Hindawi
Advances in Mathematical Physics
Volume 2023, Article ID 6283436, 11 pages
https://doi.org/10.1155/2023/6283436

Research Article

Optical Solitons and Single Traveling Wave Solutions for the Fiber Bragg Gratings with Generalized Anticubic Nonlinearity

Dan Chen and Zhao Li

College of Computer Science, Chengdu University, Chengdu 610106, China

Correspondence should be addressed to Dan Chen; mathdanc@163.com

Received 21 September 2022; Revised 22 December 2022; Accepted 6 April 2023; Published 15 April 2023

Academic Editor: Eugen Radu

This paper retrieves the vector-coupled version of the generalized anticubic nonlinearity model in fiber Bragg gratings. With the help of the trial equation approach and the complete discriminant system for polynomial, nine families of the optical solitons solutions and single traveling wave solutions for the fiber Bragg gratings with generalized anticubic nonlinearity are obtained. Under specific parameter values, three-dimensional diagrams, two-dimensional diagrams, density plots, and contour plots of the obtained solutions are displayed. Moreover, the solutions obtained in the paper further demonstrated their accurate physical behaviors.

1. Introduction

The concept of optical soliton was first proposed by Hasegawa and Tappert in 1973. Later, Mollenaure et al. observed temporal solitons in optical fibers in the laboratory for the first time. Since then, the research of optical soliton communication has started. In 2014, researchers first created optical soliton state in Kippenberg's Laboratory and informed that optical solitons can be generated by nonlinear processes. Optical soliton is a special type of ultrashort pulse, which can keep its shape, amplitude, and velocity unchanged for long time and distance, such as across the globe for trans-continental and transoceanic distances, when it propagates in optical fiber. Manufacturability of optical solitons can not only bring surprising ultrahigh data rate but also greatly reduce energy consumption of light source in an optical communication system. Therefore, optical solitons have a very important value in the field of modern communication and nonlinear optics [1]. Also, the study of optical solitons has attracted considerable attention. While the fiber Bragg gratings (FBGs) have been considered the outstanding sensor components, which can be used in the optical fiber communication system, fiber laser, and other fields. The optical nonlinearity of FBGs have been studied by integral technique [2, 3], such as Kerr law, log-law, triple-power law, power-law, dual-power law, quadratic-cubic law, cubic-quintic-septic law, anticubic law, parabolic-law and parabolic-nonlocal law. Optical solitons can transmit stably in fiber grating throughout the proper selection of input power and initial pulse width. Therefore, the study is a very important topic.

The governing generalized anticubic (AC) nonlinearity is represented as [4–6]

$$iq_t + aq_{xx} + \left(\frac{b_1}{|q|^{2n+2}} + b_2|q|^{2n} + b_3|q|^{2n+2}\right)q = 0. \quad (1)$$

Here, i is imaginary unit and $i = \sqrt{-1}$. $q(x, t)$ is the complex-valued function that expresses pulses transmitting across the fibers. In Equation (1), the first term is the linear temporal evolution, whereas the coefficient a represents chromatic dispersion (CD), the constant coefficients $b_j (j = 1, 2, 3)$ present self-phase modulation (SPM), and n is the power-law nonlinearity parameter such that $-1 < n < 3$. According to the reported literature [7], the majority of results are from numerical. In [8], Zayed et al. obtained the optical solitons of Equation (1) with three cases at $n = 1, 2, 3$ by implementing extended auxiliary equation method. Obviously, the solutions obtained in [8] mainly concentrate on the Jacobi elliptic functions solutions.

It is impossible to derive the model equations in birefringent fibers or Bragg gratings with a general value of n; therefore, we consider the case of $n = 1$ as permitted by the stability regime.

$$iq_t + aq_{xx} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4\right)q = 0. \tag{2}$$

Then, the generalized AC splits into two components, and the vector-coupled model is expressed as (see [8])

$$\begin{cases} iu_t + a_1 v_{xx} + \dfrac{f_1 u}{b_1|u|^4 + c_1|u|^2|v|^2 + d_1|v|^4} + \left(\xi_1|u|^2 + \eta_1|v|^2\right)u + \left(\theta_1|u|^4 + \gamma_1|u|^2|v|^2 + \delta_1|v|^4\right)u + i\alpha_1 u_x + \beta_1 v = 0, \\ iv_t + a_2 u_{xx} + \dfrac{f_2 v}{b_2|v|^4 + c_2|v|^2|u|^2 + d_2|u|^4} + \left(\xi_2|v|^2 + \eta_2|u|^2\right)v + \left(\theta_2|v|^4 + \gamma_2|v|^2|u|^2 + \delta_2|u|^4\right)v + i\alpha_2 v_x + \beta_2 u = 0, \end{cases} \tag{3}$$

where $a_k, b_k, c_k, d_k, f_k, \xi_k, \eta_k, \theta_k, \gamma_k, \delta_k, \alpha_k$, and β_k for $k = 1, 2$ are constants. The coefficients b_k, ξ_k, and θ_k are from SPM and the coefficients c_k, d_k, η_k, and δ_k are associated with crossphase modulation effects.

The rest of this article is listed as follows. In Section 2, under the traveling wave transformations and the trial equation method, Equation (3) is changed into planar dynamic system. In Section 3, all optical soliton solutions of Equation (3) are acquired via the complete discrimination system technique. In Section 4, we summarize the results of the current work.

2. Mathematical Analysis

For Equation (3), the following traveling wave transformation should be considered first (see [8]):

$$\begin{aligned} u(x, t) &= \phi_1(\zeta)\exp\left[i\psi(x, t)\right], v(x, t) \\ &= \phi_2(\zeta)\exp\left[i\psi(x, t)\right], \zeta \\ &= x - \rho t, \psi(x, t) \\ &= -kx + \omega t + H_0, \end{aligned} \tag{4}$$

where ϱ, k, ω, and H_0 are all nonzero constants. ϱ represents the velocity of soliton. k represents the frequency of soliton. ω represents the wave number. H_0 represents phase constant. $\phi_1(\zeta)$, $\phi_2(\zeta)$, and $\psi(x, t)$ are real functions.

Substituting (4) into the above system (3), separating the real part from the imaginary part, we obtain the formula:

$$\begin{aligned} &a_1\phi_2'' - (\omega - k\alpha_1)\phi_1 + \left(\beta_1 - a_1k^2\right)\phi_2 \\ &\quad + \frac{f_1\phi_1}{b_1\phi_1^4 + c_1\phi_1^2\phi_2^2 + d_1\phi_2^4} \\ &\quad + \left(\xi_1\phi_1^2 + \eta_1\phi_2^2\right)\phi_1 \\ &\quad + \left(\theta_1\phi_1^4 + \gamma_1\phi_1^2\phi_2^2 + \delta_1\phi_2^4\right)\phi_1 = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &a_2\phi_1'' - (\omega - k\alpha_2)\phi_2 + \left(\beta_2 - a_2k^2\right)\phi_1 \\ &\quad + \frac{f_2\phi_2}{b_2\phi_2^4 + c_2\phi_2^2\phi_1^2 + d_2\phi_1^4} \\ &\quad + \left(\xi_2\phi_2^2 + \eta_2\phi_1^2\right)\phi_2 \\ &\quad + \left(\theta_2\phi_2^4 + \gamma_2\phi_2^2\phi_1^2 + \delta_2\phi_1^4\right)\phi_2 = 0, \end{aligned} \tag{6}$$

$$(\nu - \alpha_1)\phi_1' + 2a_1k\phi_2' = 0, \tag{7}$$

$$2a_2k\phi_1' + (\nu - \alpha_2)\phi_2' = 0, \tag{8}$$

respectively.

Let $\phi_2(\zeta) = \lambda_2\phi_1(\zeta)$, where $\lambda_2(\lambda_2 \neq 1)$ is a nonzero constant. Thus, Equations (5)–(8) can be rewritten as

$$\begin{aligned} &a_1\lambda_2\phi_1'' - \left[\omega - k\alpha_1 - \lambda_2\left(\beta_1 - a_1k^2\right)\right]\phi_1 \\ &\quad + \frac{f_1}{\left(b_1 + \lambda_2^2c_1 + \lambda_2^4d_1\right)\phi_1^3} \\ &\quad + \left(\xi_1 + \lambda_2^2\eta_1\right)\phi_1^3 + \left(\theta_1 + \lambda_2^2\gamma_1 + \lambda_2^4\delta_1\right)\phi_1^5 = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &a_2\phi_1'' - \left[\lambda_2(\omega - k\alpha_2) - \left(\beta_2 - a_2k^2\right)\right]\phi_1 \\ &\quad + \frac{\lambda_2f_2}{\left(\lambda_2^4b_2 + c_2\lambda_2^2 + d_2\right)\phi_1^3} \\ &\quad + \lambda_2\left(\lambda_2^2\xi_2 + \eta_2\right)\phi_1^3 \\ &\quad + \lambda_2\left(\lambda_2^4\theta_2 + \lambda_2^2\gamma_2 + \delta_2\right)\phi_1^5 = 0, \end{aligned} \tag{10}$$

$$(\nu - \alpha_1 + 2a_1k\lambda_2)\phi_1' = 0, \tag{11}$$

$$\left[2a_2k + \lambda_2(\nu - \alpha_2)\right]\phi_1' = 0. \tag{12}$$

Thus, we obtain the following constraint condition from (11) and (12)

$$2a_1k\lambda_2^2 + (\alpha_2 - \alpha_1)\lambda_2 - 2a_2k = 0. \tag{13}$$

Therefore,

$$\lambda_2 = \frac{-(\alpha_2 - \alpha_1) \pm \sqrt{(\alpha_2 - \alpha_1)^2 + 16a_1a_2k^2}}{4a_1k}, \tag{14}$$

where $a_1a_2 > 0$. So, Equations (9) and (10) have the same form under the constraint conditions:

$$\begin{gathered} a_1\lambda_2 = a_2, \\ \xi_1 + \lambda_2^2\eta_1 = \lambda_2\left(\lambda_2^2\xi_2 + \eta_2\right), \\ \theta_1 + \lambda_2^2\gamma_1 + \lambda_2^4\delta_1 = \lambda_2\left(\lambda_2^4\theta_2 + \lambda_2^2\gamma_2 + \delta_2\right), \\ f_1\left(\lambda_2^4b_2 + c_2\lambda_2^2 + d_2\right) = \lambda_2 f_2\left(b_1 + \lambda_2^2c_1 + \lambda_2^4d_1\right), \\ \omega - k\alpha_1 - \lambda_2\left(\beta_1 - a_1k^2\right) = \lambda_2(\omega - k\alpha_2) - \left(\beta_2 - a_2k^2\right). \end{gathered} \tag{15}$$

Also, Equation (9) can be converted to (16)

$$\phi_1^3\phi_1'' + \rho_1 - \rho_2\phi_1^4 + \rho_3\phi_1^6 + \rho_4\phi_1^8 = 0, \tag{16}$$

where $\rho_1 = f_1/(a_1\lambda_2(b_1 + \lambda_2^2c_1 + \lambda_2^4d_1))$, $\rho_2 = (\omega - k\alpha_1 - \lambda_2(\beta_1 - a_1k^2))/(a_1\lambda_2)$, $\rho_3 = (\xi_1 + \lambda_2^2\eta_1)/(a_1\lambda_2)$, and $\rho_4 = (\theta_1 + \lambda_2^2\gamma_1 + \lambda_2^4\delta_1)/(a_1\lambda_2)$ satisfying $a_1\lambda_2(b_1 + \lambda_2^2c_1 + \lambda_2^4d_1) \neq 0$.

Taking the transformation $\phi_1(\zeta) = U^{1/2}(\zeta)$, where $U(\zeta)$ is a new function of ζ. Thus, Equation (16) can be rewritten as

$$\left(U'\right)^2 - 2UU'' - 4\rho_1 + 4\rho_2U^2 - 4\rho_3U^3 - 4\rho_4U^4 = 0. \tag{17}$$

According to the polynomial trial equation method of the rank homogeneous equation, we take the following trial equation [9]:

$$U'' = \sum_{i=0}^{m} A_iU^i, \tag{18}$$

where A_i are constants.

Multiplying both sides of Equation (18) by U' and integrating it yields [10]

$$\left(U'\right)^2 = \sum_{i=0}^{m} \frac{2A_m}{m+1}U^{m+1} + C_0, \tag{19}$$

where C_0 is the integral constant.

Plugging (18) and (19) into (17), we can get a polynomial about U, denoted by $G(U)$. Using the balance principle, we get $m = 3$. Let all the coefficients of $G(U)$ be zero, and a system of algebraic equations is obtained. We can calculate that A_0 is an arbitrary constant, $A_1 = 4\rho_2, A_2 = -3\rho_3, A_3 = -(8/3)\rho_4$, and $C_0 = 4\rho_1$.

Consequently, Equation (19) can be rewritten as

$$\left(U'\right)^2 = -\frac{4}{3}\rho_4U^4 - 2\rho_3U^3 + 4\rho_2U^2 + 2A_0U + 4\rho_1. \tag{20}$$

In Equation (20), if $-(4/3)\rho_4 > 0$, that is, $\rho_4 < 0$, taking the transformation

$$W = \left(-\frac{4}{3}\rho_4\right)^{1/4}U - \frac{1}{2}\rho_3\left(-\frac{4}{3}\rho_4\right)^{-3/4}, \zeta_1 = \left(-\frac{4}{3}\rho_4\right)^{1/4}\zeta. \tag{21}$$

So, we have

$$\left(W'\right)^2 = F(W) = W^4 + PW^2 + QW + R, \tag{22}$$

where $P = 4\rho_2(-(4/3)\rho_4)^{-1/2}$, $Q = (2A_0 - ((9\rho_3^3 + 48\rho_2\rho_3\rho_4)/16\rho_4^2))(-(4/3)\rho_4)^{-1/4}$ and $R = (3^4\rho_3^4/4^5\rho_4^3) + (9\rho_2\rho_3^2/16\rho_4^2) - (3A_0\rho_3/4\rho_4) + 4\rho_1$.

If $-(4/3)\rho_4 < 0$, that is, $\rho_4 > 0$, then we take the transformation

$$W = \left(\frac{4}{3}\rho_4\right)^{1/4}U + \frac{1}{2}\rho_3\left(\frac{4}{3}\rho_4\right)^{-3/4}, \zeta_1 = \left(\frac{4}{3}\rho_4\right)^{1/4}\zeta. \tag{23}$$

So, we have

$$\left(W'\right)^2 = -F(W) = -\left(W^4 + PW^2 + QW + R\right), \tag{24}$$

where $P = -4\rho_2((4/3)\rho_4)^{-1/2}$, $Q = (((9\rho_3^3 + 48\rho_2\rho_3\rho_4)/16\rho_4^2) - 2A_0)((4/3)\rho_4)^{-1/4}$ and $R = -(3^4\rho_3^4/4^5\rho_4^3) - (9\rho_2\rho_3^2/16\rho_4^2) + (3A_0\rho_3/4\rho_4) - 4\rho_1$.

Denote

$$\varepsilon = \begin{cases} 1, & \rho_4 < 0, \\ -1, & \rho_4 > 0. \end{cases} \tag{25}$$

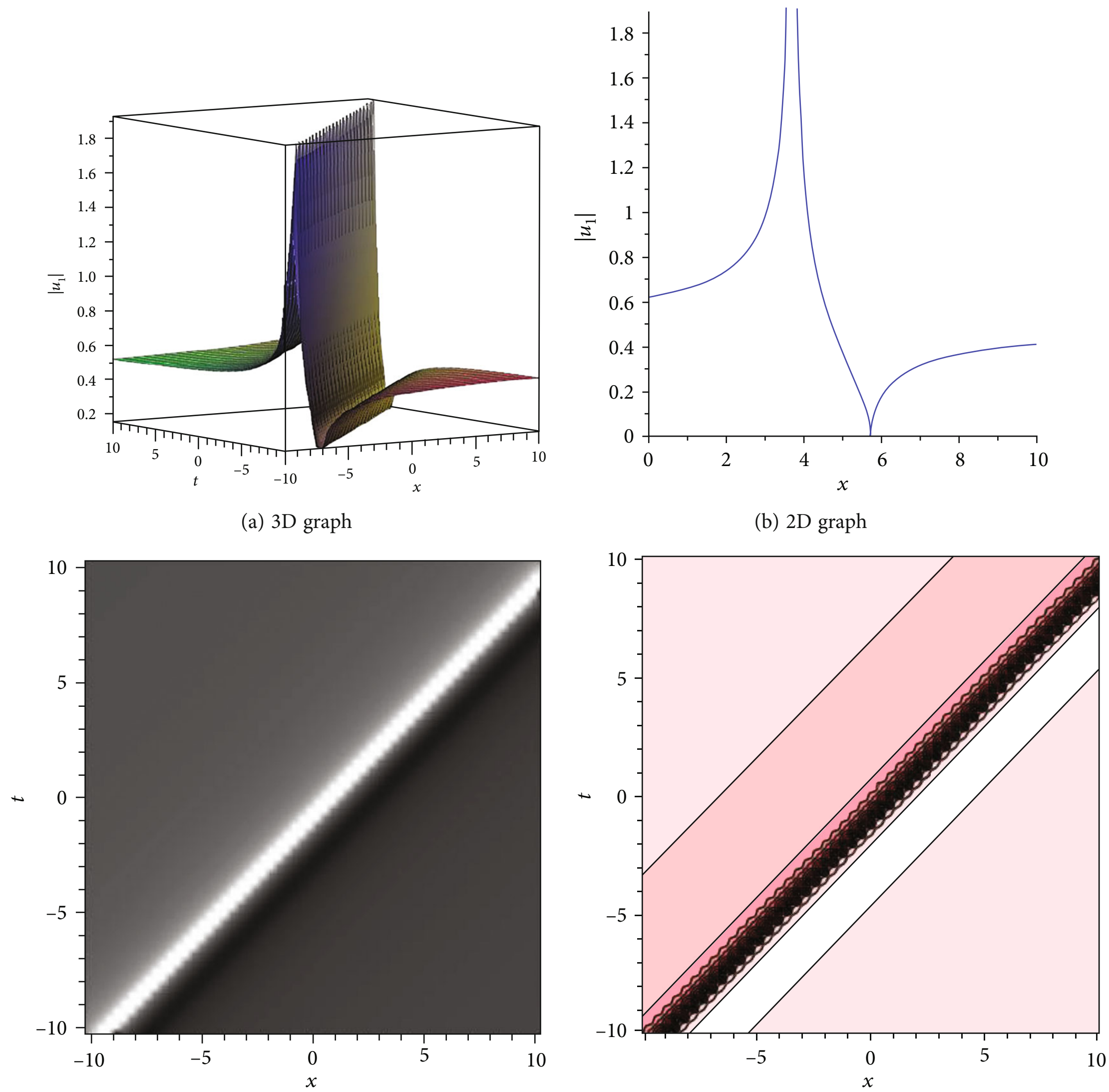
Then, Equation (20) can be written in the following integral form:

$$\pm(\zeta_1 - \zeta_0) = \int \frac{dW}{\sqrt{\varepsilon F(W)}} = \int \frac{dW}{\sqrt{\varepsilon(W^4 + PW^2 + QW + R)}}, \tag{26}$$

where ζ_0 is the integral constant.

Since Yang et al. have introduced the algorithm [11] for distinguishing the roots of polynomials, many solitons have been constructed in recent years. The discriminant system of quartic polynomials of $F(W) = W^4 + PW^2 + QW + R$ is

$$\begin{aligned} &D_1 = 4, D_2 = -P, D_3 = -2P^3 - 9Q^2 + 8PR, \\ &D_4 = -P^3Q^2 - \frac{27}{4}Q^4 + 4P^4R + 36PQ^2R - 32P^2R^2 + 64R^3, \\ &E_2 = 9Q^2 - 32PR. \end{aligned} \tag{27}$$



(a) 3D graph

(b) 2D graph

(c) Density plot

(d) Contour plt

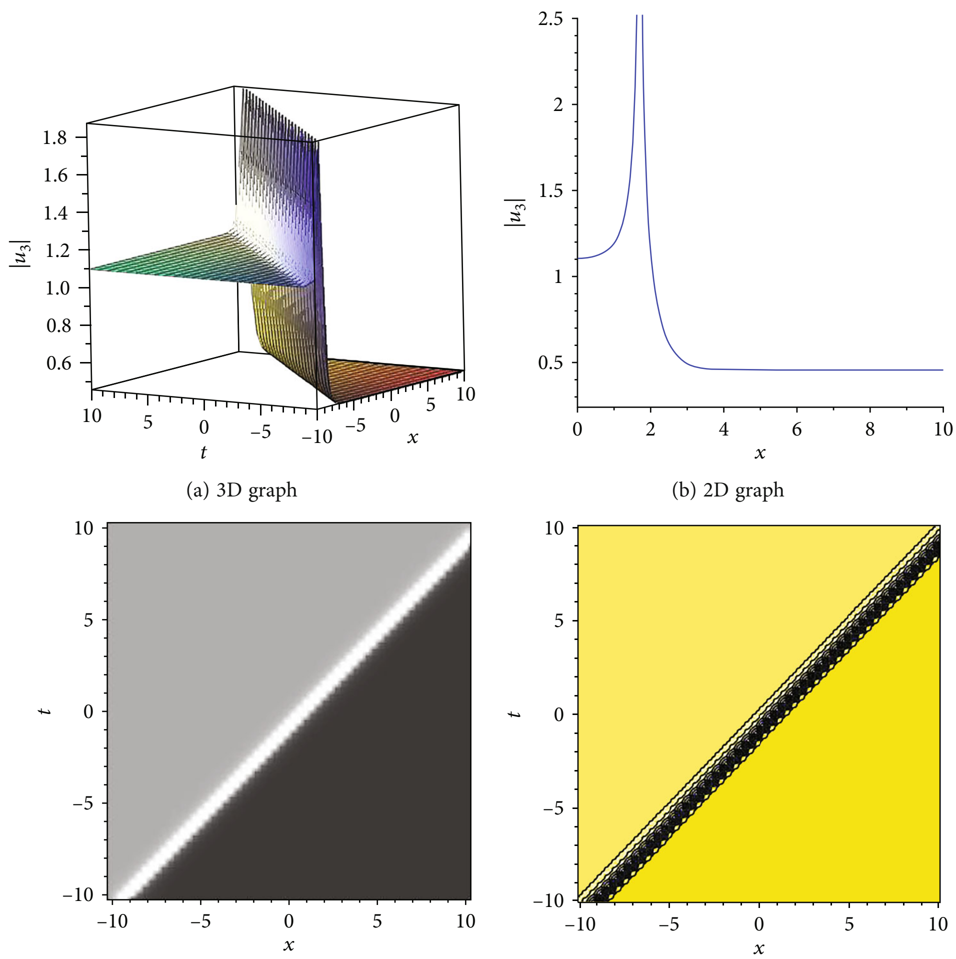
FIGURE 1: Module length graphs of $u_1(t,x)$ with $a_1 = b_1 = c_1 = d_1 = 1, f_1 = 15/256, \xi_1 = \eta_1 = -1, \theta_1 = \gamma_1 = \delta_1 = 1, \alpha_1 = -1, \beta_1 = -1, a_2 = b_2 = c_2 = d_2 = -1, f_2 = 15/256, \xi_2 = \eta_2 = 1, \theta_2 = \gamma_2 = \delta_2 = -1, \alpha_2 = 3$, and $\beta_2 = 1$.

Integrating formula (26), we obtain optical solitons and exact traveling wave solutions [12–17] of Equation (3) under nine cases. Details will be given in the next section.

3. Optical Wave Patterns

Case 1. When $D_2 = D_3 = D_4 = 0$, that is, $F(W) = W^4$. If $\rho_4 < 0$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_1(x,t) = \left(-\dfrac{1}{(-(4/3)\rho_4)^{1/2}(x - \varrho t) - (-(4/3)\rho_4)^{1/4}\zeta_0} - \dfrac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \\ v_1(x,t) = \lambda_2 \left(-\dfrac{1}{(-(4/3)\rho_4)^{1/2}(x - \varrho t) - (-(4/3)\rho_4)^{1/4}\zeta_0} - \dfrac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right]. \end{cases} \tag{28}$$



(a) 3D graph

(b) 2D graph

(c) Density plot

(d) Contour plot

Figure 2: Module length graphs of $u_3(t, x)$ with $a_1 = 1, b_1 = 3, c_1 = 4, d_1 = 9, f_1 = 5, \xi_1 = \eta_1 = -2, \theta_1 = \gamma_1 = \delta_1 = 1, \alpha_1 = -1, \beta_1 = 1/2, a_2 = -1, b_2 = -3, c_2 = -4, d_2 = -9, f_2 = 5, \xi_2 = \eta_2 = 2, \theta_2 = \gamma_2 = \delta_2 = -1, \alpha_2 = 3$, and $\beta_2 = 1/2$.

When $a_1 = b_1 = c_1 = d_1 = 1, f_1 = 15/256, \xi_1 = \eta_1 = -1, \theta_1 = \gamma_1 = \delta_1 = 1, \alpha_1 = -1, \beta_1 = -1, a_2 = b_2 = c_2 = d_2 = -1, f_2 = 15/256, \xi_2 = \eta_2 = 1, \theta_2 = \gamma_2 = \delta_2 = -1, \alpha_2 = 3$, and $\beta_2 = 1$, we draw the grah of $|u_1(x, t)|$ as shown in Figure 1.

Case 2. When $D_2 < 0, D_3 = D_4 = 0$, $F(W)$ has a pair of conjugate complex roots of multiplicities two, that is, $F(W) = [(W - \sigma_1)^2 + \sigma_2^2]^2$, where σ_1, σ_2 are real numbers and $\sigma_2 > 0$. If $\varepsilon = 1$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_2(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\sigma_1 \tan\left[\sigma_1\left(\left(-\frac{4}{3}\rho_4\right)^{1/4}(x - \varrho t) - \zeta_0\right)\right] + \sigma_2\right) - \frac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \\ v_2(x,t) = \lambda_2\left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\sigma_1 \tan\left[\sigma_1\left(\left(-\frac{4}{3}\rho_4\right)^{1/4}(x - \varrho t) - \zeta_0\right)\right] + \sigma_2\right) - \frac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right]. \end{cases} \tag{29}$$

Case 3. When $D_2 > 0, D_3 = D_4 = 0, E_2 > 0$, $F(W)$ has two real roots of multiplicities two, that is, $F(W) = (W - \sigma_3)^2 (W - \sigma_4)^2$, where σ_3, σ_4 are real numbers and $\sigma_3 > \sigma_4$. If $\varepsilon = 1$.

Case 3.1. When $W > \sigma_3$ or $W < \sigma_4$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_3(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\frac{\sigma_4-\sigma_3}{2}\left[\coth\frac{(\sigma_3-\sigma_4)\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}{2}-1\right]+\sigma_4\right)-\frac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right], \\ v_3(x,t) = \lambda_2\left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\frac{\sigma_4-\sigma_3}{2}\left[\coth\frac{(\sigma_3-\sigma_4)\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}{2}-1\right]+\sigma_4\right)-\frac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right]. \end{cases} \tag{30}$$

When $a_1 = 1, b_1 = 3, c_1 = 4, d_1 = 9, f_1 = 5, \xi_1 = \eta_1 = -2, \theta_1 = \gamma_1 = \delta_1 = 1, \alpha_1 = -1, \beta_1 = 1/2, a_2 = -1, b_2 = -3, c_2 = -4, d_2 = -9, f_2 = 5, \xi_2 = \eta_2 = 2, \theta_2 = \gamma_2 = \delta_2 = -1, \alpha_2 = 3$, and $\beta_2 = 1/2$, we draw the graph of $|u_3(x,t)|$ as shown in Figure 2.

Case 3.2. When $\sigma_4 < W < \sigma_4$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_4(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\frac{\sigma_4-\sigma_3}{2}\left[\tanh\frac{(\sigma_3-\sigma_4)\left((-(4/3)\varrho_4)^{1/4}(x-\rho t)-\zeta_0\right)}{2}-1\right]+\sigma_4\right)-\frac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right], \\ v_4(x,t) = \lambda_2\left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\frac{\sigma_4-\sigma_3}{2}\left[\tanh\frac{(\sigma_3-\sigma_4)\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}{2}-1\right]+\sigma_4\right)-\frac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right]. \end{cases} \tag{31}$$

Case 4. When $D_2 > 0, D_3 > 0, D_4 = 0$, $F(W)$ has two real roots and real roots with multiplicities two, that is, $F(W) = (W - \sigma_5)^2 (W - \sigma_6)(W - \sigma_7)$, where $\sigma_5, \sigma_6, \sigma_7$ are real numbers satisfying $\sigma_6 > \sigma_7$.

Case 4.1. If $\varepsilon = 1$.

Case 4.1.1. When $\sigma_5 > \sigma_6$ and $W < \sigma_7$, or $\sigma_5 < \sigma_7$ and $W < \sigma_6$, we can calculate (26) and get the solutions of Equation (3):

$$\pm(\zeta_1 - \zeta_0) = \frac{1}{(\sigma_5-\sigma_6)(\sigma_5-\sigma_7)} \ln \frac{\left[\sqrt{(W-\sigma_6)(\sigma_7-\sigma_5)} - \sqrt{(W-\sigma_7)(\sigma_6-\sigma_5)}\right]^2}{|W-\sigma_5|}. \tag{32}$$

Case 4.1.2. When $\sigma_6 > \sigma_5 > \sigma_7$, we can calculate (26) and get the solutions of Equation (3)

$$\pm(\zeta_1 - \zeta_0) = \frac{1}{(\sigma_6-\sigma_5)(\sigma_5-\sigma_7)} \arcsin \frac{(W-\sigma_6)(\sigma_5-\sigma_7) + (W-\sigma_7)(\sigma_5-\sigma_6)}{|(W-\sigma_5)(\sigma_6-\sigma_7)|}. \tag{33}$$

Case 4.2. If $\varepsilon = -1$.

Case 4.2.1. When $\sigma_5 > \sigma_6$ and $W < \sigma_7$, or $\sigma_5 < \sigma_7$ and $W < \sigma_6$, we can calculate (26) and get the solutions of Equation (3)

$$\pm(\zeta_1 - \zeta_0) = \frac{1}{(\sigma_6-\sigma_5)(\sigma_5-\sigma_7)} \ln \frac{\left[\sqrt{(-W+\sigma_6)(\sigma_7-\sigma_5)} - \sqrt{(W-\sigma_7)(\sigma_5-\sigma_6)}\right]^2}{|W-\sigma_5|}. \tag{34}$$

Case 4.2.2. When $\sigma_6 > \sigma_5 > \sigma_7$, we can calculate (26) and get the solutions of Equation (3)

$$\pm(\zeta_1 - \zeta_0) = \frac{1}{(\sigma_5-\sigma_6)(\sigma_5-\sigma_7)} \arcsin \frac{(-W+\sigma_6)(\sigma_5-\sigma_7) + (W-\sigma_7)(\sigma_6-\sigma_5)}{|(W-\sigma_5)(\sigma_6-\sigma_7)|}. \tag{35}$$

Case 5. When $D_2 > 0, D_3 = D_4 = E_2 = 0$, $F(W)$ has real roots of multiplicities three and real roots with multiplicities one, that is, $F(W) = (W - \sigma_8)^3 (W - \sigma_9)$, where σ_8, σ_9 are real numbers.

Case 5.1. If $\varepsilon = 1$, when $W > \sigma_8$ and $W > \sigma_9$, or $W < \sigma_8$ and $W < \sigma_9$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_9(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\dfrac{4(\sigma_8-\sigma_9)}{(\sigma_9-\sigma_8)^2\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)^2-4}+\sigma_8\right)-\dfrac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right], \\ v_9(x,t) = \lambda_2\left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\left(\dfrac{4(\sigma_8-\sigma_9)}{(\sigma_9-\sigma_8)^2\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)^2-4}+\sigma_8\right)-\dfrac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right]. \end{cases} \tag{36}$$

Case 5.2. If $\varepsilon = -1$, when $W > \sigma_8$ and $W > \sigma_9$, or $W < \sigma_8$ and $W < \sigma_9$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{10}(x,t) = \left(\left(\frac{4}{3}\rho_4\right)^{-1/4}\left(\dfrac{4(\sigma_8-\sigma_9)}{-(\sigma_9-\sigma_8)^2\left((4/3\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)^2-4}+\sigma_8\right)-\dfrac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right], \\ v_{10}(x,t) = \lambda_2\left(\left(\frac{4}{3}\rho_4\right)^{-1/4}\left(\dfrac{4(\sigma_8-\sigma_9)}{-(\sigma_9-\sigma_8)^2\left(((4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)^2-4}+\sigma_8\right)-\dfrac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right]. \end{cases} \tag{37}$$

Case 6. When $D_2D_3 < 0, D_4 = 0$, $F(W)$ has real roots of multiplicities two and a pair of conjugate complex roots, that is, $F(W) = (W-\sigma_{10})^2[(W-\sigma_{11})^2+\sigma_{12}^2]$. If $\varepsilon = 1$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{11}(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\dfrac{\left[e^{\pm\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}-Z_1\right]+\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}(2-Z_1)}{\left[e^{\pm\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}-Z_1\right]^2-1}-\dfrac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right], \\ v_{11}(x,t) = \lambda_2\left(\left(-\frac{4}{3}\rho_4\right)^{-1/4}\dfrac{\left[e^{\pm\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}-Z_1\right]+\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}(2-Z_1)}{\left[e^{\pm\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}\left((-(4/3)\rho_4)^{1/4}(x-\varrho t)-\zeta_0\right)}-Z_1\right]^2-1}-\dfrac{3\rho_3}{8\rho_4}\right)^{1/2}\exp\left[i(-kx+\omega t+H_0)\right], \end{cases} \tag{38}$$

which is a solitary wave solution and where $Z_1 = (\sigma_{10} - 2\sigma_{11})/\sqrt{(\sigma_{10}-\sigma_{11})^2+\sigma_{12}^2}$.

Case 7. When $D_2 > 0, D_3 > 0, D_4 > 0$, $F(W)$ has four distinct real roots, that is, $F(W) = (W-\sigma_{13})(W-\sigma_{14})(W-\sigma_{15})(W-\sigma_{16})$, where $\sigma_{13}, \sigma_{14}, \sigma_{15}$, and σ_{16} are real numbers satisfying $\sigma_{13} > \sigma_{14} > \sigma_{15} > \sigma_{16}$.

Case 7.1. If $\varepsilon = 1$.

Case 7.1.1. When $W > \sigma_{13}$ or $W < \sigma_{16}$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{12}(x,t) = \left(\left(-\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{14}(\sigma_{13}-\sigma_{16})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{13}(\sigma_{14}-\sigma_{16})}{(\sigma_{13}-\sigma_{16})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{14}-\sigma_{16})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right], \\ v_{12}(x,t) = \lambda_2 \left(\left(-\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{14}(\sigma_{13}-\sigma_{16})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{13}(\sigma_{14}-\sigma_{16})}{(\sigma_{13}-\sigma_{16})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{14}-\sigma_{16})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right]. \end{cases} \tag{39}$$

Case 7.1.2. When $\sigma_{15} < W < \sigma_{14}$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{13}(x,t) = \left(\left(-\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{16}(\sigma_{14}-\sigma_{15})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{15}(\sigma_{14}-\sigma_{16})}{(\sigma_{14}-\sigma_{15})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{14}-\sigma_{16})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right], \\ v_{13}(x,t) = \lambda_2 \left(\left(-\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{16}(\sigma_{14}-\sigma_{15})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{15}(\sigma_{14}-\sigma_{16})}{(\sigma_{14}-\sigma_{15})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left((-(4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{14}-\sigma_{16})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right]. \end{cases} \tag{40}$$

Case 7.2. If $\varepsilon = -1$.

Case 7.2.1. When $\sigma_{13} > W > \sigma_{14}$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{14}(x,t) = \left(\left(\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{15}(\sigma_{13}-\sigma_{14})\mathbf{sn}^2\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})}/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{14}(\sigma_{13}-\sigma_{15})}{(\sigma_{13}-\sigma_{14})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{13}-\sigma_{15})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right], \\ v_{14}(x,t) = \lambda_2 \left(\left(\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{15}(\sigma_{13}-\sigma_{14})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{14}(\sigma_{13}-\sigma_{15})}{(\sigma_{13}-\sigma_{14})\mathbf{sn}^2\left(\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})} \right)/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{13}-\sigma_{15})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right]. \end{cases} \tag{41}$$

Case 7.2.2. When $\sigma_{15} > W > \sigma_{16}$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{15}(x,t) = \left(\left(\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{13}(\sigma_{15}-\sigma_{16})\mathbf{sn}^2\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})}/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{16}(\sigma_{15}-\sigma_{13})}{(\sigma_{15}-\sigma_{16})\mathbf{sn}^2\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})}/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{15}-\sigma_{13})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right], \\ v_{15}(x,t) = \lambda_2 \left(\left(\frac{4}{3}\rho_4 \right)^{-1/4} \frac{\sigma_{13}(\sigma_{15}-\sigma_{16})\mathbf{sn}^2\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})}/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - \sigma_{16}(\sigma_{15}-\sigma_{13})}{(\sigma_{15}-\sigma_{16})\mathbf{sn}^2\left(\left(\sqrt{(\sigma_{13}-\sigma_{15})(\sigma_{14}-\sigma_{16})}/2 \right) \left(((4/3)\rho_4)^{1/4}(x-\varrho t) - \zeta_0 \right), Z_2 \right) - (\sigma_{15}-\sigma_{13})} - \frac{3\rho_3}{8\rho_4} \right)^{1/2} \exp\left[i(-kx+\omega t+H_0) \right], \end{cases} \tag{42}$$

where $Z_2^2 = ((\sigma_{13} - \sigma_{14})(\sigma_{15} - \sigma_{16}))/((\sigma_{13} - \sigma_{15})(\sigma_{14} - \sigma_{16}))$.

Case 8. When $D_4 < 0, D_2D_3 \geq 0$, $F(W)$ has two different real roots and a pair of conjugate complex roots, that is, $F(W) = (W - \sigma_{17})(W - \sigma_{18})[(W - \sigma_{19})^2 + \sigma_{20}^2]$, where $\sigma_{17}, \sigma_{18}, \sigma_{19}$, and σ_{20} are real numbers, $\sigma_{17} > \sigma_{18}$, and $\sigma_{19}, \sigma_{20} > 0$.

Take the transformation $W = (\mu_1 \cos \Phi + \mu_2)/(\mu_3 \cos \Phi + \mu_4)$, where $\mu_1 = (1/2)(\sigma_{17} + \sigma_{18})\mu_3 - (1/2)(\sigma_{17} - \sigma_{18})\mu_4$, $\mu_2 = (1/2)(\sigma_{17} + \sigma_{18})\mu_4 - (1/2)(\sigma_{17} - \sigma_{18})\mu_3$, $\mu_3 = \sigma_{17} - \sigma_{19} - (\sigma_{20}/Z_3)$, $\mu_4 = \sigma_{17} - \sigma_{19} - \sigma_{20}Z_3$, $E = (\sigma_{20}^2 + (\sigma_{17} - \sigma_{19})(\sigma_{18} - \sigma_{19}))/(\sigma_{20}(\sigma_{17} - \sigma_{18}))$, and $Z_3 = E \pm \sqrt{E^2 + 1}$.

Case 8.1. If $\varepsilon = 1$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{16}(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4} \dfrac{\mu_1 \mathbf{cn}\left(\left(\sqrt{-2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_2}{\mu_3 \mathbf{cn}\left(\left(\sqrt{-2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_4} - \dfrac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \\ v_{16}(x,t) = \lambda_2 \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4} \dfrac{\mu_1 \mathbf{cn}\left(\left(\sqrt{-2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_2}{\mu_3 \mathbf{cn}\left(\left(\sqrt{-2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_4} - \dfrac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right]. \end{cases} \tag{43}$$

Case 8.2. If $\varepsilon = -1$, the solutions of Equation (3) are given in the following form

$$\begin{cases} u_{17}(x,t) = \left(\left(\frac{4}{3}\rho_4\right)^{-1/4} \dfrac{\mu_1 \mathbf{cn}\left(\left(\sqrt{2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left(((4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_2}{\mu_3 \mathbf{cn}\left(\left(\sqrt{2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left(((4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_4} - \dfrac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \\ v_{17}(x,t) = \lambda_2 \left(\left(\frac{4}{3}\rho_4\right)^{-1/4} \dfrac{\mu_1 \mathbf{cn}\left(\left(\sqrt{2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\right)\left(((4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_2}{\mu_3 \mathbf{cn}\left(\sqrt{2\sigma_{20}Z_3(\sigma_{17} - \sigma_{18})}/2Z_4Z_3\left((4/3\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_4\right) + \mu_4} - \dfrac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \end{cases} \tag{44}$$

which are elliptic double periodic function solutions and where $Z_4^2 = 1/(1 + Z_3^2)$.

Case 9. When $D_4 > 0, D_2D_3 \leq 0$, $F(W)$ has two pairs of conjugate complex roots, that is, $F(W) = [(W - \sigma_{21})^2 + \sigma_{22}^2][(W - \sigma_{23})^2 + \sigma_{24}^2]$, where $\sigma_{21}, \sigma_{22}, \sigma_{23}$, and σ_{24} are real numbers and $\sigma_{22} > \sigma_{24} > 0$.

If $\varepsilon = 1$, we take the transformation $W = (\mu_5 \tan \Theta + \mu_6)/(\mu_7 \tan \Theta + \mu_8)$, where $\mu_5 = \sigma_{21}\mu_7 + \sigma_{22}\mu_8$, $\mu_6 = \sigma_{21}\mu_8 - \sigma_{22}\mu_7$, $\mu_7 = -\sigma_{22} - (\sigma_{24}/Z_5)$, $\mu_8 = \sigma_{21} - \sigma_{23}$, $E = ((\sigma_{21} - \sigma_{23})^2 + \sigma_{22}^2 + \sigma_{24}^2)/2\sigma_{22}\sigma_{24}$, and $Z_5 = E + \sqrt{E^2 - 1}$, the solutions of Equation (3) are given in the following form:

$$\begin{cases} u_{18}(x,t) = \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4} \dfrac{\mu_5 \mathbf{sn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right) + \mu_6 \mathbf{cn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right)}{\mu_7 \mathbf{sn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right) + \mu_8 \mathbf{cn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right)} - \dfrac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \\ v_{18}(x,t) = \lambda_2 \left(\left(-\frac{4}{3}\rho_4\right)^{-1/4} \dfrac{\mu_5 \mathbf{sn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right) + \mu_6 \mathbf{cn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right)}{\mu_7 \mathbf{sn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right) + \mu_8 \mathbf{cn}\left(Z_7\left((-(4/3)\rho_4)^{1/4}(x - \varrho t) - \zeta_0\right), Z_6\right)} - \dfrac{3\rho_3}{8\rho_4}\right)^{1/2} \exp\left[i(-kx + \omega t + H_0)\right], \end{cases} \tag{45}$$

Table 1: Comparison between my results and literature [8].

Type of solutions	Results: Our results	Results: Results of literature [8]
Exponential function solutions	$u_{11}(t,x)$,	
Trigonometric function solutions	$u_2(t,x)$,	Solutions (79), (87)
Rational function solutions	$u_1(x,t)$, $u_9(t,x)$,$u_{10}(t,x)$	
Hyperbolic function solutions	$u_3(t,x)$, $u_4(t,x)$	Solutions (75), (76),(82),
Jacobi elliptic function solutions	$u_{12}(t,x) \sim u_{18}(t,x)$	Solutions (73), (74), (77), (78), (80), (83), (84)-(86), (88), (89)
Implicit function solutions	Solutions (3.5)-(3.8)	

where $Z_7 = \sigma_{24}\sqrt{(\mu_7^2+\mu_8^2)(Z_5^2\mu_7^2+\mu_8^2)}/(\mu_7^2+\mu_8^2)$.

Compared with the reported in the existing literature [8], we have obtained exponential function solutions, trigonometric function solutions, rational function solutions, hyperbolic function solutions, and implicit function solutions(see Table 1).

4. Conclusion

This paper studied optical solitons with generalized anticubic nonlinearity in fiber Bragg gratings with one case at $n = 1$. The complete discriminant system for polynomial method combining with the trial equation method yielded a plethora of solutions. In addition to the Jacobian function solutions reported in the existing literature [7, 8], we have obtained exponential function solutions, trigonometric function solutions, rational function solutions, hyperbolic function solutions, and soliton solutions. All such studies will yield marvelous and novel results that will be gradually disclosed. In the future work, we will still focus on the study of optical soliton solutions of coupled nonlinear partial differential equations. Moreover, we will further analyze the dynamic behavior of such equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This project is supported by the Scientific Research Funds of Chengdu University under grant no. 2081920034.

References

[1] Z. Li, X. Xie, and C. Jin, "Phase portraits and optical soliton solutions of coupled nonlinear Maccari systems describing the motion of solitary waves in fluid flow," *Results in Physics*, vol. 41, article 105932, 2022.

[2] Y. Yldrm, "Optical solitons to Chen-Lee-Liu model with trial equation approach," *Optik*, vol. 183, pp. 849–853, 2019.

[3] Y. Li, X. Geng, B. Xue, and R. Li, "Darboux transformation and exact solutions for a four-component Fokas-Lenells equation," *Results in Physics*, vol. 31, article 105027, 2021.

[4] E. M. Zayed, M. E. Alngar, M. El-Horbaty et al., "Chirped and chirp-free optical solitons having generalized anti-cubic nonlinearity with a few cutting-edge integration technologies," *Optik*, vol. 206, article 163745, 2020.

[5] N. A. Kudryashov, "A generalized model for description of propagation pulses in optical fiber," *Optik*, vol. 189, pp. 42–52, 2019.

[6] A. Biswas, A. Sonmezoglu, M. Ekici, A. S. Alshomrani, and M. R. Belic, "Optical solitons with Kudryashov's equation by F-expansion," *Optik*, vol. 199, article 163338, 2019.

[7] E. M. E. Zayed, R. M. A. Shohib, A. Biswas et al., "Optical solitons with fiber Bragg gratings and dispersive reflectivity having parabolic-nonlocal combo nonlinearity via three prolific integration architectures," *Optik*, vol. 208, article 164065, 2020.

[8] E. M. E. Zayed, R. M. A. Shohib, A. Biswas et al., "Optical solitons in fiber Bragg gratings with generalized anti-cubic nonlinearity by extended auxiliary equation," *Chinese Journal of Physics*, vol. 65, pp. 613–628, 2020.

[9] C. Liu, "Trial equation method to nonlinear evolution equations with rank inhomogeneous: mathematical discussions and its applications," *Communications in Theoretical Physics*, vol. 45, no. 2, pp. 219–223, 2006.

[10] C. Liu, "A new equation method and its applications," *Communications in Theoretical Physics*, vol. 54, pp. 3395–3397, 2006.

[11] L. Yang, X. Hou, and Z. Zeng, "A complete discriminant system of polynomials," *Science China Technological Sciences*, vol. 26, pp. 424–441, 1996.

[12] M. S. Ullah, M. Z. Ali, H. O. Roshid, A. R. Seadawy, and D. Baleanu, "Collision phenomena among lump, periodic and soliton solutions to a (2+1)-dimensional Bogoyavlenskii's breaking soliton model," *Physics Letters A*, vol. 397, article 127263, 2021.

[13] Z. Li, "Bifurcation and traveling wave solution to fractional Biswas-Arshed equation with the beta time derivative," *Chaos, Solitons & Fractals*, vol. 160, article 112249, 2022.

[14] M. S. Osman, K. U. Tariq, A. Bekir et al., "Investigation of soliton solutions with different wave structures to the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation," *Communications in Theoretical Physics*, vol. 72, no. 3, article 035002, 2020.

[15] C. Park, R. I. Nuruddeen, K. K. Ali, L. Muhammad, M. S. Osman, and D. Baleanu, "Novel hyperbolic and exponential ansatz methods to the fractional fifth-order Korteweg-de Vries

equations," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[16] K. K. Ali, M. A. Abd El Salam, E. M. Mohamed, B. Samet, S. Kumar, and M. S. Osman, "Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using Chebyshev series," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[17] Y. Saliou, S. Abbagari, A. Houwe et al., "W-shape bright and several other solutions to the (3+1)-dimensional nonlinear evolution equations," *Modern Physics Letters B*, vol. 35, no. 30, article 2150468, 2021.