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Research Article

A Solution of the Complex Fuzzy Heat Equation in Terms of Complex Dirichlet Conditions Using a Modified Crank–Nicolson Method

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Complex fuzzy sets (CFSs) have recently emerged as a potent tool for expanding the scope of fuzzy sets to encompass wider ranges within the unit disk in the complex plane. This study explores complex fuzzy numbers and introduces their application for the first time in the literature to address a complex fuzzy partial differential equation that involves a complex fuzzy heat equation under Hukuhara differentiability. The researchers utilize an implicit finite difference scheme, namely the Crank–Nicolson method, to tackle complex fuzzy heat equations. The problem's fuzziness arises from the coefficients in both amplitude and phase terms, as well as in the initial and boundary conditions, with the Convex normalized triangular fuzzy numbers extended to the unit disk in the complex plane. The researchers take advantage of the properties and benefits of CFS theory in the proposed numerical methods and subsequently provide a new proof of the stability under CFS theory. A numerical example is presented to demonstrate the proposed approach's reliability and feasibility, with the results showing good agreement with the exact solution and relevant theoretical aspects.

1. Introduction

Zadeh introduced the theory of fuzzy sets, which is a useful tool to handle uncertainty and vagueness in mathematical models, leading to a better understanding of real-life phenomena. Many real-life problems can be formulated as mathematical models that involve differential equations. However, the classical (Crisp) quantities in these equations can often be uncertain and imprecise. To account for this, fuzzy quantities can be used instead, resulting in what are known as fuzzy differential equations. Recently, there has been a growing interest in the analysis and applications of fuzzy differential equations, as they have found considerable use in fields such as mathematical physics [1], engineering [2], medicine [3], and others [4–8].

The fuzzy partial differential equation is commonly utilized to explain the behavior of dynamic phenomena in which imprecision or indeterminacy is present. This includes fuzzy heat conduction and fuzzy particle diffusion, with the fuzzy heat equation being one of the most important fuzzy parabolic partial differential equations for describing how a fuzzy quantity such as heat diffuses through a given area [9–21]. While exact analytical solutions for fuzzy heat equations may be challenging to obtain, numerical techniques are needed to achieve the solution. In recent decades, there have been numerous studies on solutions to the fuzzy heat equation. Allahviranloo [22] developed finite difference methods for solving fuzzy heat and wave equations, which were demonstrated to be effective by numerical examples. Ahmadi and Kiani [23] extended the differential transformation method to solve fuzzy partial differential equations under strongly generalized differentiability, which was found to be a straightforward and efficient method for obtaining analytical–numerical solutions. Wang and Qiu [24] proposed a fuzzy numerical technique based on the finite difference method for solving heat conduction problems that incorporate uncertainties in initial/boundary conditions and physical parameters. Bayrak and Can [25] considered the concept of generalized differentiability to solve fuzzy parabolic partial differential equations using the finite difference method, demonstrating its efficiency and simplicity.

The aforementioned studies employed fuzzy sets theory to solve governing equations, where the range of values falls within [0, 1]. Fuzzy sets have recently been utilized in medical applications to tackle complex biological systems and develop algorithmic solutions. To enhance the representation of information in accordance with the human brain, the range of membership function in complex fuzzy sets (CFSs) has been expanded from [0, 1] to the unit disk in the complex plane. This expansion allows for a more detailed representation of information while retaining its full meaning. The human mind typically processes meaningful information from vast amounts of data and produces reasonable solutions, but it can be influenced by various phases/factors that affect thinking and decision-making. Therefore, we propose that the use of CFS provides a suitable foundation for summarizing and extracting information from large datasets that impact the human brain under different phases/factors related to task performance.

Buckley et al. [26–29] first introduced the concept of fuzzy complex numbers (FCNs) in 1987. FCNs incorporate complex numbers into the support of a fuzzy set, resulting in a new type of fuzzy set with complex-valued membership functions. This idea was further extended in 2002 with the introduction of CFSs [30], which generalize the membership function of fuzzy sets from the unit interval [0, 1] to the unit disk in the complex plane. CFSs have been widely studied and applied by many researchers [31–43]. The innovation of CFS lies in its ability to represent both uncertainty and periodicity semantics simultaneously without losing the full meaning of the data. The concept of phase degrees was developed to classify similar data that are measured in different phases or levels. The complex fuzzy membership grade can be represented using polar and Cartesian forms with two fuzzy components, where the amplitude and phase terms of the complex numbers lie in the range [0, 1] [35]. The CFS reduces to a traditional fuzzy set when the phase membership is not considered [44]. To address the limitations and restrictions of CFS, Tamir and Kandel [34] developed an axiomatic for propositional complex fuzzy logic.

As known, the fuzzifications of the fuzzy heat equation are represented by the value in (real numbers) [0, 1]. The modified Crank–Nicolson method for solving the fuzzy complex heat equation improves accuracy by considering fuzzy complex temperature values at the half-time step, effectively capturing uncertainties and imprecisions in a more precise manner compared to the classical Crank–Nicolson method [45–48]. The aim of this paper is focused on the fuzzification of complex fuzzy heat equation using the modified Crank–Nicolson method, where represented by two values (amplitude and phase terms) in the unit disk in the complex plane to provide the generalization and more accuracy for the solution of fuzzy heat equation by putting in the account of a new periodicity semantics that appears in the complex fuzzy information [30, 43].

2. Heat Equation in Complex Fuzzy Environment

Consider the general form of 1D complex fuzzy heat equation [49] as follows:

$$\begin{aligned} \frac{\partial \tilde{u}(x,t)}{\partial t} &= \tilde{D}(x,t)\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + \tilde{b}(x,t), \quad 0<x<l, t>0 \\ \tilde{u}(x,0) &= \tilde{f}(x),\ \tilde{u}(0,t) = \tilde{g}(t),\ \tilde{u}(l,t) = \tilde{z}(t), \end{aligned} \tag{1}$$

where $\tilde{u}(x,t)$ is the complex fuzzy unknown function of the crisp variable x and t. $\frac{\partial \tilde{u}(x,t)}{\partial t}$, $\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2}$ are first and second complex fuzzy partial derivatives with $\tilde{D}(x,t)$ and $\tilde{b}(x,t)$ are the complex fuzzy functions. $\tilde{u}(0,x)$ is the complex fuzzy initial condition. $\tilde{u}(0,t)$ and $\tilde{u}(l,t)$ are the complex fuzzy boundary conditions. where the range of complex functions lies in the unit disk in the complex plane.

In Equation (1), the complex fuzzy functions $\tilde{D}(x)$, $\tilde{b}(x)$, $\tilde{f}(x)$, $\tilde{g}(t)$, and $\tilde{z}(t)$ being complex fuzzy convex numbers which are defined as follows [34]:

$$\begin{aligned} \tilde{D}(x,t) &= \tilde{q}_1 e^{i\alpha\widetilde{w_1}} s_1(x,t) \\ \tilde{b}(x,t) &= \tilde{q}_2 e^{i\alpha\widetilde{w_2}} s_2(x,t) \\ \tilde{f}(x) &= \tilde{q}_3 e^{i\alpha\widetilde{w_3}} s_3(x) \\ \tilde{g}(t) &= \tilde{q}_4 e^{i\alpha\widetilde{w_4}} s_4(t) \\ \tilde{z}(t) &= \tilde{q}_5 e^{i\alpha\widetilde{w_5}} s_5(t), \end{aligned} \tag{2}$$

where $s_1(x,t)$, $s_2(x,t)$, $s_3(x)$, $s_4(t)$, and $s_5(t)$ are the crisp functions of the crisp variable x and t with $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{q}_3$, $\tilde{q}_4$, $\tilde{q}_5$, $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$, $\tilde{w}_4$, and $\tilde{w}_5$ being fuzzy convex numbers.

The fuzzification of Equation (1) for all $r \in [0,1]$ is as follows [50]:

$$[\tilde{u}(x,t)]_{r,\theta} = \underline{u}(x,t;r,\theta), \overline{u}(x,t;r,\theta). \tag{3}$$

$$\left[\frac{\partial \tilde{u}(x,t)}{\partial t}\right]_{r,\theta} = \frac{\partial \underline{u}(x,t;r,\theta)}{\partial t}, \frac{\partial \overline{u}(x,t;r,\theta)}{\partial t}. \tag{4}$$

$$\left[\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2}\right]_{r,\theta} = \frac{\partial^2 \underline{u}(x,t;r,\theta)}{\partial x^2}, \frac{\partial^2 \overline{u}(x,t;r,\theta)}{\partial x^2}. \tag{5}$$

$$\left[\tilde{D}(x,t)\right]_{r,\theta} = \underline{D}(x,t;r,\theta), \overline{D}(x,t;r,\theta). \tag{6}$$

$$\left[\tilde{b}(x,t)\right]_{r,\theta} = \underline{b}(x,t;r,\theta), \overline{b}(x,t;r,\theta). \tag{7}$$

$$[\tilde{u}(x,0)]_{r,\theta} = \underline{u}(x,0;r,\theta), \overline{u}(x,0;r,\theta). \tag{8}$$

$$[\tilde{u}(0,t)]_{r,\theta} = \underline{u}(0,t;r,\theta), \overline{u}(0,t;r,\theta). \tag{9}$$

$$[\tilde{u}(l,t)]_{r,\theta} = \underline{u}(l,t;r,\theta), \overline{u}(l,t;r,\theta). \tag{10}$$

$$\left[\tilde{f}(x)\right]_{r,\theta} = \underline{f}(x;r,\theta), \overline{f}(x;r,\theta). \tag{11}$$

$$\begin{aligned} [\tilde{g}(t)]_{r,\theta} &= \underline{g}(t;r,\theta), \overline{g}(t;r,\theta) \\ [\tilde{z}(t)]_{r,\theta} &= \underline{z}(t;r,\theta), \overline{z}(t;r,\theta). \end{aligned} \tag{12}$$

$$\begin{aligned} \left[\tilde{D}(x,t)\right]_{r,\theta} &= \left[\underline{q}_1(r), \overline{q}_1(r)\right] e^{i\alpha\left[\underline{w}_1(\theta), \overline{w}_1(\theta)\right]} s_1(x,t) \\ \left[\tilde{b}(x,t)\right]_{r,\theta} &= \left[\underline{q}_2(r), \overline{q}_2(r)\right] e^{i\alpha\left[\underline{w}_2(\theta), \overline{w}_2(\theta)\right]} s_2(x,t) \\ \left[\tilde{f}(x)\right]_{r,} &= \left[\underline{q}_3(r), \overline{q}_3(r)\right] e^{i\alpha\left[\underline{w}_3(\theta), \overline{w}_3(\theta)\right]} s_3(x) \\ [\tilde{g}(t)]_{r,\theta} &= \left[\underline{q}_4(r), \overline{q}_4(r)\right] e^{i\alpha\left[\underline{w}_4(\theta), \overline{w}_4(\theta)\right]} s_4(\mathrm{t}) \\ [\tilde{z}(t)]_{r,\theta} &= \left[\underline{q}_5(r), \overline{q}_5(r)\right] e^{i\alpha\left[\underline{w}_5(\theta), \overline{w}_5(\theta)\right]} s_5(\mathrm{t}). \end{aligned} \tag{13}$$

The complex membership function is defined by using the fuzzy extension principle (ref based on complex).

$$\begin{aligned} \underline{u}(x,t;r,\theta) &= \min\{\tilde{u}(\tilde{\mu}(r,\theta),t)\,)|\tilde{\mu}(r,\theta) \in \tilde{u}(x,t;r,\theta)\} \\ \overline{u}(x,t;r,\theta) &= \max\{\tilde{u}(\tilde{\mu}(r,\theta),t)|\tilde{\mu}(r,\theta) \in \tilde{u}(x,t;r,\theta)\}. \end{aligned} \tag{14}$$

Now by substitute Equations (3)–(13) into Equation (1) to get the follows:

$$\begin{aligned} \frac{\partial \underline{u}(x,t;r,\theta)}{\partial t} &= \left[\underline{q}_1(r) e^{i\alpha \underline{w}_1(\theta)} s_1(x,t)\right] \frac{\partial^2 \underline{u}(x,t;r,\theta)}{\partial x^2} + \underline{q}_2(r) e^{i\alpha \underline{w}_2(\theta)} s_2(x,t) \\ \underline{u}(x,0;r,\theta) &= \underline{q}_3(r) e^{i\alpha \underline{w}_3(\theta)} s_3(x) \\ \underline{u}(0,t;r,\theta) &= \underline{q}_4(r) e^{i\alpha \underline{w}_4(\theta)} s_4(t), \underline{u}(l,t;r,\theta) = \underline{q}_5(r) e^{i\alpha \underline{w}_5(\theta)} s_5(t). \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial \overline{u}(x,t;r,\theta)}{\partial t} &= \left[\overline{q}_1(r) e^{i\alpha \overline{w}_1(\theta)} s_1(x,t)\right] \frac{\partial^2 \overline{u}(x,t;r,\theta)}{\partial x^2} + \overline{q}_2(r) e^{i\alpha \overline{w}_2(\theta)} s_2(x,t) \\ \overline{u}(x,0;r,\theta) &= \overline{q}_3(r) e^{i\alpha \overline{w}_3(\theta)} s_3(x) \\ \overline{u}(0,t;r,\theta) &= \overline{q}_4(r) e^{i\alpha \overline{w}_4(\theta)} s_4(t), \overline{u}(l,t;r,\theta) = \overline{q}_5(r) e^{i\alpha \overline{w}_5(\theta)} s_5(t). \end{aligned} \tag{16}$$

Equations (15) and (16) represent the general lower and upper forms of complex fuzzy heat equations, respectively.

3. Crank–Nicolson Method for Solution of Complex Fuzzy Heat Equation

This section adapts and uses a central difference approximation for the first-order time derivative and central difference approximation at time level $j+\frac{1}{2}$ for the second-order space derivative to solve the complex fuzzy heat equation.

The partial time derivative $\frac{\partial \underline{u}(x,t;r,\theta)}{\partial t}, \frac{\partial \overline{u}(x,t;r,\theta)}{\partial t}$ is discretised as follows:

$$\frac{\partial \underline{u}_{i,j}(x,t;r,\theta)}{\partial t} = \frac{\underline{u}_{i,j+1}(x,t;r,\theta) - \underline{u}_{i,j}(x,t;r,\theta)}{\Delta^t}. \tag{17}$$

$$\frac{\partial \overline{u}_{i,j}(x,t;r,\theta)}{\partial t} = \frac{\overline{u}_{i,j+1}(x,t;r,\theta) - \overline{u}_{i,j}(x,t;r,\theta)}{\Delta^t}. \tag{18}$$

Also, the second partial derivatives $\partial^2 \frac{\underline{u}\, i,j(x,t;r,\theta)}{dx^2}$, $\frac{\partial^2 \overline{u}_{i,j}(x,t;r,\theta)}{dx^2}$ can be defined as follows:

$$\frac{\partial^2 \underline{u}_{i,j}(x,t;r,\theta)}{dx^2} = \frac{\underline{u}_{i+1,j+\frac{1}{2}}(x,t;r,\theta) - 2\underline{u}_{i,j+\frac{1}{2}}(x,t;r,\theta) + \underline{u}_{i-1,j+\frac{1}{2}}(x,t;r,\theta)}{\Delta x^2}. \tag{19}$$

$$\frac{\partial^2 \overline{u}_{i,j}(x,t;r,\theta)}{dx^2} = \frac{\overline{u}_{i+1,j+\frac{1}{2}}(x,t;r,\theta) - 2\overline{u}_{i,j+\frac{1}{2}}(x,t;r,\theta) + \overline{u}_{i-1,j+\frac{1}{2}}(x,t;r,\theta)}{\Delta x^2}. \tag{20}$$

Now substitute Equations (17)–(20) into Equations (15) and (16), respectively, to obtain the following:

$$\frac{\underline{u}_{i,j+1}(x,t;r,\theta)-\underline{u}_{i,j}(x,t;r,\theta)}{\Delta^t} = \underline{D}(x,t;r,\theta)\frac{\underline{u}_{i+1,j+\frac{1}{2}}(x,t;r,\theta)-2\underline{u}_{i,j+\frac{1}{2}}(x,t;r,\theta)+\underline{u}_{i-1,j+\frac{1}{2}}(x,t;r,\theta)}{\Delta x^2}+\underline{b}(x,t;r,\theta). \tag{21}$$

$$\frac{\overline{u}_{i,j+1}(x,t;r,\theta)-\overline{u}_{i,j}(x,t;r,\theta)}{\Delta^t} = \overline{D}(x,t;r,\theta)\frac{\overline{u}_{i+1,j+\frac{1}{2}}(x,t;r,\theta)-2\overline{u}_{i,j+\frac{1}{2}}(x,t;r,\theta)+\overline{u}_{i-1,j+\frac{1}{2}}(x,t;r,\theta)}{\Delta x^2}+\overline{b}(x,t;r,\theta). \tag{22}$$

By assume that $\widetilde{s}(r,\theta)=\frac{\widetilde{D(x,t;r,\theta)\Delta t}}{\Delta x^2}$ and then Equations (21) and (22) are simplified to get the generally lower and upper solution for the complex fuzzy heat equation for all r, $\theta\in[0,1]$ as follows:

$$(2+2s)\underline{u}_{i,j+1}(x,t;r,\theta)-s\Big(\underline{u}_{i+1,j+1}(x,t;r,\theta)+\underline{u}_{i-1,j+1}(x,t;r,\theta)\Big) = (2-2s)\underline{u}_{i,j}(x,t;r,\theta)+s\Big(\underline{u}_{i+1,j}(x,t;r,\theta)+\underline{u}_{i-1,j}(x,t;r,\theta)\Big) + \Delta^t\underline{b}(x,t;r,\theta). \tag{23}$$

$$(2+2s)\overline{u}_{i,j+1}(x,t;r,\theta)-s\big(\overline{u}_{i+1,j+1}(x,t;r,\theta)+\overline{u}_{i-1,j+1}(x,t;r,\theta)\big) = (1-2s)\overline{u}_{i,j}(x,t;r,\theta)+s\big(\overline{u}_{i+1,j}(x,t;r,\theta)+\overline{u}_{i-1,j}(x,t;r,\theta)\big) + \Delta^t\overline{b}(x,t;r,\theta). \tag{24}$$

4. The Stability of Crank–Nicolson for Fuzzy Complex Heat Equation

Theorem 1. The Crank–Nicolson method in Equation (23) for complex fuzzy heat equation is unconditionally stable.

Proof. Let $\widetilde{\varepsilon}_i^0$ represent the fuzzy error of the discretization of the initial condition. □

Let $\widetilde{u}_i^0=\widetilde{u}_i^{\prime 0}-\widetilde{\varepsilon}_i^0$, $\widetilde{u}_i^n$, and $\widetilde{u}_i^{\prime n}$ refer to numerical solution of Equation (15) in terms to the initial data $\widetilde{f}_i^0$ and $\widetilde{f}_i^{\prime 0}$, respectively.

Let $\left[\widetilde{u}_{i+1}^n(x,\ t)\right]_{r,\theta}=\overline{u}(r,\theta)-\underline{u}(r,\theta)$, where $r,\theta\in[0,1]$.

The fuzzy absolute error is established by the following form:

The fuzzy error equations for Equation (12) are as follows:

$$[\widetilde{\varepsilon}_i^n]_{r,\theta}=\left[\widetilde{u}_i^{\prime n}-\widetilde{u}_i^n\right]_{r,\theta},\ n=1,2,\ldots,X\times M,\ i=1,2,\ldots,X-1. \tag{25}$$

$$(2+2s)\widetilde{\varepsilon}_i^{n+1}-s\big(\widetilde{\varepsilon}_{i+1}^{n+1}+\widetilde{\varepsilon}_{i-1}^{n+1}\big)=(2-2s)\widetilde{\varepsilon}_i^n+s\big(\widetilde{\varepsilon}_{i+1}^n+\widetilde{\varepsilon}_{i-1}^n\big). \tag{26}$$

$\widetilde{\varepsilon}_0^n=\widetilde{\varepsilon}_X^n=0$, $n=1,2,\ldots,T\times M$. Let $\widetilde{\varepsilon}_i^n=[\widetilde{\varepsilon}_1^n,\widetilde{\varepsilon}_2^n,\ldots,\widetilde{\varepsilon}_{X-1}^n]$, and introduce the following fuzzy norm:

$$\|\widetilde{\varepsilon}^n\|_2^2=\sum_{i=1}^{X-1}h|\widetilde{\varepsilon}_i^n|^2. \tag{27}$$

Suppose that $\widetilde{\varepsilon}_i^n$ can be expressed in the form

$$\widetilde{\varepsilon}_i^n=\widetilde{\lambda}^n e^{\sqrt{-\theta_i}},\ \text{where}\ \widetilde{\theta}_i=qih. \tag{28}$$

Substituting Equation (28) into Equation (26) to obtain the following:

$$(2+2s)\widetilde{\lambda}^{n+1}e^{\sqrt{-\theta_i}}-s\Big(\widetilde{\lambda}^{n+1}e^{\sqrt{-\theta_{i+1}}}+\widetilde{\lambda}^{n+1}e^{\sqrt{-\theta_{i-1}}}\Big) = (2-2s)\widetilde{\lambda}^n e^{\sqrt{-\theta_i}}+s\Big(\widetilde{\lambda}^n e^{\sqrt{-\theta_{i+1}}}+\widetilde{\lambda}^n e^{\sqrt{-\theta_{i-1}}}\Big). \tag{29}$$

Divide Equation (29) on $\widetilde{\lambda}^n e^{\sqrt{-\theta i}}$ to obtain the following:

$$(2+2s)\widetilde{\lambda}-s\Big(\widetilde{\lambda}e^{\sqrt{-\theta_i}}+\widetilde{\lambda}e^{-\sqrt{-\theta_i}}\Big) = (2-2\widetilde{s})+\widetilde{s}\Big(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\Big), \tag{30}$$

Since $\Big(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\Big)=\cos\theta=1-2\sin^2\left(\frac{\theta}{2}\right)$ and substituted in Equation (30) to get the following:

$$\widetilde{\lambda}=\left|\frac{1-2\,s\sin^2\left(\frac{\theta}{2}\right)}{1+2\,s\sin^2\left(\frac{\theta}{2}\right)}\right|. \tag{31}$$

Since the maximum of $\sin^2\left(\frac{\theta}{2}\right)=1$, we obtain $|\widetilde{s}|\le 1$. So the Crank–Nicolson method is unconditionally stable.

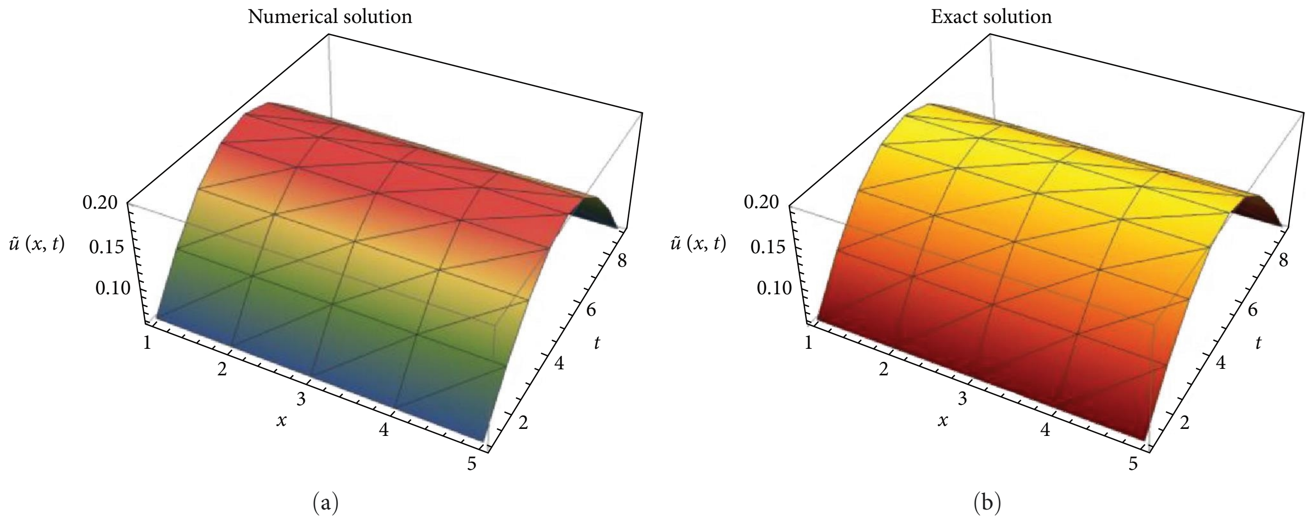

Figure 1: (a) Fuzzy numerical and (b) fuzzy exact solution of Equation (32) at $\theta = 0.2$ and $r = 0$.

5. Numerical Example

The fuzzy heat equation by Allahviranloo and Taheri [51] is generalized by adding the phase term, as discussed in the previous sections, to be a complex heat equation as follows:

$$\frac{\partial \widetilde{U}(x,t)}{\partial t} - \frac{\partial^2 \widetilde{U}(x,t)}{\partial x^2} = 0,\ 0<x<1, t>0, \tag{32}$$

with the boundary conditions $\widetilde{u}(0,t) = \widetilde{u}(1,t) = 0$ and initial condition

$\widetilde{u}(x,0) = \widetilde{f}(x)$, where complex fuzzy function $\widetilde{f}(x)$ is defined as follows:

$$\widetilde{f}(x) = \widetilde{k}e^{i2\pi\widetilde{w}}\cos\left(\pi x - \frac{\pi}{2}\right), \tag{33}$$

where

$$\left[\widetilde{k}\right]_r = [-1,0,1]_r = [r-1, 1-r]. \tag{34}$$

$$[\widetilde{w}]_\theta = [-1,0,1]_\theta = [\theta-1, 1-\theta]. \tag{35}$$

$$\begin{aligned}\left[\widetilde{E}\right]_{r,\theta} &= \left|\widetilde{U}(t,x;r,\theta) - \widetilde{u}(t,x;r,\theta)\right.\\ &= \begin{cases}\left[\underline{E}\right]_{r,\theta} = \left|\underline{U}(t,x;r,\theta) - \underline{u}(t,x;r,\theta)\right| \\ \left[\overline{E}\right]_{r,\theta} = \left|\overline{U}(t,x;r,\theta) - \overline{u}(t,x;r,\theta)\right|\end{cases}.\end{aligned} \tag{36}$$

The exact solution of Equation (32) is defined as follows:

$$\widetilde{u}(x,t;r) = \widetilde{\alpha}e^{i2\pi\widetilde{w}-\pi t}\cos\left(\pi x - \frac{\pi}{2}\right). \tag{37}$$

Figures 1–4 and Tables 1 and 2 show that the obtained numerical solutions by the implicit Crank–Nicolson method possess a high degree of congruence with the exact solution at $x = 0.9$, $t = 0.05$ for all $r, \theta \in [0,1]$. Furthermore, both the exact solution and numerical solutions to the proposed schemes take on the shape of a triangular fuzzy number for both the real part and imaginary part, which satisfies the properties of complex fuzzy numbers [46–48]. Additionally, from Tables 1 and 2, it can be noted that the accuracy of numerical results depends upon the value of phase term θ which satisfies with our theoretical analysis and demonstrates the significance and impact of adding the phase term. It takes into consideration that the numerical solution of the fuzzy heat equation is obtained from the complex fuzzy heat by substituting $\theta = 0$ and 1.

6. Conclusions

In this paper, the FCN has been applied to solve the complex fuzzy heat equation based on the Crank–Nicolson method. The fuzziness of the problem appears in the initial and boundary conditions as well as coefficients in both amplitude and phase terms simultaneously. The obtained results using the Crank–Nicolson scheme satisfy the complex fuzzy number properties by taking the triangular fuzzy number shape for both the real part and imaginary part and have an accuracy of order $O(\Delta t + \Delta x^2)$. The stability of the proposed approach is discussed to show that Crank–Nicolson scheme is unconditionally stable. Furthermore, it was found that the complex fuzzy approach is general and computationally efficient to transfer the information that happens periodically. The presented approach may be extended to solve several linear and nonlinear complex fuzzy partial differential equations, and this will be investigated in detail at a later stage.

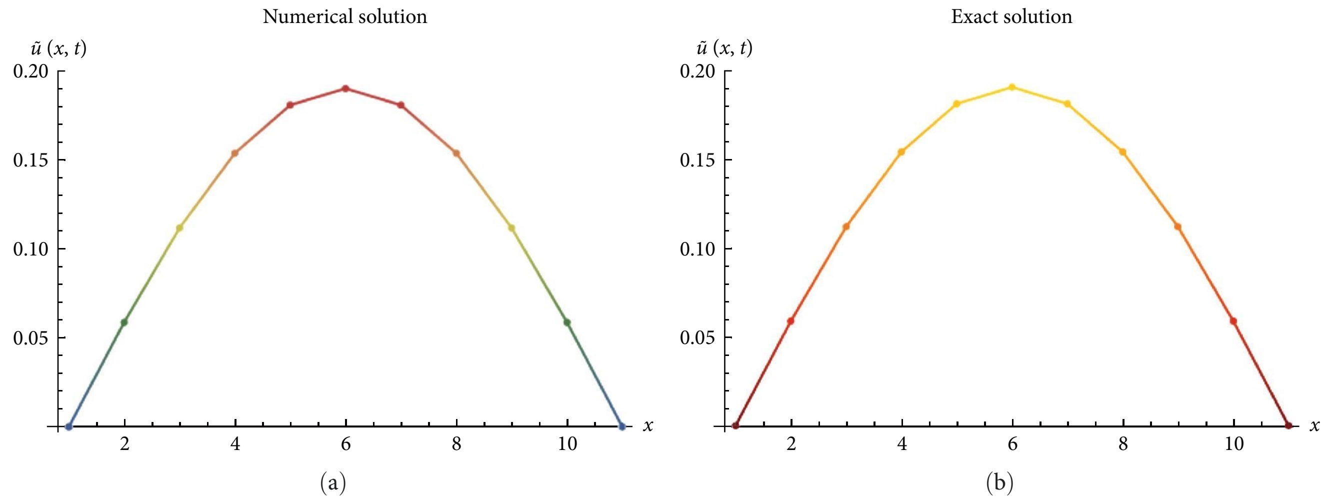

Figure 2: (a) Fuzzy numerical and (b) fuzzy exact solution of Equation (32) at $t = 5, \theta = 0.2$, and $r = 0$.

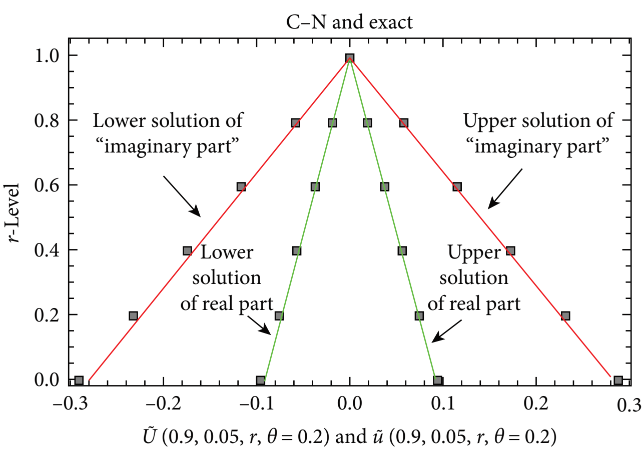

Figure 3: The exact and numerical solution of Equation (32) by C–N at $t = 0.05$, $x = 0.9$, and $\theta = 0.2$ for all $r \in [0, 1]$.

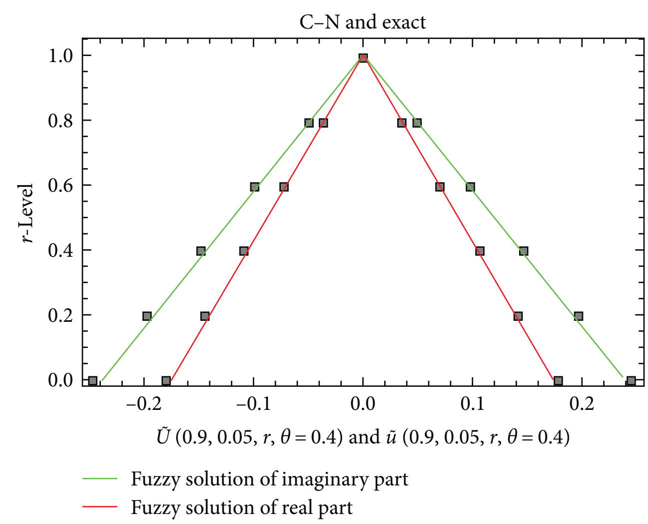

Figure 4: The exact and numerical solution of Equation (32) by C–N at $t = 0.05$, $x = 0.9$, and $\theta = 0.4$ for all $r \in [0, 1]$.

Table 1: Numerical solution of Equation (32) by C–N at $t = 0.05$ and $x = 0.9$ for all $r, \theta \in [0, 1]$.

		Lower solution		Upper solution	
θ	r	$\underline{u}(0.9, 0.05; r, \theta)$	$\underline{E}(0.9, 0.05; r, \theta)$	$\overline{u}(0.9, 0.05; r, \theta)$	$\overline{E}(0.9, 0.05; r, \theta)$
0.2	0	$-0.09093 - 0.27985\,i$	$0.00307 + 0.00946\,i$	$0.09093 + 0.27985\,i$	$0.00307 + 0.00946\,i$
	0.2	$-0.07274 - 0.22388\,i$	$0.00246 + 0.00756\,i$	$0.07274 + 0.22388\,i$	$0.00246 + 0.00756\,i$
	0.4	$-0.05455 - 0.16791\,i$	$0.00184 + 0.00567\,i$	$0.05455 + 0.16791\,i$	$0.00184 + 0.00567\,i$
	0.6	$-0.03637 - 0.1119\,i$	$0.00123 + 0.00378\,i$	$0.03637 + 0.1119\,i$	$0.00123 + 0.00378\,i$
	0.8	$-0.01819 - 0.05597\,i$	$0.00061 + 0.0019\,i$	$0.01819 + 0.05597\,i$	$0.00061 + 0.0019\,i$
	1	0	0	0	0
0.4	0	$0.23805 - 0.17295\,i$	$0.00804 + 0.00584\,i$	$-0.23805 + 0.17295\,i$	$0.00804 + 0.00584\,i$
	0.2	$0.19044 - 0.13836\,i$	$0.00643 + 0.00467\,i$	$-0.19044 + 0.13836\,i$	$0.00643 + 0.00467\,i$
	0.4	$0.14283 - 0.10377\,i$	$0.00483 + 0.00351\,i$	$-0.14283 + 0.10377\,i$	$0.00483 + 0.00351\,i$
	0.6	$0.09522 - 0.06918\,i$	$0.00322 - 0.06682\,i$	$-0.09522 + 0.06918\,i$	$0.00322 - 0.06682\,i$
	0.8	$0.04761 - 0.03459\,i$	$0.00162 + 0.00118\,i$	$-0.04761 + 0.03459\,i$	$0.00162 + 0.00118\,i$
	1	0	0	0	0

Table 2: Numerical solution of Equation (32) by C–N at $t = 0.05$ and $x = 0.9$ for all $r, \theta \in [0, 1]$.

		Lower solution		Upper solution	
θ	r	$\underline{u}(0.9, 0.05; r, \theta)$	$\underline{E}(0.9, 0.05; r, \theta)$	$\overline{u}(0.9, 0.05; r, \theta)$	$\overline{E}(0.9, 0.05; r, \theta)$
0.6	0	$0.23805 + 0.17295\,i$	$0.00804 + 0.00584\,i$	$-0.23805 - 0.17295\,i$	$0.00804 + 0.00584\,i$
	0.2	$0.19044 + 0.13836\,i$	$0.00643 + 0.00467\,i$	$-0.19044 - 0.13836\,i$	$0.00643 + 0.00467\,i$
	0.4	$0.14283 + 0.10377\,i$	$0.00483 + 0.00351\,i$	$-0.14283 - 0.10377\,i$	$0.00483 + 0.00351\,i$
	0.6	$0.09522 + 0.06918\,i$	$0.00322 - 0.06682\,i$	$-0.09522 - 0.06918\,i$	$0.00322 - 0.06682\,i$
	0.8	$0.04761 + 0.03459\,i$	$0.00162 + 0.00118\,i$	$-0.04761 - 0.03459\,i$	$0.00162 + 0.00118\,i$
	1	0	0	0	0
0.8	0	$-0.09093 + 0.27985\,i$	$0.00307 + 0.00946\,i$	$0.09093 - 0.27985\,i$	$0.00307 + 0.00946\,i$
	0.2	$-0.07274 + 0.22388\,i$	$0.00246 + 0.00756\,i$	$0.07274 - 0.22388\,i$	$0.00246 + 0.00756\,i$
	0.4	$-0.05455 + 0.16791\,i$	$0.00184 + 0.00567\,i$	$0.05455 - 0.16791\,i$	$0.00184 + 0.00567\,i$
	0.6	$-0.03637 + 0.1119\,i$	$0.00123 + 0.00378\,i$	$0.03637 - 0.1119\,i$	$0.00123 + 0.00378\,i$
	0.8	$-0.01819 + 0.05597\,i$	$0.00061 + 0.0019\,i$	$0.01819 - 0.05597\,i$	$0.00061 + 0.0019\,i$
	1	0	0	0	0

Data Availability

The data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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