Hindawi
Advances in Mathematical Physics
Volume 2023, Article ID 7565988, 5 pages
https://doi.org/10.1155/2023/7565988

Research Article

Position Vectors of the Natural Mate and Conjugate of a Space Curve

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Received 6 September 2022; Revised 26 December 2022; Accepted 22 March 2023; Published 8 May 2023

Academic Editor: Ghulam Rasool

The concept of the natural mate and the conjugate curves associated to a smooth curve in Euclidian 3-space were introduced initially by Dashmukh and others. In this paper, we give some extra results that add more properties of the natural mate and the conjugate curves associated with a smooth space curve in $\mathbb{E}^3$. The position vectors of the natural mate and the conjugate of a given smooth curve are investigated. Also, using the position vector of the natural mate, the necessary and sufficient condition for a smooth given curve to be a Bertrand curve is introduced. Moreover, a new characterization of a general helix is introduced.

1. Introduction

The differential geometry of curves and surfaces is an ancient topic in differential geometry, but it is still an active area of research. This is because of its applications in several fields such as computer graphics, computer vision, medical imaging, physics, and aerospace. A helix plays a crucial role in many applications in engineering and also, in DNA structures. In fact, a DNA molecule can be described by double helix. Also, it has been observed that in a molecular model of the DNA there are two side-by-side in opposing direction helices linked by hydrogen bonds (cf. [1]). The rectifying curves are used to analyze joint kinematics (cf. [2, 3]). The Salkowski curves are useful in constructing closed curves with constant curvature and continuous torsion such as knotted curves (cf. [4]). The Salkowski curves are examples of slant helice with constant curvatures.

In differential geometry, curves and their Frenet frames play central roles for creating special surfaces (c.f [5–10]). The Frenet frame associated with a regular curve in $\mathbb{E}^3$, which is a moving frame along the curve, forms an orthonormal basis for the Euclidean space $\mathbb{E}^3$ at each point of the given curve. This allows geometers to analyze a curve and to study the position vector of the given curve and other curves. The terminologies of natural mate and conjugate associated with a smooth curve were introduced and studied in [11]. Mainly, some relationships between a given curve and its natural mates were investigated in [11] as well as the necessary and sufficient conditions for the natural mate associated with a given Frenet curve to be a spherical curve, a helix, or a curve with a constant curvature. The most natural geometric object in differential geometry of curves in Euclidian 3-space is a position vector. The position vector is very important, owing to its applications in mathematics, engineering, physics, and other natural sciences.

In this paper, we investigate the position vectors of the natural mate and the conjugate of a given space curve using the Frenet frame of the given curve as a basis for $\mathbb{E}^3$. The position vectors of the natural mate and conjugate curves will be useful for studying the surfaces generated by these curves such as ruled and translation surfaces. In Section 2 of this paper, we review some basic concepts of space curves which will be used in the rest of this study. In Section 3, we study the position vectors of natural mate and conjugate of a unit speed curve with nonvanishing curvature and torsion. Using the position vector of the natural mate, we give the necessary and sufficient condition for a given curve with nonvanishing curvature and torsion to be a Bertrand curve. Also, a new characterization for a general helix is proven.

2. Preliminaries

In this section, we review some basic concepts of the differential geometry of curves in Euclidean 3-space, and for more detail, we refer the reader to [2, 3, 11–16]. First, we start with the definition of a smooth space curve. A parametrized curve α in $\mathbb{E}^3$ is a map $\alpha : I \longrightarrow \mathbb{E}^3$, where I is a real interval, given by $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ such that $\alpha_1(t), \alpha_2(t)$, and $\alpha_3(t)$ are smooth functions on I. α is a regular curve if $\alpha'(t) \neq 0$ for all $t \in I$. The unit tangent vector, the unit principal normal vector, and the unit binormal vector of a regular curve α are defined by $T = \alpha'/\|\alpha'\|$, $N = (\alpha' \wedge \alpha''/\|\alpha' \wedge \alpha''\|) \wedge (\alpha'/\|\alpha'\|)$, and $B = \alpha' \wedge \alpha''/\|\alpha' \wedge \alpha''\|$, respectively. At any point of α, there are three planes spanned by the vectors N, B, T, B, and T, N, these planes are the normal plane, the rectifying plane, and the osculating plane, respectively. The curvature and the torsion of a regular curve α are given by $\kappa = \|\alpha' \wedge \alpha''\|/\|\alpha'\|^3$ and $\tau = \det(\alpha', \alpha'', \alpha''')/\|\alpha' \wedge \alpha''\|^2$, respectively, and α is called a Frenet curve if $\kappa > 0$ and $\tau \neq 0$. The Serret Frenet apparatus associated to α is given by $\{\kappa, \tau, T, N, B\}$.

If $\alpha'(t) = 1$ for all $t \in I$, then α is called a unit speed curve and the Frenet- Serret equations are given by

$$\begin{cases} T' = \kappa N, \\ N' = -\kappa T + \tau B, \\ B' = -\tau N. \end{cases} \tag{1}$$

In the rest of this section, we state definitions of some special curves.

Definition 1. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a helix if its tangent makes a fixed angle with a fixed direction.

Definition 2. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a slant helix if its principal normal makes a fixed angle with a fixed direction.

Definition 3. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a rectifying curve if it lies in the rectifying plane at each point.

It has been obtained by Chen in [3] that the distance squared function of a rectifying curve is a quadratic polynomial in its arc length.

Definition 4. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a spherical curve if it lies in a sphere.

For a spherical curve, it is obvious to obtain that its distance from the center of the sphere, which the curve lies on, is equal to the radius of the sphere. This will play a role in the proof of Theorem 10.

Definition 5. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a smooth space curve. Then, α is called a Salkowski curve if it has constant curvature and nonconstant torsion.

Definition 6. Given a unit speed curve α with nonvanishing curvature and torsion. The natural mate of α is defined by $\beta = \int (N) ds$. If α has negative torsion, then its conjugate is given by $\bar{\alpha} = \int (B) ds$.

3. Natural and Conjugate Mates Associated with a Smooth Space Curve

In this section, we give the position vectors of natural and conjugate curves. Using position vectors of the mentioned curves, we give more brand results which carry interesting relationship between a given smooth curve and its associated natural and conjugate curves.

Theorem 7. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed Frenet curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$. The natural mate β of α is given by*

$$\beta = \left(\int (\kappa h) ds \right) T + hN - \left(\int (\tau h) ds \right) B, \tag{2}$$

where $h = dd'$, d is the distance function of β, and d' is the derivative of d with respect to s.

Proof. Since the unit tangent of β is given by $T_\beta = N$, we have

$$\beta = \int (T_\beta) ds = \int (N) ds = gT + hN + lB. \tag{3}$$

Now, differentiating equation (3), we get

$$N = \left(g' - \kappa h \right) T + \left(h' + \kappa g - \tau l \right) N + \left(l' + \tau h \right) B. \tag{4}$$

Thus, from equation (4), we have the following equations:

$$\begin{cases} g' - \kappa h = 0, \\ h' + \kappa g - \tau l = 1, \\ l' + \tau h = 0. \end{cases} \tag{5}$$

Therefore, we have

$$\begin{cases} g = \int (\kappa h) ds, \\ l = -\int (\tau h) ds. \end{cases} \tag{6}$$

Our task now is to find h. The distance squared function, d^2, of β is given by

$$d^2 = g^2 + h^2 + l^2. \tag{7}$$

Now, differentiating equation (7), we get

$$dd' = gg' + hh' + ll'. \tag{8}$$

Hence, using equations (5) and (6) in equation (8), we obtain $h = dd'$ which completes the proof. □

Now, as an application of Theorem 7, we give a criterion for a Bertrand curve with a neat proof. First, we state the following definition and a well-known result regarding the Bertrand curve.

Definition 8. A curve $\gamma : I \longrightarrow \mathbb{E}^3$ is called a Bertrand curve if there is another curve $\bar{\gamma}$, different from γ, and a bijection η between γ and $\bar{\gamma}$ such that γ and $\bar{\gamma}$ have the same principal normal at each pair of corresponding points under η.

The following theorem is a well-know result, and it can be found in many books of elementary differential geometry of curves and surfaces.

Theorem 9. *A curve $\gamma : I \longrightarrow \mathbb{E}^3$ with $\kappa \neq 0$ and $\tau \neq 0$ is called a Bertrand curve if and only if it satisfies the condition*

$$a\kappa + b\tau = 1, \tag{9}$$

where a and b are constants.

Now, we give a criterion for a Bertrand curve in term of natural mate.

Theorem 10. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with nonzero curvature and nonzero torsion and Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ and β be its natural mate. Then, the following assertions are equivalent:*

(1) α is a Bertrand curve

(2) β is a spherical curve

(3) β lies in the rectifying plane of α

Proof. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with $\kappa \neq 0$ and $\tau \neq 0$. Then, from Theorem 7, the naturel mate of α is given by

$$\beta = \left(\int (\kappa h) ds\right) T + hN - \left(\int (\tau h) ds\right) B, \tag{10}$$

where $h = dd'$ and d is the distance function of β. Now, β is a spherical curve if and only if d is a positive nonzero constant if and only if

$$\beta = c_1 T - c_2 B, \tag{11}$$

where c_1 and c_2 are constants (i.e., β lies in the rectifying plane of α), if and only if

$$N = (c_1\kappa + c_2\tau)N, \tag{12}$$

if and only if $c_1\kappa + c_2\tau = 1$ if and only if α is a Bertrand curve. □

Remark 11. Theorem 10 gives a method to create a spherical curve using a Bertrand curve. In [9], a method to create a Bertrand curve using a spherical curve with Sabban frame was introduced. Also, in [17], some methods to create special curves such as helix, slant helix, Bertrand curves, and Mannheim curves were introduced.

Corollary 12. *If α is a Salkowski curve, then its natural mate is given by*

$$\beta = \frac{1}{\kappa} T. \tag{13}$$

In what follows, we give an example of the natural mate of a Salkowski curve, and for more detail in Salkowski curve, we refer the reader to [4].

Example 13. Let $\alpha : I \longrightarrow \mathbb{E}^3$, be a Salkowski curve given by $\alpha = (\alpha_1, \alpha_2, \alpha_2)$ where,

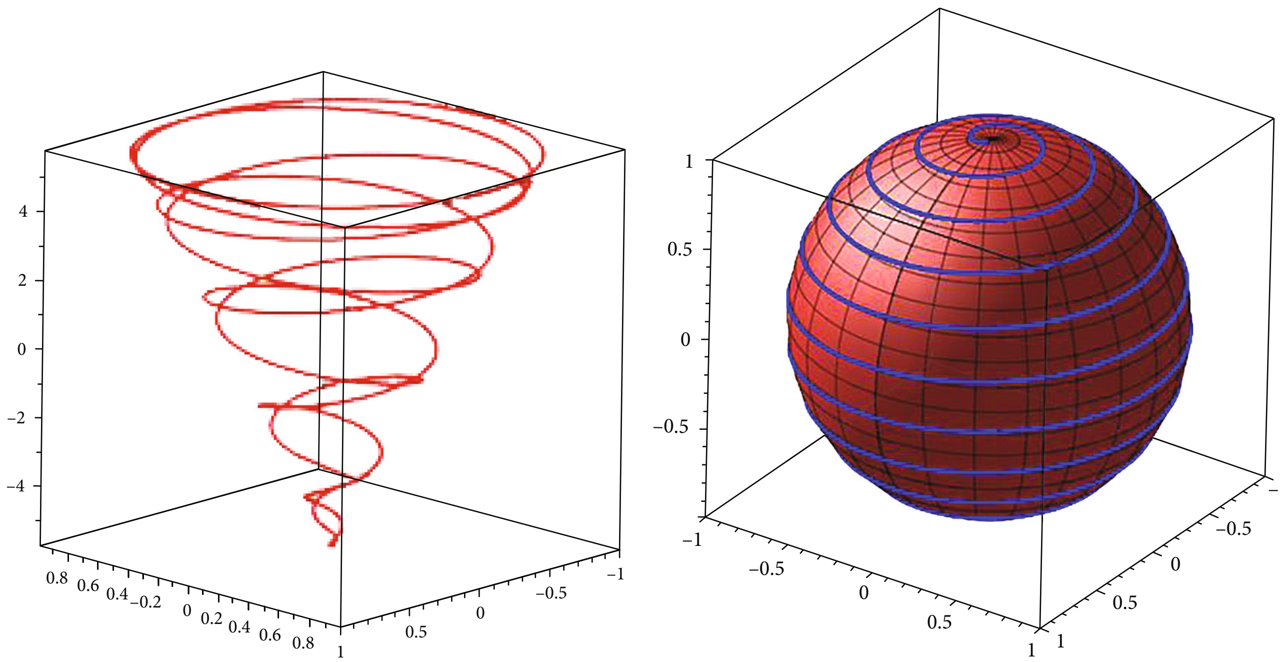
$$\begin{aligned}
\alpha_1 &= \frac{1}{\sqrt{1+m^2}}\left(-\frac{1-n}{4+8m}\sin(1+2t) - \frac{1+n}{4-8m}\sin(1-2t) - \frac{1}{2}\sin(t)\right),\\
\alpha_2 &= \frac{1}{\sqrt{1+m^2}}\left(\frac{1-n}{4+8m}\cos(1+2t) + \frac{1+n}{4-8m}\cos(1-2t) + \frac{1}{2}\cos(t)\right),\\
\alpha_3 &= \frac{1}{\sqrt{1+m^2}}\left(\frac{1}{4m}\cos(2nt)\right),
\end{aligned} \tag{14}$$

where $n = m/\sqrt{1+m^2}$ and $m \neq 0, \pm 1/\sqrt{3}$. This curve was investigated by Salkowski in 1909.

This curve has $\kappa = 1$ and torsion $\tau = -\tan(mt/\sqrt{m^2+1})$. The unit tangent of α is given by $T = (T_1, T_2, T_3)$ where

$$\begin{aligned}
T_1 &= -\cos(t)\cos(nt) - n\sin(t)\sin(nt),\\
T_2 &= n\cos(t)\sin(nt) - \cos(nt)\sin(t),\\
T_3 &= \frac{n}{m}\sin(nt).
\end{aligned} \tag{15}$$

Now, using Theorem 10, the natural mate β of α is given by $\beta = T$. Now, we draw pictures for α and its natural mate when $m = 1/23$ as shown in Figure 1. In this, Figure 1(a) is the curve α and (b) is the natural mate of α. It can be



(a) The curve α

(b) The blue curve is β lying on the unit sphere centered at the origin

Figure 1: The curve α and its natural mate β when $m = 1/23$.

observed from (b) that the natural mate of α lies on a unit sphere.

It can be easily observed from Theorem 10 that if α is a circular helix or a Salkowski curve, then its natural mate always lies on the rectifying planes of α.

In the coming theorem, we give a new criterion for a general helix.

Theorem 14. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with nonvanishing curvature and torsion. Then, α is a general helix if and only if there exists a fixed direction orthogonal to its natural mate.*

Proof. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with nonvanishing curvature and torsion, and β be its natural mate. First, assume that α is a general helix, then there exists a fixed direction makes a constant angle with its tangent. Let $\mathscr{U}$ be a unit constant vector lies on that direction, then $T \cdot \mathscr{U} = \cos\theta = \text{constant}$, and $B \cdot \mathscr{U} = \sin\theta = \text{constant}$. Now, using Theorem 7, we have

$$\beta \cdot \mathscr{U} = \cos\theta \int (\kappa h) ds - \sin\theta \int (\tau h) ds. \tag{16}$$

Since α is a helix, then $\cos\theta \int \kappa h ds - \sin\theta \int (\tau h) ds = 0$. Therefore, $\beta \cdot \mathscr{U} = 0$, which means that $\mathscr{U}$ is orthogonal to β.

Conversely, assume that there exists a fixed direction orthogonal to β. Let $\mathscr{U}$ be a unit constant vector lies on that direction, then $\beta \cdot \mathscr{U} = 0$; therefore, $N \cdot \mathscr{U} = 0$ which implies that $T \cdot \mathscr{U} = \text{constant}$, which means that α is a general helix. □

Remark 15. In [7, 11], it has been proved that a smooth curve with nonvanishing curvature and torsion is a general helix if and only if its natural mate is a plane curve. Using this fact and the result in Theorem 14, it can be concluded that the axis of a helix is always normal to the plane containing its natural mate.

Now, we give the position vector for conjugate.

Theorem 16. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a Frenet curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ with negative torsion. The conjugate mate $\bar{\alpha}$ of α is given by*

$$\bar{\alpha} = \left(\int (\kappa h_1) ds \right) T + h_1 N + \left(s - \int (\tau h_1) ds \right) B, \tag{17}$$

where $h_1 = (1/\tau)(1 - d_1'^2 - d_1 d_1'')$ and d_1 is the distance function of $\bar{\alpha}$.

Proof. Since the unit tangent of $\bar{\alpha}$ is given by $T_{\bar{\alpha}} = B$, we have

$$\bar{\alpha} = \int (T_{\bar{\alpha}}) ds = \int (B) ds = g_1 T + h_1 N + l_1 B. \tag{18}$$

Now, differentiating equation (18) w.r.t the arc length, we get

$$B = \left(g_1' - \kappa h_1 \right) T + \left(h_1' + \kappa g_1 - \tau l_1 \right) N + \left(l_1' + \tau h_1 \right) B. \tag{19}$$

Thus, from equation (19), we have the following equations:

$$\begin{cases} g_1' - \kappa h_1 = 0, \\ h_1' + \kappa g_1 - \tau l_1 = 0, \\ l_1' + \tau h_1 = 1. \end{cases} \tag{20}$$

Therefore, we have

$$\begin{cases} g_1 = \int (\kappa h_1) ds, \\ l_1 = s - \int (\tau h_1) ds. \end{cases} \tag{21}$$

Our task now is to find h_1. The distance squared function, d_1^2, of $\bar{\alpha}$ is given by

$$d_1^2 = g_1^2 + h_1^2 + l_1^2. \tag{22}$$

Now, differentiating equation (22), we get

$$dd' = g_1 g_1' + h_1 h_1' + l_1 l_1'. \tag{23}$$

Hence, using equations (20) and (21) in equation (23), we obtain $dd' = s - \int(\tau h_1)$ which implies that $h_1 = (1/\tau)(1 - d_1'^2 - d_1 d_1'')$ which completes the proof. □

From Theorem 16, we have the following corollary.

Corollary 17. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ with negative torsion. Then,*

(1) If $\bar{\alpha}$ is a spherical curve, then $h_1 = 1/\tau$

(2) If $\bar{\alpha}$ is a rectifying curve, then $h_1 = 0$

Proof. If $\bar{\alpha}$ is a spherical curve, then its distance function d is a constant. Thus, $h_1 = 1/\tau$. If $\bar{\alpha}$ is a rectifying curve, then it has been proved in [3] that its distance function d_1 satisfies $d_1^2(s) = s^2 + c_1 s + c_2$ for some constants c_1 and c_2. Therefore, $h_1 = 0$. □

4. Conclusion

The position vector of a curve in the Euclidian 3-space is the most natural geometric object. It is important in many applications in several areas such as mathematics, engineering, and natural sciences. Throughout this paper, we study the position vectors of the natural and conjugate mates associated with a given smooth space curve in Euclidian 3-space. The position vectors of the natural and the conjugate mates associated with a given smooth curve are very useful for studying these curves. Also, the position vector of the natural mate associated to a given curve is used to prove a new criterion for a helix and Bertrand curves. Moreover, it can be easy to study the surfaces generated by the natural and conjugate mates associated with a smooth curve using their position vectors.

Data Availability

No external data has been used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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