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Research Article

On Bernstein's Problem of Complete Parabolic Hypersurfaces in Warped Products

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We study constant mean curvature hypersurfaces constructed over the fiber M^n of warped products $I \times_f M^n$. In this setting, assuming that the sign of the angle function does not changed along the hypersurfaces, we infer the uniqueness of such hypersurfaces by applying a parabolicity criterion. As an application, we get some Bernstein type theorems.

1. Introduction

In this paper, we investigate the uniqueness results in a certain class of Riemannian manifolds, that is, the warped products. In the sense of [1], the warped products $I \times_f M^n$, where the base $I \subset \mathbb{R}$ is an interval, the n-dimensional Riemannian manifold M^n is a fiber, and $f : I \longrightarrow \mathbb{R}^+$ is a warping function (for further details, see Section 2). Before presenting details on our work, we give a brief overview of some articles concerning ours.

In [2], Montiel proved that any compact orientable constant mean curvature hypersurfaces in warped products $\mathbb{R} \times_\rho M^n$ that can be written as a graph over M^n must be a slice, under the assumption that the Ricci curvature Ric^M of M^n and the function ρ satisfy the convergence condition $\mathrm{Ric}^M \geq (n-1) \sup (\rho'^2 - \rho\rho'')$. Later on, Alas and Dajczer [3] studied the constant mean curvature hypersurfaces in warped product spaces. In this setting, if the hypersurface is compact, they extended the previous results by Montiel. Afterwards, some of these generalizations hold for complete hypersurfaces. In recent years, by using the Omori-Yau generalized maximum principle for complete hypersurfaces and supposing suitable assumptions, some researchers proved that such a hypersurface must be a slice. For instance, in [4], Caminha and de Lima studied complete graphs of constant mean curvature in the hyperbolic and steady-state spaces, and they obtained some rigidity theorems for such graphs. Later, Aquino and de Lima [5] extended the results in [4] to complete constant mean curvature graphs in warped products under appropriate convergence condition. In [6], Cavalcante et al. considered the Bernstein type properties of complete two-sided hypersurfaces in weighted warped products; they established sufficient conditions which guarantee that such a hypersurface must be a slice. Furthermore, [7] obtained uniqueness results for complete hypersurfaces in Riemannian warped products whose fiber has parabolic universal covering. More recently, by the weak Omori-Yau's maximum principle, the author [8] proved new Bernstein type results of complete constant weighted mean curvature hypersurfaces in weighted warped products $I \times_\rho M^n_f$.

This paper is organized as follows: in Section 2, we introduce some basic facts for hypersurfaces in warped products. Section 3 is devoted to compute the Laplacian of the angle function Θ which we will define in Section 2. Moreover, using the parabolicity criterion, we establish the uniqueness results concerning constant mean curvature hypersurfaces. As a consequence of this previous study, we prove some Bernstein type results for constant mean curvature entire graphs in warped products.

2. Preliminaries

Throughout this paper, we consider the warped products $\bar{M}^{n+1} = I \times_f M^n$, where M^n is a connected oriented n-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an interval with a positive definite metric dt^2, $f : I \longrightarrow \mathbb{R}^+$ is a positive smooth function, and the product manifold $I \times M^n$ is endowed with the Riemannian metric

$$\langle , \rangle = \pi_I^* \left(dt^2\right) + f(\pi_I)^2 \pi_M^* (\langle , \rangle_M), \tag{1}$$

where π_I and π_M denote the projections onto I and M, respectively. Such the resulting space is said to be a *warped product* in [1], Chapter 7, with *fiber* $(M^n, \langle , \rangle_M)$, *base* (I, dt^2), and *warping function* f. Moreover, an immersion $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ of an n-dimensional manifold Σ^n is called a *hypersurface*. Furthermore, the induced metric through ψ on Σ^n is also denoted by $\langle , \rangle$.

In fact that each leaf $M^n_{t_0} = \{t_0\} \times M^n$ of the foliation $t_0 \in I \longrightarrow M^n_{t_0}$ of $\bar{M}^{n+1}$ by complete hypersurfaces has constant mean curvature $H = f'(t_0)/f(t_0)$. Here, we say that $M^n_{t_0} = \{t_0\} \times M^n$ is a *slice* of $\bar{M}^{n+1}$. Thus, a slice is minimal if and only if $f'(t_0) = 0$.

Observe that the vector field $K = f(\pi_I)\partial_t$ is a closed conformal vector field in $\bar{M}^{n+1}$, that is,

$$\bar{\nabla}_X K = f'(\pi_I) X, \tag{2}$$

where $\partial_t = \partial/\partial t$ is a unit vector field tangent to I, $X \in \mathfrak{X}(\bar{M}^{n+1})$, and $\bar{\nabla}$ is the Levi-Civita connection in $\bar{M}^{n+1}$.

In this paper, we consider the hypersurfaces $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ oriented a unit normal vector field N, and such hypersurfaces are called two-sided hypersurfaces. In what follows, we will study its *angle* (*or support*) *function* $\Theta = \langle N, \partial_t \rangle$ and *height function* $\tau = (\pi_I)|_{\Sigma^n}$.

Let ∇ be the Levi-Civita connection in Σ^n. A direct computation shows that

$$\bar{\nabla} \pi_I = \left\langle \bar{\nabla} \pi_I, \partial_t \right\rangle \partial_t = \partial_t. \tag{3}$$

Thus, the gradient of τ is given by

$$\nabla \tau = \left(\bar{\nabla} \pi_I\right)^T = \partial_t^T = -\Theta N + \partial_t, \tag{4}$$

where $(\cdot)^T$ is the tangential component of a vector field in $\mathfrak{X}(\bar{M}^{n+1})$ along Σ^n. Moreover,

$$|\nabla \tau|^2 = -\Theta^2 + 1, \tag{5}$$

where $|\cdot|$ is the norm of a vector field on Σ^n.

3. Parametric Uniqueness Results

In order to establish our uniqueness results in warped products $\bar{M}^{n+1}$, we need to compute the Laplacian of the angle function Θ to obtain a bounded on the Laplacian of the function $\log(1 + \Theta)$.

Lemma 1. *Let $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ be a hypersurface in a warped product $\bar{M}^{n+1} = I \times_f M^n$. Then, the angle function Θ of Σ^n satisfies*

$$\begin{aligned} \Delta\Theta = & -n\langle \nabla H, \partial t \rangle - n\frac{f'(\tau)}{f(\tau)} H\left(1 + \Theta^2\right) + 2\frac{f'(\tau)}{f(\tau)} \langle A\nabla\tau, \nabla\tau \rangle \\ & - \Theta|A|^2 - \frac{f''(\tau)}{f(\tau)} \Theta |\nabla\tau|^2 - \frac{f'(\tau)^2}{f(\tau)^2} \Theta\left(n - 3|\nabla\tau|^2\right) \\ & - \Theta\left(Ric^M(N^*, N^*) + (n-1)(\log f)''(\tau)|\nabla\tau|^2\right), \end{aligned} \tag{6}$$

where Ric^M is the Ricci curvature tensor of M^n and $N^* = N - \langle N, \partial_t \rangle \partial_t$ stands for the projection of the vector field N onto M^n.

Proof. The Gauss and Weingarten formulas of $\psi : \Sigma^n \longrightarrow I \times_f M^n$ are

$$\bar{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle N, \tag{7}$$

$$AX = -\bar{\nabla}_X N, \tag{8}$$

where $X, Y \in \mathfrak{X}(\Sigma)$ and $A : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ is the shape operator of Σ^n corresponding to N.

Now, taking the tangential component in (2), by (7) and (8), we can easily get that

$$\nabla_X K^T = f(\tau) A\Theta X + f'(\tau) X, \tag{9}$$

where $f'(\tau) = f' \circ \tau$ and $K^T = f(\tau)\partial_t^T = K - \langle K, N \rangle N$ is the tangential component of K along Σ^n. Therefore, it follows from (9) that

$$\nabla\Theta = -A\nabla\tau - \frac{f'(\tau)}{f(\tau)} \Theta \nabla\tau. \tag{10}$$

Moreover, by (4) and (9), we conclude that

$$\nabla_X \nabla\tau = \frac{f'(\tau)}{f(\tau)} X + \Theta A X - \frac{f'(\tau)}{f(\tau)} \langle X, \nabla\tau \rangle \nabla\tau. \tag{11}$$

Let $E_1, \cdots, E_n$ be a local orthonormal frame on $\mathfrak{X}(\Sigma^n)$; from (10), we know that

$$\Delta\Theta = -\sum_{i=1}^n \left\langle \nabla_{E_i}(A\nabla\tau), E_i \right\rangle - \sum_{i=1}^n \left\langle \nabla_{E_i}\left(\frac{f'(\tau)}{f(\tau)} \Theta \nabla\tau\right), E_i \right\rangle. \tag{12}$$

In fact, for every $X, Y \in \mathfrak{X}(\Sigma^n)$, we have that

$$(\nabla_X A)(Y) = \nabla A(Y, X) = \nabla_X(AY) - A(\nabla_X Y), \tag{13}$$

where $\nabla_X A$ denotes the covariant derivative of A. From (12) and (13), we can rewrite (12) as

$$\Delta\Theta = -\sum_{i=1}^{n}\left\langle\left(\nabla_{E_i}A\right)\nabla\tau, E_i\right\rangle - \sum_{i=1}^{n}\left\langle\nabla_{E_i}\nabla\tau, AE_i\right\rangle - \frac{f''(\tau)}{f(\tau)}\Theta|\nabla\tau|^2 + 2\frac{f'(\tau)^2}{f(\tau)^2}\Theta|\nabla\tau|^2 + \frac{f'(\tau)}{f(\tau)}\langle A\nabla\tau,\nabla\tau\rangle - \frac{f'(\tau)}{f(\tau)}\Theta\sum_{i=1}^{n}\left\langle\nabla_{E_i}\nabla\tau, E_i\right\rangle. \tag{14}$$

Recall that the Codazzi equation of $\psi : \Sigma^n \longrightarrow I \times_f M^n$ is

$$\left\langle\bar{R}(X, Y)N, Z\right\rangle = \langle(\nabla_X A)Y, Z\rangle - \langle(\nabla_Y A)X, Z\rangle, \tag{15}$$

or, equivalently,

$$\left(\bar{R}(X, Y)N\right)^T = (\nabla_Y A)X - (\nabla_X A)Y, \tag{16}$$

where $\bar{R}$ denotes the curvature tensor of $\bar{M}^{n+1}$. Therefore, using (11) and (15), we conclude from (14) that

$$\Delta\Theta = -\sum_{i=1}^{n}\langle(\nabla_{\nabla\tau}A)E_i, E_i\rangle - \sum_{i=1}^{n}\left\langle\bar{R}(E_i,\nabla\tau)N, E_i\right\rangle + \frac{f'(\tau)}{f(\tau)}\langle A\nabla\tau,\nabla\tau\rangle - \Theta|A|^2 - n\frac{f'(\tau)}{f(\tau)}H - \frac{f''(\tau)}{f(\tau)}\Theta|\nabla\tau|^2 + 2\frac{f'(\tau)^2}{f(\tau)^2}\Theta|\nabla\tau|^2 + \frac{f'(\tau)}{f(\tau)}\langle A\nabla\tau,\nabla\tau\rangle + \frac{f'(\tau)^2}{f(\tau)^2}\Theta|\nabla\tau|^2 - n\frac{f'(\tau)}{f(\tau)}H\Theta^2 - n\frac{f'(\tau)^2}{f(\tau)^2}\Theta. \tag{17}$$

Here, we know the general fact that

$$\mathrm{tr}(\nabla_X A) = \langle\nabla\mathrm{tr}A, X\rangle = n\langle\nabla H, X\rangle, \tag{18}$$

for every $X \in \mathfrak{X}(\Sigma^n)$. Then, it follows from (9) that

$$\Delta\Theta = -n\langle\nabla H,\nabla\tau\rangle + \bar{\mathrm{Ric}}(N,\nabla\tau) - \Theta|A|^2 + 2\frac{f'(\tau)}{f(\tau)}\langle A\nabla\tau,\nabla\tau\rangle - n\frac{f'(\tau)}{f(\tau)}H\left(1+\Theta^2\right) - \frac{f''(\tau)}{f(\tau)}\Theta|\nabla\tau|^2 - \frac{f'(\tau)^2}{f(\tau)^2}\Theta\left(n - 3|\nabla\tau|^2\right), \tag{19}$$

where $\bar{\mathrm{Ric}}$ stands for the Ricci curvature tensor of $\bar{M}^{n+1}$. On the other hand, using Corollary 7.43 of [1], we have that

$$\bar{\mathrm{Ric}}(N,\nabla\tau) = \mathrm{Ric}^M(N^*, (\nabla\tau)^*) - (n-1)(\log f)''(\tau)\Theta|\nabla\tau|^2, \tag{20}$$

where Ric^M is the Ricci curvature tensor of the fiber M^n and $N^* = N - \langle N, \partial_t\rangle\partial_t$ and $(\nabla\tau)^* = \nabla\tau - \langle\nabla\tau, \partial_t\rangle\partial_t$ denote the projections of the vector field N and $\nabla\tau$ onto M^n, respectively. Moreover, from (4), we obtain

$$(\nabla\tau)^* = -\langle N, \partial_t\rangle N^* = -\Theta N^*. \tag{21}$$

Therefore, from (20) and (21), we get

$$\bar{\mathrm{Ric}}(N,\nabla\tau) = -\Theta\left(\mathrm{Ric}^M(N^*, N^*) + (n-1)(\log f)''(\tau)|\nabla\tau|^2\right). \tag{22}$$

Finally, substituting (22) into (19), by a direct computation, we can obtain (6). □

We recall that a complete Riemannian manifold is parabolic in the sense that any positive superharmonic function on the Riemannian manifold must be constant (see [9]). In this setting, we obtain some uniqueness results via parabolicity criterion.

Theorem 2. *Let $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1} (n \geq 3)$ be a complete parabolic hypersurface with constant mean curvature in a warped product $\bar{M}^{n+1} = I \times_f M^n$ whose fiber M^n has nonnegative Ricci curvature. Assume that the warping function satisfies $(\log f)''(\tau) \geq 0$. If $Hf'(\tau) \geq 0$ and $\Theta^2 \geq 1/4$, then Σ^n is totally geodesic with constant warping function. In addition, if the fiber M^n has positive Ricci curvature, then Σ^n is a totally geodesic minimal slice.*

Proof. From (10), we obtain that

$$|\nabla\Theta|^2 = \frac{f'(\tau)^2}{f(\tau)^2}\Theta^2|\nabla\tau|^2 + |A(\nabla\tau)|^2 + 2\frac{f'(\tau)}{f(\tau)}\Theta\langle A\nabla\tau,\nabla\tau\rangle. \tag{23}$$

Therefore, using (6) and (23), we deduce that

$$\Delta\log(1+\Theta) = \frac{\Delta\Theta}{1+\Theta} - \frac{|\nabla\Theta|^2}{(1+\Theta)^2} = -nH\frac{1+\Theta^2}{1+\Theta}\frac{f'(\tau)}{f(\tau)} + \frac{2}{(1+\Theta)^2}\frac{f'(\tau)}{f(\tau)}\langle A\nabla\tau,\nabla\tau\rangle - \frac{\Theta}{1+\Theta}|A|^2 - \frac{|A(\nabla\tau)|^2}{(1+\Theta)^2} - n\frac{\Theta}{1+\Theta}\frac{f'(\tau)^2}{f(\tau)^2} + \frac{2\Theta^2+3\Theta}{(1+\Theta)^2}\frac{f'(\tau)^2}{f(\tau)^2}|\nabla\tau|^2 - \frac{\Theta}{1+\Theta}\frac{f''(\tau)}{f(\tau)}|\nabla\tau|^2 - \frac{\Theta}{1+\Theta}\left(\mathrm{Ric}^M(N^*, N^*) + (n-1)(\log f)''(\tau)|\nabla\tau|^2\right). \tag{24}$$

Moreover, using Young's inequality, we have

$$2\frac{f'(\tau)}{f(\tau)}\langle A\nabla\tau,\nabla\tau\rangle \le \frac{f'(\tau)^2}{f(\tau)^2}|\nabla\tau|^2 + |A(\nabla\tau)|^2. \tag{25}$$

So, we can estimate

$$\begin{aligned}\Delta\log(1+\Theta) \le{}& -nH\frac{1+\Theta^2}{1+\Theta}\frac{f'(\tau)}{f(\tau)} + \frac{1}{1+\Theta}\\ &\cdot\left[1-(n-2)\Theta-\Theta^2-2\Theta^3\right]\frac{f'(\tau)^2}{f(\tau)^2}\\ &-\frac{\Theta}{1+\Theta}|A|^2 - \frac{\Theta}{1+\Theta}\\ &\cdot\left(\mathrm{Ric}^M(N^*,N^*)+(n-1)(\log f)''(\tau)|\nabla\tau|^2\right).\end{aligned} \tag{26}$$

Using the hypothesis of Theorem 2, we conclude that $\log(1+\Theta)>0$ and $\Delta\log(1+\Theta)\le 0$, which suffices to show that $\log(1+\Theta)=\log(1+\Theta_0)$ is constant, So, $\Delta\log(1+\Theta)=0$. From (26), we know that

$$\begin{aligned} f'(\tau)&=0,\\ |A|&=0.\end{aligned} \tag{27}$$

It follows that the hypersurface Σ^n is totally geodesic and $f(\tau)$ is constant. In addition, if the fiber M^n has positive Ricci curvature, from (26), we have that $N^*(p)=0$ at any $p\in\Sigma^n$; that is, $\nabla\tau=0$ on Σ^n, and then, Σ^n is a totally geodesic slice. Moreover, we note that the mean curvature H of a slice in a warped product $I\times_f M^n$ is given by

$$H=\frac{f'(\tau)}{f(\tau)}=0. \tag{28}$$

Therefore, Σ^n is a totally geodesic minimal slice. □

Remark 3. For the two-dimensional case, we can set $\Theta^2\ge 0.44$. More precisely, for any dimension, we can chose $\Theta^2\ge\alpha$, where α is the only real (positive) root of $P(t)=1-(n-2)t-t^2-2t^3$; this root goes to zero as $n\longrightarrow\infty$.

On the other hand, if the warped products $I\times_f M^n$ satisfy the following convergence condition:

$$\mathrm{Ric}^M \ge (n-1)\sup_I\left(f'^2-f''f\right), \tag{29}$$

we have the following.

Theorem 4. *Let $\psi:\Sigma^n\longrightarrow\bar{M}^{n+1}$ be a complete parabolic hypersurface with constant mean curvature in a warped product $\bar{M}^{n+1}=I\times_f M^n$ which satisfies the convergence condition (29). Assume that the warping function satisfies $f''(\tau)\ge 0$. If $Hf'(\tau)\ge 0$ and $\Theta\ge 1/3$, then Σ^n is totally geodesic with constant warping function. In addition, if the inequality (29) is strict, then Σ^n is a totally geodesic minimal slice.*

Proof. By an analogous way in the proof of Theorem 2, we can estimate

$$\begin{aligned}\Delta\log(1+\Theta)\le{}& -nH\frac{1+\Theta^2}{1+\Theta}\frac{f'(\tau)}{f(\tau)} - \frac{\Theta}{1+\Theta}|A|^2-\Theta(1-\Theta)\\ &\cdot\frac{f''(\tau)}{f(\tau)} - \left[(n+2)\Theta^2+(n-1)\Theta-1\right]\\ &\cdot\frac{f'(\tau)^2}{f(\tau)^2} - \frac{\Theta}{1+\Theta}\\ &\cdot\left(\mathrm{Ric}^M(N^*,N^*)+(n-1)(\log f)''(\tau)|\nabla\tau|^2\right).\end{aligned} \tag{30}$$

Considering condition (29), it follows that $\mathrm{Ric}^M(N^*,N^*)+(n-1)(\log f)''(\tau)|\nabla\tau|^2\ge 0$. Moreover, since $\Theta\ge 1/3$, we get $(n+2)\Theta^2+(n-1)\Theta-1\ge 0$. Under the assumptions of Theorem 4, from (30), we conclude that $\Delta\log(1+\Theta)\le 0$, taking into account that the hypersurface Σ^n is parabolic. Via parabolicity criterion, we get that $\Theta=\Theta_0$ is constant, $f'(\tau)=0$, and $|A|=0$. Therefore, we have that the hypersurface Σ^n is totally geodesic and $f(\tau)$ is constant.

Moreover, if (29) is strict, by a similar reasoning as in the proof of Theorem 2, we get that Σ^n is a totally geodesic minimal slice. □

Remark 5. Note that by weakening the assumptions of Theorem 2 and Theorem 4, the uniqueness result of warped product does not hold. In fact, the sphere S^n immersed in a slice of $\mathbb{R}\times_f S^n$ with $f(t)=e^t$ satisfies all hypothesis of both theorems except that $Hf'(\tau)\ge 0$, since the mean curvature of the sphere is negative. It is a slice, but it fails to be totally geodesic.

Moreover, if the fiber M^n has nonnegative Ricci curvature, assume that the warping function satisfies $(\log f)''(\tau)\ge 0$which automatically implies that the hypersurface obeys the convergence condition (29). Thus, Theorem 4 extends Theorem 2.

It should also be noticed that the similar idea has been used to obtain the rigidity of hypersurfaces in warped products (see [10], Theorem 4.11). Nevertheless, we take a different approach to prove our main uniqueness results (Theorems 2 and 4) of constant mean curvature hypersurfaces in warped products.

4. Bernstein Type Problems

In the nonparametric case of hypersurfaces in the Riemannian manifold, there is a very celebrated result known as the Bernstein theorem. The original Bernstein theorem is that each complete minimal surface in $\mathbb{R}^3$ that can be written as the graph of a function on $\mathbb{R}^2$ must be a plane. Later, Chern [11] built a different proof of the original Bernstein theorem. In [12], Simons extended the Bernstein theorem to Euclidean space $\mathbb{R}^{n+1}$ $(n \leq 7)$ in which any complete minimal hypersurface in $\mathbb{R}^{n+1}$ must be a hyperplane with $n \leq 7$, the result which is obtained by successive efforts of Almgren [13], Fleming [14], and De Giorgi [15]. However, for $n \geq 8$, Bombieri et al. [16] constructed a counterexample. In recent years, many researchers made great efforts to extend these Bernstein type theorems to a wide variety of ambient spaces.

As a consequence of the parametric case, in this section, we will solve the Bernstein problem in warped products. Therefore, we study the graph over $(M^n, \langle,\rangle_M)$ in the warped product $\bar{M}^{n+1} = I \times_f M^n$, given by

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\} \subset \bar{M}^{n+1}, \tag{31}$$

where $\Omega \subset M^n$ denote a connected domain of M^n and $u : \Omega \longrightarrow I$ is a smooth function on Ω. Moreover, the graph inherits from $\bar{M}^{n+1}$ a metric, which is defined by

$$\langle,\rangle = du^2 + f(u)^2 \langle,\rangle_M. \tag{32}$$

Note that if M^n is complete and $\inf_{\Sigma^n} f(u) > 0$, then $\Sigma^n(u)$ is complete. Moreover, for any point $p \in M^n$, we have $\tau(u(p), p) = u(p)$. Therefore, u and τ may be naturally identified on $\Sigma^n(u)$.

Moreover, if $\Omega = M^n$, then the graph $\Sigma^n(u)$ is entire graph. In this case, the unit normal vector field on $\Sigma^n(u)$ is

$$N = \frac{1}{f(u)\sqrt{f(u)^2 + |Du|^2}} \left(f(u)^2 \partial_t - Du\right). \tag{33}$$

Thus, the mean curvature function $H(u)$ of $\Sigma^n(u)$ associated to N is

$$H(u) = \operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 + |Du|^2}}\right) - \frac{f'(u)}{n\sqrt{f(u)^2 + |Du|^2}}\left(n - \frac{|Du|^2}{f(u)^2}\right). \tag{34}$$

In the following, we will apply the uniqueness results of constant mean curvature hypersurfaces obtained in Section 3 to prove new Bernstein type results for the constant mean curvature hypersurface equation:

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 + |Du|^2}}\right) - \frac{f'(u)}{\sqrt{f(u)^2 + |Du|^2}}\left(n - \frac{|Du|^2}{f(u)^2}\right) = H. \tag{35}$$

Theorem 6. *Let $\Sigma(u)$ be a parabolic entire graph in $I \times_f M^n$ $(n \geq 3)$, where $f : I \longrightarrow \mathbb{R}^+$ is a smooth function which satisfies $(\log f)''(u) \geq 0$. Then, the only bounded entire solutions to equation (35) with $|Du| \leq \sqrt{3} f(u)$, for $H \in \mathbb{R}$ and such that $Hf'(u) \geq 0$ on a complete Riemannian manifold M^n with positive Ricci curvature, are the constant functions $u = t_0$, with $t_0 \in I$ such that $f'(t_0) = 0$.*

Proof. A straightforward computation yields

$$|\nabla \tau|^2 = 1 - \Theta^2 = \frac{|Du|^2}{f(u)^2 + |Du|^2}. \tag{36}$$

It follows that from the constraints $|Du| \leq \sqrt{3} f(u)$ and (36), we have that

$$\frac{1}{4} < \Theta^2 \leq 1. \tag{37}$$

Finally, applying Theorem 2, the proof ends. □

Furthermore, we can use condition $\mathrm{Ric}^M > (n-1)\sup_I (f'^2 - f''f)$ in Theorem 4 to prove a Bernstein type result in an analogous way.

Theorem 7. *Let $\Sigma(u)$ be a parabolic entire graph in $I \times_f M^n$, where $f : I \longrightarrow \mathbb{R}^+$ is a smooth function which satisfies $f''(u) \geq 0$. Then, the only bounded entire solutions to equation (35) with $|Du| \leq 2\sqrt{2} f(u)$, for $H \in \mathbb{R}$ and such that $Hf'(u) \geq 0$ on a complete Riemannian manifold M^n satisfies the convergence condition (29), are the constant functions $u = t_0$, with $t_0 \in I$ such that $f'(t_0) = 0$.*

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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