not.

4. Lagrangians for Wave Equations

The Lagrangian formalism requires prior knowledge of a Lagrangian from which a dynamical equation is derived. Typically, the Lagrangians are presented without explaining their origin because there are no methods to derive them from first principles; however, for some systems, a Lagrangian can be constructed by accounting for the invariance of physical laws, the invariance of a physical system under consideration, and the structure of its equations (linear or nonlinear, driven or undriven, damped or undamped, etc.). Historically, most equations of physics were established first, and only then their Lagrangians were found, often by guessing. Once the Lagrangians are known, the process of finding the resulting equations is straightforward requiring the substitution of these Lagrangians into the Euler-Lagrange (E-L) equation. Despite some progress in deriving Lagrangians for physical systems described by ordinary differential equations (ODEs) (e.g., [26–31]), similar work for partial differential equations (PDEs) has only limited applications (e.g., [9, 32, 33]).

Let $L(\phi, \partial_t\phi, \nabla\phi)$, where $\partial_t = \partial/\partial t$, be a Lagrangian that satisfies the E-L equation

$$\frac{\partial L}{\partial \phi} - \partial_t\left(\frac{\partial L}{\partial(\partial_t\phi)}\right) - \nabla\cdot\left(\frac{\partial L}{\partial(\nabla\phi)}\right) = 0. \tag{12}$$

Substituting $L(\phi, \partial_t\phi, \nabla\phi)$ into Equation (12) gives the required dynamical equation if, and only if, the Lagrangian is a priori known. In case the equation is given first, its Lagrangian must be constructed in such a way that when substituted into Equation (12), the desired dynamical equation is obtained; this is the Lagrangian formalism.

Since the SWE given by Equation (6) is hyperbolic, its Lagrangian can be constructed [9, 32], and the result is

$$L_{swe}(\partial_t\phi, \nabla\phi) = \frac{1}{2}\left[c_w^{-2}(\partial_t\phi)^2 - (\nabla\phi)^2\right]. \tag{13}$$

It is easy to verify that the substitution of the Lagrangian $L = L_{\mathrm{swe}}(\partial_t\phi, \nabla\phi)$ into Equation (12) gives the required symmetric wave-like equation.

The Schrödinger-like wave equation given by Equation (7) is parabolic; thus, its Lagrangian must be of a special form involving both ϕ and its complex conjugate ϕ^* [32]. The form of this Lagrangian is

$$L_{Sch}(\phi, \phi^*, \partial_t\phi, \partial_t\phi^*, \nabla\phi, \nabla\phi^*) = \frac{i}{2}(\phi^*\partial_t\phi - \phi\partial_t\phi^*) - \frac{c_w^2}{\omega}(\nabla\phi^*)\cdot(\nabla\phi). \tag{14}$$

This Lagrangian gives the Schrödinger-like wave equation when substituted into the E-L equation for the variations in ϕ^*. On the other hand, the variations in ϕ lead to the complex conjugate Schrödinger-like wave equation, which becomes important when the probability density $|\phi|^2$ is required. However, in theories of classical waves, $|\phi|^2$ does not play any significant role as it does in QM [5].

To find the Lagrangian for the new asymmetric wave equation given by Equation (8), the Lagrangian for the Schrödinger-like wave equation must be modified as

$$\begin{aligned} L_{asy}(\phi, \phi^*, \partial_t\phi, \partial_t\phi^*, \nabla\phi, \nabla\phi^*) &= (\partial_t\phi^*)(\partial_t\phi) \\ &+ + \frac{i}{2}c_w^2[\phi^*(\mathbf{k}\cdot\nabla\phi) - \phi(\mathbf{k}\cdot\nabla\phi^*)]. \end{aligned} \tag{15}$$

Then, this Lagrangian is substituted into the E-L equation

$$\frac{\partial L_{asy}}{\partial \phi^*} - \partial_t\left(\frac{\partial L_{asy}}{\partial(\partial_t\phi^*)}\right) - (\mathbf{k}\cdot\nabla)\cdot\left(\frac{\partial L_{asy}}{\partial(\mathbf{k}\cdot\nabla\phi^*)}\right) = 0, \tag{16}$$

and the new asymmetric wave equation (see Equation (8)) is obtained.

The presented results demonstrate that Lagrangians exist for the symmetric and asymmetric wave equations and that these equations are local. Therefore, the last requirement for a wave equation to be called fundamental is its Galilean invariance, which is now investigated.

5. Galilean Invariance of Wave Equations

5.1. Known Fundamental Equations of Nonrelativistic Physics. Let S and S' be two inertial frames moving with respect to each other with the velocity $\mathbf{v}$ = const, which allows writing a boost as $\mathbf{x} = \mathbf{x}' + \mathbf{v}t'$ with $t' = t$. Then, the Galilean metric in space is $ds^2 = d\mathbf{x}\cdot d\mathbf{x} = dx^2 + dy^2 + dz^2$ with $ds^2 = ds'2$, and in time $dt^2 = dt'2$. By performing the Galilean transformations (translations in space and time, rotations, and boosts) that form the Galilean group of the metric [10] or the extended Galilean group $\mathcal{G}_e$ [12], Galilean invariance of the metrics can be verified. The invariance means that the forms of the metrics remain the same for all Galilean observers.

Similarly, for a dynamical equation to be Galilean invariant, it is required that the form of the equation remains the same in all inertial frames. This means that the coefficients of this equation must also be the same in all inertial frames. With the equation retaining its form, the solutions of this equation are also the same for all Galilean observers. The simplest example is the second-order ODE describing the law of inertia, whose invariance with respect to all transformations that form the Galilean group of the metrics is well-known [3, 10]. It is also known that the Lagrangian of the law of inertia is not Galilean invariant [2, 3, 10]; however, it was recently shown that Galilean invariance of the Lagrangian can be restored by using the so-called null Lagrangians [34]. Thus, the law of inertia is a fundamental equation of classical mechanics.

As shown above, space and time in Galilean relativity are separated and obey different metrics. Therefore, for dynamical equations to be Galilean invariant they must be asymmetric in time and space derivatives. The ODE describing the law of inertia is asymmetric as it does not have any space derivative. However, among the wave equations obtained in this paper and given by Equations (6), (7), and (8), the SWE is symmetric and the two other wave equations are asymmetric. As a result, *the SWE is not Galilean invariant* and, thus, *it is not fundamental.* In other words, for classical particles, the law of inertia is the fundamental equation, but there is no corresponding fundamental equation for classical waves. The main objective of this paper is to find such an equation and apply it to wave theories.

The Schödinger equation of QM is asymmetric and its Galilean invariance is well-known, requiring a phase factor, whose form is frame-dependent [5, 12, 16–18]. The existence of this phase factor makes the wavefunction to be different for each Galilean observer, which may imply that the equation is not Galilean invariant. However, the presence of the phase factor in the solutions does not violate Galilean invariance because in QM only the square of the absolute value of the wavefunction is the measurable quantity, and this quantity remains the same for all Galilean observers. Thus, the Schrödinger equation for free quantum particles is a fundamental equation of QM. Similarly, the Lévy-Leblond equation for its spinor wavefunction is Galilean invariant and a fundamental equation of QM [6, 12]. Moreover, the Lagrangians for the Schrödinger and Lévy-Leblond equations are Galilean invariant.

Having demonstrated that the law of inertia and the Schrödinger and Lévy-Leblond equations are the fundamental equations of nonrelativistic physics, and the SWE cannot be a fundamental equation for classical waves, it remains now to determine whether the Schrödinger-like and new asymmetric equations are fundamental.

5.2. Schrödinger-Like Wave Equation. Applying the Galilean transformations to Equation (7), the transformed Schrödinger-like wave equation can be written as

$$\left[i\frac{\partial}{\partial t'} + \frac{c_w'2}{\omega'}\nabla'2\right]\phi'\left(t',\mathbf{x}'\right) = 0, \tag{17}$$

where the original and transformed wave functions are related by

$$\phi(t,\mathbf{x}) = \phi\left(t',\mathbf{x}' + \mathbf{v}t'\right) = \phi'\left(t',\mathbf{x}'\right)e^{i\eta\left(t',\mathbf{x}'\right)}, \tag{18}$$

with the phase factor given by

$$\eta\left(t',\mathbf{x}'\right) = \frac{\omega'}{2c_w'^2}\left(\mathbf{v}\cdot\mathbf{x}' + v^2t'/2\right). \tag{19}$$

For the obtained transformed Schrödinger-like wave equation to be Galilean invariant, it is also required that $c_w^2/\omega = c_w'2/\omega'$. This condition is satisfied when

$$\mathbf{k}' = \mathbf{k} - \frac{\omega}{2c_w^2}\mathbf{v}, \tag{20}$$

$$\omega' = \omega\left(1 + \frac{v^2}{4c_w^2}\right) - \mathbf{k}\cdot\mathbf{v}. \tag{21}$$

The above results demonstrate that the Schrödinger-like wave equation preserves its form in all inertial frames if, and only if, the wavefunction transforms according to Equation (18), and Equations. (20) and (21) are satisfied. The existence of the phase factor given by Equation (19), which is a frame-dependent quantity, is well-known, and its presence does not violate Galilean invariance of the Schödinger equation in QM because of its requirement that only $|\phi(t,\mathbf{x})|^2 = |\phi'(t',\mathbf{x}')|^2$ must be valid for all Galilean observers [5, 12, 16–18].

For classical waves, the wavefunction $\phi(t,\mathbf{x})$ represents one of the physical variables describing a wave; thus, to get the same wave description by all Galilean observers, the solutions for $\phi(t,\mathbf{x})$ and $\phi'(t',\mathbf{x}')$ must be the same in all inertial frames. However, they are not because of the presence of the phase factor (see Equation (18)) that is different in different inertial frames. As a result, Galilean observers describe waves differently in their inertial frames, which means that the Schrödinger-like equation for classical waves is not Galilean invariant, and therefore it is not fundamental.

5.3. New Asymmetric Wave Equation. Let $\phi(t,\mathbf{x})$ be the wavefunction of Equation (8) and $\phi(\mathbf{x}',t')$ be the transformed wavefunction. After performing the Galilean transformations, Equation (8) becomes

$$\left[\frac{\partial^2}{\partial t'2} - ic_w'2\mathbf{k}'\cdot\nabla'\right]\phi\left(t',\mathbf{x}'\right) = \left[2\left(\mathbf{v}\cdot\nabla'\right)\frac{\partial}{\partial t'} - \left(\mathbf{v}\cdot\nabla'\right)^2\right]\phi\left(t',\mathbf{x}'\right). \tag{22}$$

A comparison of this equation to Equation (8) shows that its LHS is of the same form as the new asymmetric wave equation if, and only if, the RHS is zero. Let $\phi'(t',\mathbf{x}')$ be the wavefunction that satisfies the RHS of Equation (22). As already demonstrated [24], the solution to the RHS of Equation (22) is any function $\phi'(t',\mathbf{x}') = \phi'(\mathbf{r}')$, where $\mathbf{r}' = \mathbf{x}' + \mathbf{v}t'/2$.

Then, with $\phi(t',\mathbf{x}') = \phi'(t',\mathbf{x}') = \phi'(\mathbf{r}')$, the LHS of Equation (22) can be written as

$$\left[\frac{d^2}{d\mathbf{r}'2} - i\left(\frac{2c_w'}{v}\right)^2\mathbf{k}'\cdot\frac{d}{d\mathbf{r}'}\right]\phi'\left(\mathbf{r}'\right) = 0. \tag{23}$$

Using the Galilean transformations, $\mathbf{r}' = \mathbf{x} - \mathbf{v}t/2 \equiv \mathbf{r}$, and Equation (8) becomes

$$\left[\frac{d^2}{d\mathbf{r}^2} - i\left(\frac{2c_w}{v}\right)^2\mathbf{k}\cdot\frac{d}{d\mathbf{r}}\right]\phi(\mathbf{r}) = 0, \tag{24}$$

where $\phi(\mathbf{r}) = \phi(t,\mathbf{x}) = \phi(t',\mathbf{x}') = \phi'(t',\mathbf{x}') = \phi'(\mathbf{r}')$. Then, Equations (23) and (24) are of the same form if, and only if, $c_w' = c_w$ and $\mathbf{k}' = \mathbf{k}$. To show that these conditions are valid, let the phase of a wave in the inertial frame S' be given by

$$\mathbf{k}'\cdot\mathbf{r}' = \mathbf{k}'\cdot\mathbf{x}' + \frac{1}{2}\left(\mathbf{k}'\cdot\mathbf{v}\right)t' = \mathbf{k}'\cdot\mathbf{x}' + \frac{1}{2}\left(\frac{\omega'}{c_w'}\right)\left(\widehat{\mathbf{k}'}\cdot\hat{\mathbf{v}}\right)vt', \tag{25}$$

where $\hat{\mathbf{k}}$ and $\hat{\mathbf{v}}$ are unit vectors corresponding to $\mathbf{k}$ and $\mathbf{v}$, respectively. It must also be noted that the dispersion relation $\omega' = k'c_w'$ was used to obtain the wave phase.

After the Galilean transformations, Equation (25) reduces to

$$\mathbf{k}'\cdot\mathbf{r}' = \mathbf{k}'\cdot\mathbf{x} - \frac{1}{2}\left(\frac{\omega'}{c_w'}\right)\left(\widehat{\mathbf{k}'}\cdot\hat{\mathbf{v}}\right)vt, \tag{26}$$

which represents the forward waves in an inertial frame S' (see Equation (11)). However, since $(\hat{\mathbf{k}}'\cdot\hat{\mathbf{v}}) = \cos\theta'$ can be either positive or negative, $\mathbf{k}' \cdot \mathbf{r}'$ may also describe the backward waves if $\cos\theta' < 0$. This can be fixed by writing

$$\mathbf{k}' \cdot \mathbf{r}' = \mathbf{k}' \cdot \mathbf{x} \pm \frac{1}{2}\left(\frac{\omega'}{c_w'}\right)\left|\widehat{\mathbf{k}'} \cdot \hat{\mathbf{v}}\right| vt, \tag{27}$$

where the + and − signs correspond to the backward and forward waves, respectively.

On the other hand, the wave phase in an inertial frame S is

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{x} \pm \frac{1}{2}\left(\frac{\omega}{c_w}\right)\left|\hat{\mathbf{k}} \cdot \hat{\mathbf{v}}\right| vt, \tag{28}$$

with the + and − signs corresponding to the backward and forward waves, respectively.

The requirement of Galilean invariance is that the wave phases are the same in all inertial frames, which means that $\mathbf{k}' \cdot \mathbf{r}' = \mathbf{k} \cdot \mathbf{r}$. Hence,

$$\left(\mathbf{k}' - \mathbf{k}\right) \cdot \mathbf{x} \pm \frac{1}{2}\left[\frac{\omega'}{c_w'}\left|\widehat{\mathbf{k}'} \cdot \hat{\mathbf{v}}\right| - \frac{\omega}{c_w}\left|\hat{\mathbf{k}} \cdot \hat{\mathbf{v}}\right|\right] vt = 0, \tag{29}$$

which is only satisfied when $\mathbf{k}' = \mathbf{k}$ and $\omega'/c_w' = \omega/c_w$.

With $\mathbf{k}' = \mathbf{k} = \text{const}$ (see Equation (2)), Equations (22) and (23) can be written in the following form:

$$\frac{d}{d\left(\mathbf{k}' \cdot \mathbf{r}'\right)}\left[\frac{d}{d\left(\mathbf{k}' \cdot \mathbf{r}'\right)} - i\left(\frac{2c_w'}{v}\right)^2\right]\phi'\left(\mathbf{k}' \cdot \mathbf{r}'\right) = 0, \tag{30}$$

$$\frac{d}{d(\mathbf{k} \cdot \mathbf{r})}\left[\frac{d}{d(\mathbf{k} \cdot \mathbf{r})} - i\left(\frac{2c_w}{v}\right)^2\right]\phi(\mathbf{k} \cdot \mathbf{r}) = 0. \tag{31}$$

Since $\mathbf{k}' \cdot \mathbf{r}' = \mathbf{k} \cdot \mathbf{r}$, $\phi'(\mathbf{k}' \cdot \mathbf{r}') = \phi(\mathbf{k} \cdot \mathbf{r})$, and $c_w' = c_w$, the above equations are of the same form, they are Galilean invariant, and this invariance does not require any phase factor. However, the form of Equation (31) is very different from that of the original new asymmetric equation given by Equation (8), which means that in order for this equation to be fundamental, the existence of its Lagrangian must be established.

5.4. Fundamental Wave Equation for Classical Waves. The Galilean invariant equation (Equation (31)) is an ordinary differential equation, whose Lagrangian can be found by one of the methods previously developed for ODEs (e.g., [26–31]). The Lagrangian for Equation (31) can be written as

$$L_{as}(d_{kr}\phi, \mathbf{k} \cdot \mathbf{r}) = \frac{1}{2}[d_{kr}\phi(\mathbf{k} \cdot \mathbf{r})]^2 e^{-4i(\mathbf{k} \cdot \mathbf{r})c_w^2/v^2}, \tag{32}$$

where $d_{kr} = d/d(\mathbf{k} \cdot \mathbf{r})$. The derived Lagrangian depends on the wave phase $\mathbf{k} \cdot \mathbf{r}$ that involves both $\mathbf{x}$ and t.

In CM, the dependence of Lagrangians on t implies that the total energy of a dynamical system is not conserved and, as a result, the energy function must be calculated [3, 4]. For physical systems with their Lagrangians explicitly time-dependent, the exponentially decaying or increasing terms are present, like in the well-known Caldirola-Kanai Lagrangian [35, 36], originally written for the Bateman oscillator [37, 38]. However, the Lagrangian given by Equation (32) is of a different form as its exponential term is periodic in $\mathbf{k} \cdot \mathbf{r}$ instead. Since the first term on the RHS in Equation (32) represents the wave kinetic energy, the exponential term shows that this energy is required to oscillate in time and space in the Lagrangian so that the correct wave equation is obtained. This is a new phenomenon in classical waves and, thus, $L_{\text{new}}(d_{kr}\phi, \mathbf{k} \cdot \mathbf{r})$ forms a separate class among all Lagrangians known in physics.

To demonstrate that the Lagrangian $L_{\text{as}}(d_z\phi, z)$ is Galilean invariant, the Galilean transformations are applied, and the following transformed Lagrangian is found

$$L_{\text{as}}'\left(d_{\text{kr}}'\phi', \mathbf{k}' \cdot \mathbf{r}'\right) = \frac{1}{2}\left[d_{\text{kr}}'\phi'\left(\mathbf{k}' \cdot \mathbf{r}'\right)\right]^2 e^{-4i\left(\mathbf{k}' \cdot \mathbf{r}'\right)c_w'{}^2/v^2}, \tag{33}$$

where $d_{\text{kr}}' = d/d(\mathbf{k}' \cdot \mathbf{r}')$. The transformed Lagrangian is of the same form as the original one given by Equation (32) because $\mathbf{k} \cdot \mathbf{r} = \mathbf{k}' \cdot \mathbf{r}'$, $\phi(\mathbf{k} \cdot \mathbf{r}) = \phi'(\mathbf{r}' \cdot \mathbf{k}')$, and $c_w^2/v^2 = c_w'^2/v^2$. Therefore, the Lagrangian $L_{\text{as}}(d_{\text{kr}}\phi, \mathbf{k} \cdot \mathbf{r})$ is Galilean invariant.

After substituting the Lagrangian given by Equation (32) into the E-L equation

$$d_{\text{kr}}\left(\frac{dL_{\text{as}}}{d(d_{\text{kr}}\phi)}\right) - \frac{dL_{\text{as}}}{d\phi} = 0, \tag{34}$$

the following equation is obtained

$$\left[\frac{d^2\phi}{d(\mathbf{k} \cdot \mathbf{r})^2} - i\left(\frac{2c_w}{v}\right)^2\frac{d\phi}{d(\mathbf{k} \cdot \mathbf{r})}\right]e^{-4i(\mathbf{k} \cdot \mathbf{r})c_w^2/v^2} = 0. \tag{35}$$

Since $e^{-4i(\mathbf{k} \cdot \mathbf{r})c_w^2/v^2} \neq 0$, the terms in the square brackets must be zero, which gives Equation (31). This shows that in addition to being local and Galilean invariant, Equation (31) can also be derived from the Lagrangian given by Equation (32). With its Lagrangian known and Galilean invariance of the Lagrangian verified, Equation (31) is the new fundamental asymmetric wave equation or simply the fundamental wave equation (FWE). By being fundamental, the FWE gives the most comprehensive description of free classical waves, as it accounts for the Doppler effect, the forward and backward waves, and makes the wave speed to be the same in all inertial frames.

The wave speed c_w is constant for all Galilean observers, and since $v = \text{const}$, the coefficient $4c_w^2/v^2 = \text{const}$. This is an interesting result. It shows that this coefficient plays a similar role for classical waves in Galilean relativity as the speed of light c plays in the special theory of relativity (STR) for electromagnetic (EM) waves. However, while $c = \text{const}$ is the basic principle of nature and the foundation of STR, the coefficient $4c_w^2/v^2 = \text{const}$ is the necessary condition for

Galilean invariance, and its validity is guaranteed by the existence of the FWE and by selecting the wave phase as the variable representing the waves.

Thus, the main result of this paper is that classical waves propagating at speeds $c_w << c$ may "mimic" the behavior of EM waves in STR when they are described by the FWE. For this reason, it is suggested that these waves be called the *basic classical waves* in Galilean Relativity.

6. Applications to Acoustic Wave Propagation

6.1. Freely Propagating Acoustic Waves. Acoustic waves propagate freely in uniform media, and the solutions of the SWE that describe such propagation are given by Equations (9) and (10), with the wave frequency ω and the wave vector $\mathbf{k}$ being frame-dependent (the Doppler effect); this means that Galilean observers see plane waves with their frequencies and wave vectors being different in their respective inertial frames moving with constant velocity $\mathbf{v}$. Therefore, the SWE is not Galilean invariant, and thus it is not fundamental.

Finding the solutions to the FWE given by Equation (31) is straightforward. After two integrations, it yields

$$\phi(\mathbf{k}\cdot\mathbf{r}) = c_1 e^{i\theta_s} + c_2, \quad (36)$$

where c_1 and c_2 are integration constants, and the phase of the acoustic wave is

$$\theta_s \equiv \left(\frac{2c_s}{v}\right)^2 (\mathbf{k}\cdot\mathbf{r}) = \left(\frac{2c_s}{v}\right)^2 \left[\mathbf{k}\cdot\mathbf{x} \pm \frac{1}{2}\left(\frac{v}{2c_s}\right)\left|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}\right|\omega t\right], \quad (37)$$

where $c_w \equiv c_s$ is the speed of sound. The solution for $\phi(\mathbf{k}\cdot\mathbf{r})$ describes both the forward and backward propagating acoustic waves (see Equation (11)). The conditions $\mathbf{k}\cdot\mathbf{r} = \mathbf{k}'\cdot\mathbf{r}'$ and $(2c_s/v)^2 = (2c_s'/v)^2$ guarantee that the solution is the same in all inertial frames and that it accounts for the Doppler effect. Thus, the above solution shows that its description of acoustic waves freely propagating in a uniform medium is much more comprehensive than that given by the SWE.

In the next section, the assumption of uniform media is removed and a gradient of density is included, making the background medium stratified.

6.2. Lamb's Cutoff Frequency. In his original work, Lamb [39–41] considered acoustic waves propagating in the z-direction in the background medium with gravity $\vec{g} = -g\hat{z}$ and density gradient $\rho_0(z) = \rho_{00}\exp(-z/H)$, where ρ_{00} is the gas density at the height $z = 0$, and $H = c_s^2/\gamma g$ is the density scale height, with γ denoting the ratio of specific heats. In his model, the background gas pressure p_0 and gas density ρ_0 vary with height z; however, the temperature T_0 remains constant. As a result, H = const and c_s = const.

This stratified but otherwise isothermal medium is often referred to as an *isothermal atmosphere*, and acoustic waves in this atmosphere are described by the following variables: velocity $u(t, z)$, pressure $p(t, z)$ and density $\rho(t, z)$ perturbations. The resulting acoustic wave equation (AWE) is derived for the transformed wave variables $u_1(t, z) = u(t, z)\rho_0^{1/2}$, $p_1(t, z) = p(t, z)\rho_0^{-1/2}$, and $\rho_1(t, z) = \rho(t, z)\rho_0^{-1/2}$ using the hydrodynamic equations [40–43]. The resulting wave equation can be written as

$$\left[\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial z^2} + \Omega_{ac}^2\right][u_1(t, z), p_1(t, z), \rho_1(t, z)] = 0, \quad (38)$$

where the speed of sound is $c_s = [\gamma p_0(z)/\rho_0(z)]^{1/2} = [\gamma R T_0/\mu]^{1/2}$, while the acoustic cutoff frequency $\Omega_{ac} = c_s/2H = \gamma g/2c_s$ remains constant in the entire isothermal atmosphere [39–43]. The Lamb cutoff frequency describes the effects of the atmospheric density gradient on acoustic wave propagation, and it is used to determine the wave propagation conditions (see Section 6.4). Note also that the form of the wave equation is the same for each wave variable in an isothermal atmosphere.

The fact that the form of the derived AWE remains the same at every height in an isothermal atmosphere is well-known, and it was first shown by Lamb [39–41]. However, different inertial observers see the waves differently, namely, with their different characteristic speeds, frequencies, and wave vectors. Different waves seen in different inertial frames mean that the theory of waves based on the AWE is not fundamental because it is not the same for all Galilean observers.

In numerous studies of propagation of acoustic waves that followed Lamb's work, different aspects of the wave propagation were investigated by using methods based on either global and local dispersion relations, or the WKB approximation, or finding analytical solutions to acoustic wave equations for special cases [44–46]. A method to determine the cutoff frequency for linear and adiabatic acoustic waves propagating in nonisothermal media without gravity was also developed [43] based on transformations of wave variables that lead to standard wave equations and using the oscillation theorem to determine the turning point frequencies. Physical arguments are used to select the largest of these frequencies as the Lamb cutoff frequency. In this paper, the Lamb cutoff frequency is obtained for the new fundamental wave equation.

6.3. Fundamental Wave Equation and Lamb's Cutoff Frequency. The acoustic wave equation given by Equation (38) is obtained from the hydrodynamic equations. It is easy to show that neither the Schrödinger-like wave equation nor the new asymmetric wave equation can be derived using only the hydrodynamic equations. However, both wave equations can be derived from the hydrodynamic equations if, and only if, these equations are supplemented by the eigenvalue equations. Specifically, the Schrödinger-like wave equation is obtained when the eigenvalue equation given by Equation (1) is applied to Equation (38). However, the Schrödinger-like wave equation is not fundamental (see Section 5.2); therefore, the equation will not be further considered here.

Instead, the new asymmetric equation given by Equation (7) is considered. By applying the eigenvalue equation given by Equation (2) to Equation (38), the following equation is obtained

$$\left[\frac{\partial^2}{\partial t^2} - \mathrm{ik} c_s^2 \frac{\partial}{\partial z} + \Omega_{ac}^2\right]\phi(t, z) = 0. \quad (39)$$

For the considered acoustic wave propagation along the z-axis, the label $\mathbf{k}$ of the irreps of $\mathscr{G}_e$ is identified with the wave vector and $k = \mathbf{k} \cdot \hat{z}$. In addition, the wavefunction $\phi(t, z)$ represents one of the acoustic wave variables in Equation (38).

The results presented in Section 5.3 demonstrate that the new asymmetric wave equation can be converted into a form that is Galilean invariant (see Equation (31)). Applying the results to Equation (39), the resulting wave equation is

$$\left[\frac{d^2}{d\chi^2} - i\left(\frac{2c_s}{v}\right)^2 \frac{d}{d\chi} + \left(\frac{2\Omega_{\mathrm{ac}}}{\mathrm{kv}}\right)^2\right]\phi(\chi) = 0, \quad (40)$$

where $\chi = \mathbf{k} \cdot \mathbf{r} = k(z \pm |\hat{z} \cdot \hat{v}| vt/2)$. Since $\Omega_{\mathrm{ac}} = \gamma g / 2c_s$, with $c_s = c_s'$, $k = k'$, and with γ and g being the same in all inertial frames, Equation (40), and its solutions are the same for all Galilean observers; this means that the derived equation is the FWE for the considered acoustic waves. The obtained FWE describes the effects of an isothermal atmosphere on acoustic wave propagation. Thus, Equation (40) generalizes Equation (31), which describes only freely propagating acoustic waves in a medium without any gradients.

As a result of the Galilean transformations, the term that represents Lamb's cutoff frequency is now modified by the factor 2/kv, which describes the effects of moving inertial frames on the cutoff. These effects are more prominent for smaller velocities v and wavevectors k. All the presented results are valid for $v > 0$ (see Section 5.1), which means that if S' moves with respect to S with velocity $\mathbf{v}$, then S moves with respect to S' with velocity $-\mathbf{v}$. In the case where there is only one stationary inertial frame with $v = 0$, this frame must be treated separately by using Equation (8) that is not Galilean transformed. It is also important to point out that any Galilean observer may boost its inertial frame to the wave frame by setting $v = c_s$.

6.4. Conditions for Acoustic Wave Propagation. As originally demonstrated by Lamb [39–41], the frequency Ω_{ac} uniquely determines whether acoustic waves in an isothermal atmosphere are propagating or evanescent. Since $\Omega_{\mathrm{ac}} = \mathrm{const}$ in the isothermal atmosphere, after making the Fourier transforms in time and space, the AWE (Equation (38)) gives the global dispersion relation: $(\omega^2 - \Omega_{\mathrm{ac}}^2) = k^2 c_s^2$, where ω is the wave frequency and $k = \mathbf{k} \cdot \hat{z}$ is the wave vector. The obtained dispersion relation is valid in one selected inertial frame, which is called *stationary*. In this frame, the waves are propagating when $\omega > \Omega_{\mathrm{ac}}$ and k is real, and they are nonpropagating (evanescent) when either $\omega = \Omega_{\mathrm{ac}}$ with $k = 0$ or $\omega < \Omega_{\mathrm{ac}}$ with k being imaginary.

When a Galilean observer moves with velocity v with respect to the stationary frame, then the wave frequency (the Doppler effect), wave vector, and characteristic wave speed change, which means that $(\omega' 2 - \Omega_{\mathrm{ac}}' 2) = k' 2 c_s' 2$; the dispersion relation preserves its form, but the values of the wave parameters change from one inertial frame to another. With $\Omega_{\mathrm{ac}} \neq \Omega_{\mathrm{ac}}'$, the acoustic cutoff frequency is different in different inertial frames.

The same conditions for wave propagation are obtained when the Fourier transforms in time and space are performed in the new asymmetric wave equation (Equation (39)), and the result is $(\omega^2 - \Omega_{ac}^2) = k^2 c_s^2$, which is the same as the dispersion relation obtained for the AWE. Thus, the conditions for wave propagation are also the same. However, neither the AWE given by Equation (38) nor the new asymmetric wave equation given by Equation (39) is fundamental. The only FWE is given by Equation (40). The conditions for wave propagation resulting from this equation are now determined and discussed.

To find the conditions for acoustic wave propagation in an isothermal atmosphere, the FWE given by Equation (40) must be solved. The obtained solutions $\phi_1(\chi)$ and $\phi_2(\chi)$ are

$$\phi_{1,2}(\chi) = \exp\left[\frac{i}{2}\left(\frac{2c_s}{v}\right)^2\left(1 \pm \sqrt{1 + \left(\frac{v}{c_s}\right)^2 \frac{\Omega_{\mathrm{ac}}^2}{\omega^2 - \Omega_{\mathrm{ac}}^2}}\right)\chi\right], \quad (41)$$

and their superposition gives the general solution $\phi(\chi) = C_1\phi_1(\chi) + C_2\phi_2(\chi)$. Note that the dispersion relation $k^2 c_s^2 = (\omega^2 - \Omega_{\mathrm{ac}}^2)$ was used to derive Equation (41). Using the dispersion relation, the wave phase $\chi = \mathbf{k} \cdot \mathbf{r} = k(z \pm |\hat{z} \cdot \hat{v}| vt/2)$ can be written as

$$\chi = \left(\frac{z}{c_s} \pm \frac{1}{2}\frac{v}{c_s}|\hat{z} \cdot \hat{v}| t\right)\sqrt{\omega^2 - \Omega_{\mathrm{ac}}^2}, \quad (42)$$

which allows writing the solutions given by Equation (41) in the following form

$$\theta_{1,2}(t, z) = \frac{c_s}{v}\left[\sqrt{\omega^2 - \Omega_{ac}^2} \pm \sqrt{\omega^2 - \left(1 - \frac{v^2}{c_s^2}\right)\Omega_{ac}^2}\right]\left(\frac{2z}{v} \pm |\hat{z} \cdot \hat{v}| t\right), \quad (43)$$

and the general solution becomes

$$\phi(t, z) = C_1 e^{i\theta_1(t,z)} + C_2 e^{i\theta_2(t,z)}. \quad (44)$$

This solution and its wave phases are now used to determine the conditions for the propagation of acoustic waves in an isothermal atmosphere.

The general solution given by Equation (43) shows that any real $\theta_1(t, z)$ and $\theta_2(t, z)$ describe propagating waves. On the other hand, imaginary wave phases make the general solution exponentially decay, which corresponds to nonpropagating (or evanescent) waves. There are several cases of interest that are now considered. If $\omega >> \Omega_{ac}$, then the wave phases are $\theta_1(t, z) = (2z/v \pm |\hat{z} \cdot \hat{v}|t)\omega$ and $\theta_2(t, z) = 0$, with the first phase representing a freely propagating acoustic wave along the z – axis, and the second phase is a trivial (constant) solution that shows no acoustic wave; these results are consistent with a more general (3-dimensional) solution given by Equations (36) and (38). The obtained results demonstrate that the propagation of very high-frequency acoustic waves is not affected by the stratification of the isothermal atmosphere.

The effects of medium stratification on the acoustic wave propagation become important when $\omega \gtrsim \Omega_{ac}$; in this case, the wave phase is given by Equation (43) and both solutions contribute to $\phi(z, t)$ given by Equation (44). The most interesting case is when $\omega = \Omega_{ac}$, which gives $\theta_{1,2}(t, z) = \pm(2z/v \pm |\hat{z} \cdot \hat{v}|t)\Omega_{ac}$, showing that propagating acoustic waves cease to exist as they are replaced by oscillations of the atmosphere with its natural frequency Ω_{ac}. The existence of oscillations in planetary, solar, and stellar atmospheres is well known [47–49]. The origin of solar 5 min oscillations is attributed to the acoustic waves trapped in the solar interior [48]; however, the 3 min oscillations of the solar atmosphere are driven by the propagating acoustic waves [50]. The results presented in this paper demonstrate that the propagation of acoustic waves is terminated when $\omega = \Omega_{ac}$, and that the solar atmosphere begins to oscillate with its natural frequency Ω_{ac}, which is also the cutoff frequency for acoustic waves, as it was first shown by Lamb [39–41].

Having demonstrated that acoustic wave propagation is terminated in the limit when $\omega \longrightarrow \Omega_{ac}$, this means that Ω_{ac} is the Lamb (acoustic) cutoff frequency. It must be now verified that the wave phases given by Equation (43) become imaginary for any $\omega < \Omega_{ac}$, that is, the solutions $\phi_{1,2}(t, z)$ are exponentially decaying and the waves are evanescent. If $\omega \lesssim \Omega_{ac}$, the wave phases are imaginary and given by

$$\theta_{1,2}(t, z) = i\frac{c_s}{v}\left[\sqrt{\Omega_{ac}^2 - \omega^2} \pm \sqrt{\left(1 - \frac{v^2}{c_s^2}\right)\Omega_{ac}^2 - \omega^2}\right]\left(\frac{2z}{v} \pm |\hat{z} \cdot \hat{v}|t\right). \tag{45}$$

In general, the term $[(1 - v^2/4c_s^2)\Omega_{ac}^2 - \omega^2] > 0$, but it may also become negative if $v > 2c_s$, which means that if the second term of these phases becomes imaginary, then this term would give oscillatory solutions. However, this does not affect the general solution as the exponential decay caused by the first term takes over and makes the entire solution evanescent. Similarly, when $\omega << \Omega_{ac}$, the wave phases become

$$\theta_{1,2}(t, z) = i\frac{c_s}{v}\left[\Omega_{ac} \pm \sqrt{\left(1 - \frac{v^2}{c_s^2}\right)}\Omega_{ac}\right]\left(\frac{2z}{v} \pm |\hat{z} \cdot \hat{v}|t\right), \tag{46}$$

showing that the solutions are exponentially decaying. Based on the above discussion, the obtained results are valid in both cases when $v \leq 2c_s$ as well as when $v > 2c_s$. Thus, acoustic waves of all frequencies lower than Ω_{ac} are always evanescent.

The presented results show that the FWE for acoustic waves can be derived from the hydrodynamic equations after using the eigenvalue equation given by Equation (2). As a result, the FWE directly displays the characteristic atmospheric frequency Ω_{ac} similar as the AWE does. By solving the FWE for acoustic waves, it is demonstrated that Ω_{ac} is the Lamb (acoustic) cutoff frequency that uniquely determines the conditions for the acoustic wave propagation, which is consistent with the original results presented by Lamb in 1910 [39]. However, there are main differences between the results presented in this paper and those obtained by Lamb [39–41], namely, Lamb's results are valid in only one stationary inertial frame S, which is selected to describe the waves, while the presented results are the same for all inertial observers in the Galilean space and time. In other words, for all Galilean observers, the waves have the same wave speed, frequency, and wavenumber, and their propagation conditions remain also frame-independent, which shows that the newly formulated theory of acoustic waves based on the FWE is fundamental.

In realistic physical situations where the wave speed is not constant and wave damping and nonlinearities may be present, the FWE may lose its status as being fundamental. Nevertheless, it will still remain another wave equation, which may be applicable to some physical situations involving classical waves as the Schrödinger equation has been used [19–23].

7. Conclusions

A method based on the irreps of the extended Galilean group is used to derive infinite sets of symmetric and asymmetric second-order PDEs with constant coefficients of arbitrary real values. The obtained results demonstrate that among these equations only one asymmetric equation is a new fundamental wave equation, which gives the most complete description of propagating waves as it accounts for the Doppler effect, forward and backward waves, and makes the wave speed to be the same in all inertial frames. Thus, the main result of this paper is that classical waves propagating at speeds $c_w << c$ may "mimic" the behavior of electromagnetic waves when they are described by the FWE. It is suggested that these waves be called the *basic classical waves* in Galilean Relativity because they "mimic" the behavior of EM waves in the special theory of relativity.

Contrary to the standard wave equation and the Schrödinger equation for classical waves, which are second-order PDEs, the new fundamental asymmetric wave equation discovered in this paper is an ODE. The conversion from the PDE to ODE was achieved by using wave phases as the independent wave variables that depend on both time and space. An interesting result of this paper is that only the new asymmetric equation can be converted into the fundamental wave equation and that its form resembles the law of inertia. The mathematical forms of both equations are similar; however, the new fundamental asymmetric wave equation has one extra term that allows for periodic solutions. This may suggest that the derived fundamental wave equation plays the same role for classical waves in theories of waves as the law of inertia plays for classical particles in CM.

The fundamental wave equation is applied to the propagation of acoustic waves in an isothermal atmosphere. The analysis shows that the wave propagation conditions are uniquely determined by the existence of the atmospheric natural frequency, which is identified with the acoustic cutoff frequency originally introduced by Lamb [39]. However, while Lamb's wave description and its cutoff frequency are frame-dependent, the wave description given by the new fundamental wave equation (Equation (40)) and its acoustic cutoff remains the same for all Galilean observers in their inertial frames. The presented theory of waves based on the fundamental wave equation also predicts the existence of atmospheric oscillations with the natural atmospheric frequency that are driven by the process of the propagating waves becoming evanescent when their frequencies become equal to the Lamb frequency.

Appendix

Derivation of the Eigenvalue Equations

Let us consider a set of N functions that forms a basis of an N-dimensional representation given by a set of $N \times N$ matrices A for each irrep, and for each element of the group

$$\widehat{\alpha} f_l^{(k)} = \sum_m A_{ml}(\widehat{\alpha}) f_m^{(k)}, \tag{A.1}$$

where α is one of the elements of the group, k labels the irreps, and l is one of the members of the set of N functions satisfying Equation (A.1). In addition, the sum on m is over the N members of the set, and the matrices A are unitary.

Writing Equation (A.1) for space translations a, the result is

$$\widehat{T}_{\mathbf{a}} \psi(t, \mathbf{x}) \equiv \psi(t, \mathbf{x} + \mathbf{a}) = e^{i\mathbf{k}\cdot\mathbf{a}} \psi(t, \mathbf{x}). \tag{A.2}$$

Making the Taylor series expansion of $\phi(r + a)$, one gets

$$\phi(t, \mathbf{x} + \mathbf{a}) = \exp\left[i(-i\mathbf{a} \cdot \nabla)\right] \phi(t, \mathbf{x}). \tag{A.3}$$

Comparing Equation (A.3) to Equation (A.2), the following eigenvalue equation is obtained

$$-i\nabla \phi(t, \mathbf{x}) = \mathbf{k} \phi(t, \mathbf{x}), \tag{A.4}$$

which is the eigenvalue equation given by Equation (2).

For the time translation t_0, one obtains

$$\widehat{T}_{t_0} \psi(t, \mathbf{x}) \equiv \psi(t + t_0, \mathbf{x}) = e^{-i\omega t_0} \psi(t, \mathbf{x}). \tag{A.5}$$

Comparison of this equation to the Taylor series expansion

$$\phi(t + t_0, x) = \exp\left[i(-it_0 \partial/\partial t)\right] \phi(t, x), \tag{A.6}$$

gives

$$i \frac{\partial}{\partial t} \phi(t, \mathbf{x}) = \omega \phi(t, \mathbf{x}), \tag{A.7}$$

which is the eigenvalue equation given by Equation (1).

The derived eigenvalue equations represent the necessary conditions that $\phi(t, \mathbf{x})$ transforms as one of the irreps of $T(3+1)$ [17]. The above results also show that the irreps of the group $T(3+1)$ are labeled by the real vector $\mathbf{k}$ and the real scalar ω, and there are no other restrictions on these quantities. It must also be mentioned that these labels are preserved in the irreps of the entire $\mathcal{G}_e$ because $T(3+1)$ is its invariant subgroup [12].

Data Availability

All data that supports the findings of this study is within this article.

Conflicts of Interest

The author declares no conflict of interest.

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