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Research Article

Well-Posedness and Blow-Up of Solutions for a Variable Exponent Nonlinear Petrovsky Equation

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In this article, we investigate a nonlinear Petrovsky equation with variable exponent and damping terms. First, we establish the local existence using the Faedo–Galerkin approximation method under the conditions of positive initial energy and appropriate constraints on the variable exponents $p(\cdot)$ and $q(\cdot)$. Finally, we prove a finite-time blow-up result for negative initial energy.

1. Introduction

In this work, we investigate the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u, & \Omega \times (0, T), \\ u(x,t) = \dfrac{\partial}{\partial \nu} u(x,t) = 0, & \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ $(n \in N^+)$ is a bounded domain with smooth boundary $\partial\Omega$:

$$\begin{cases} -\Delta u_{tt} \text{ is a dissipative term}, \\ -\Delta u_t \text{ is a strong damping term}, \\ \Delta u \text{ is a Laplace operator}, \\ \Delta^2 u \text{ is a biharmonic operator}, \end{cases} \tag{2}$$

and $u_0(x) \in H^2(\Omega) \cap H^4(\Omega)$, $u_1(x) \in L^2(\Omega)$ are initial conditions. $p(\cdot)$ and $q(\cdot)$ are given measurable functions on Ω, satisfying the following equations:

$$\begin{cases} 2 < p_1 \le p(x) \le p_2 < \infty, & \text{if } n \le 4, \\ 2 < p_1 \le p(x) \le p_2 < \dfrac{2n-4}{n-4}, & \text{if } n > 4, \end{cases} \tag{3}$$

and

$$\begin{cases} 2 < q_1 \le q(x) \le q_2 < \infty, & \text{if } n \le 4, \\ 2 < q_1 \le q(x) \le q_2 < \dfrac{2n-4}{n-4}, & \text{if } n > 4, \end{cases} \tag{4}$$

where

$$p_1 = ess \inf_{x\in\Omega} p(x), p_2 = ess \operatorname{esssup}_{x\in\Omega} p(x),$$
$$q_1 = ess \inf_{x\in\Omega} q(x), q_2 = ess \operatorname{esssup}_{x\in\Omega} q(x), \quad (5)$$

and the log-Hölder continuity condition for $A>0, 0<\delta<1$:

$$|p(x)-p(y)| \leq -\frac{A}{\ln|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y|<\delta. \quad (6)$$

(i) This kind of Equation (1) without variable exponent has its origin in the canonical model introduced by Petrovsky [1, 2]. Petrovsky [1, 2] type equation originated from the study of plate and beams, and it can also be used in many branches of science, such as ocean acoustics, geophysics, optics, and acoustics [3].
(ii) The problems with variable exponents arise in many branches in science such as the image processing, filtration processes in porous media, flow of electrorheological fluids, and nonlinear viscoelasticity [4–6].

In the study of Ouaoua and Boughamsa [7], they looked into the following equation:

$$u_{tt} + \Delta^2 u - \Delta u + |u_t|^{m(x)-2} u_t = |u|^{r(x)-2} u. \quad (7)$$

They showed the local existence and also proved that the local solution is global. Antontsev et al. [8] studied the following a nonlinear Petrovsky equation:

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \quad (8)$$

Under suitable assumptions on the variable exponents and initial data, they obtain local weak solutions and established a blow-up result. Tebba et al. [9] discussed a new class of nonlinear wave equation:

$$u_{tt} - \Delta u - \Delta u_{tt} + a|u_t|^{m(x)-2} u_t = b|u|^{p(x)-2} u. \quad (9)$$

Under appropriate assumptions on the variable exponents, they demonstrated the existence of a unique weak solution using the Faedo–Galerkin method. They also proved the finite time blow-up of solutions.

Moreover, numerous researchers have studied the mathematical behavior of equations using the Faedo–Galerkin and the perturbed energy method [10–14].

In this work, we are concerned the existence and blow-up of the problem (1). The obtained existence and blow-up results improve and generalize many results in the literature.

This work is composed of three sections in addition to the introduction. Part 2 presents preliminary information regarding variable exponents Lebesgue and Sobolev spaces. Additionally, we outline significant lemmas and assumptions. Part 3 focuses on proving the local existence of solutions. In Part 4, we establish the blow-up of solutions with a positive initial energy.

2. Preliminaries

Throughout this work, we present some important facts about Lebesgue and Sobolev spaces with variable exponents (see [5, 15]).

Let $r:\Omega \longrightarrow [1,\infty]$ be a measurable function, where Ω is a domain of R^n. We define the variable exponent Lebesgue space by the following equation:

$$L^{r(x)}(\Omega) = \left\{u:\Omega \longrightarrow R;\ u \text{ measurable in } \Omega : \rho_{r(\cdot)}(\lambda u) <\infty, \text{ for some } \lambda>0\right\}, \quad (10)$$

where $\rho_{r(\cdot)}(u) = \int_\Omega |u(x)|^{r(x)} dx$. Equipped with the following Luxembourg-type norm:

$$\|u\|_{r(\cdot)} = \inf\left\{\lambda>0: \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{r(x)} dx \leq 1\right\}. \quad (11)$$

The space $L^{r(\cdot)}(\Omega)$ is a Banach space.

The variable-exponent Sobolev space is defined as follows:

$$W^{1,r}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{r(\cdot)}(\Omega)\right\}. \quad (12)$$

This is a Banach space with respect to the norm $\|u\|_{W_0^{1,r}(\Omega)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)}$.

Furthermore, we set $W_0^{1,r(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,r(\cdot)}(\Omega)$. Let us note that the space $W_0^{1,r(\cdot)}(\Omega)$ has a differenet definition in the case of variable exponents.

However, under the log-Hölder continuity condition, both definitions are equivalent [5]. The space $W^{-1,r'(\cdot)}(\Omega)$, dual of $W_0^{1,r(\cdot)}(\Omega)$, is defined in the same way as the classical Sobolev spaces, where $\frac{1}{r(\cdot)} + \frac{1}{r'(\cdot)} = 1$.

Lemma 1 (Diening et al. [5]). *If:*

$$1 \leq r_1 = ess \inf_{x\in\Omega} r(x) \leq r(x) \leq r_2 = ess \sup_{x\in\Omega} r(x) < \infty, \quad (13)$$

then we have:

$$\min\left\{\|u\|_{r(\cdot)}^{r_1}, \|u\|_{r(\cdot)}^{r_2}\right\} \leq \rho_{r(\cdot)}(u) \leq \max\left\{\|u\|_{r(\cdot)}^{r_1}, \|u\|_{r(\cdot)}^{r_2}\right\}, \quad (14)$$

for any $u \in L^{r(\cdot)}(\Omega)$.

Lemma 2 (Diening et al. [5]). *Let* $m, r, s \geq 1$ *be measurable functions defined on* Ω *such that:*

$$\frac{1}{s(y)} = \frac{1}{m(y)} + \frac{1}{r(y)}, \text{ for a.e. } y \in \Omega. \tag{15}$$

If $v_1 \in L^{m(\cdot)}(\Omega)$ *and* $v_2 \in L^{r(\cdot)}(\Omega)$, *then* $v_1 v_2 \in L^{s(\cdot)}(\Omega)$, *with:*

$$\|v_1 v_2\|_{s(\cdot)} \leq 2\|v_1\|_{m(\cdot)}\|v_2\|_{r(\cdot)}. \tag{16}$$

Lemma 3 (Diening et al. [5]). *If* r *is a measurable function on* Ω *satisfying* (6), *then the embedding* $H_0^1(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is continuous and compact. Then, the embedding* $H_0^2(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ *is continuous and compact.*

As per Lemma 3, there exists a positive constant denoted as c_* that fulfills the following condition:

$$\|u\|_{p(\cdot)} \leq c_*\|\nabla u\|_2, \text{ for } u \in H_0^1(\Omega). \tag{17}$$

Lemma 4 (Komornik [16]). *Let* $F : R^+ \longrightarrow R^+$ *be a nonincreasing function and assume that there are two constants* $\alpha > 0$ *and* $C > 0$ *in the following equation:*

$$\int_t^\infty F^{\alpha+1}(s)ds \leq CF^\alpha(0)F(s), \ \forall t \in R^+. \tag{18}$$

Then, we have the following equation:

$$F(t) \leq F(0)\left(\frac{C + \alpha t}{C + \alpha C}\right)^{\frac{-1}{\alpha}}, \ \forall t \geq C. \tag{19}$$

To articulate and demonstrate our outcome, we define the subsequent functionals:

$$\begin{aligned} E(t) = &\frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\Delta u\|^2) \\ &- \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx, \end{aligned} \tag{20}$$

$$I(t) = \|\nabla u\|^2 + \|\Delta u\|^2 - \int_\Omega |u|^{q(x)}dx, \tag{21}$$

$$J(t) = \frac{1}{2}(\|\nabla u\|^2 + \|\Delta u\|^2) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx. \tag{22}$$

Lemma 5. *Let* u *be a solution of problem* (1). *Then, the energy functional satisfies the following equation:*

$$E'(t) = -\int_\Omega |\nabla u_t|^2 dx - \int_\Omega |u_t|^{p(x)}dx, \ t \in [0, T], \tag{23}$$

and

$$E(t) \leq E(0). \tag{24}$$

Proof. Multiplying the first equation in Equation (1) by u_t and integrating over Ω yields the following equation:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 + \|\Delta u_t\|^2 + \|\Delta u\|^2) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx\right) \\ &\quad = -\int_\Omega |\nabla u_t|^2 dx - \int_\Omega |u_t|^{p(x)}dx, \end{aligned} \tag{25}$$

then:

$$E'(t) = -\int_\Omega |\nabla u_t|^2 dx - \int_\Omega |u_t|^{p(x)}dx \leq 0. \tag{26}$$

Integrating Equation (26) over $(0, t)$, we obtain the following equation:

$$E(t) \leq E(0). \tag{27}$$

□

Lemma 6. *Under the assumptions of Theorem 5 and* $E(0) > 0$ *hold:*

$$I(0) > 0, \tag{28}$$

and

$$\theta_1 + \theta_2 < 1, \tag{29}$$

where

$$\theta_1 = \alpha\left\{c_{1,*}^{q_1}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_2-2}{2}}, c_{1,*}^{q_2}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_2-2}{2}}\right\},$$
$$\theta_2 = (1-\alpha)\left\{c_{2,*}^{q_1}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_2-2}{2}}, c_{2,*}^{q_2}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_2-2}{2}}\right\}, \tag{30}$$

with $0<\alpha<1$, $c_{1,*}$ *and* $c_{2,*}$ are the bests embedding constants of $H_0^2(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ *and* $H_0^2(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, *respectively,* then $I(t)>0$, *for all* $t\in[0,T]$.

Proof. Due to continuity, there exists T_*, such that:

$$I(t)\geq 0, \text{ for all } t\in[0,T_*]. \tag{31}$$

Now, for all $t\in[0,T_*]$, we have the following equation:

$$\begin{aligned}J(t) &= \frac{1}{2}(\|\nabla u\|^2+\|\Delta u\|^2) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx.\\ &\geq \frac{1}{2}(\|\nabla u\|^2+\|\Delta u\|^2) - \frac{1}{q_1}(\|\nabla u\|^2+\|\Delta u\|^2 - I(t))\\ &\geq \frac{q_1-2}{2q_1}(\|\nabla u\|^2+\|\Delta u\|^2) + \frac{1}{q_1}I(t).\end{aligned} \tag{32}$$

Using Equation (31), we obtain the following equation:

$$(\|\nabla u\|^2+\|\Delta u\|^2)\leq \frac{2q_1}{q_1-2}J(t), \text{ for all } t\in[0,T_*]. \tag{33}$$

By Lemma 5, we get the following equation:

$$\|\nabla u\|^2+\|\Delta u\|^2\leq \frac{2q_1}{q_1-2}E(t)\leq \frac{2q_1}{q_1-2}E(0). \tag{34}$$

Moreover, according to Lemma 1, we obtain the following equation:

$$\begin{aligned}\int_\Omega |u|^{q(x)}dx &\leq \max\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\}\\ &= \alpha\max\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\}\\ &\quad + (1-\alpha)\max\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\}.\end{aligned} \tag{35}$$

By the embedding of $H_0^2(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ and $H_0^2(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$, we obtain the following equation:

$$\begin{aligned}\int_\Omega |u|^{q(x)}dx &\leq \alpha\max\left\{c_{1,*}^{q_1}\|\nabla u\|_2^{q_1}, c_{1,*}^{q_2}\|\nabla u\|_2^{q_2}\right\}\\ &\quad + (1-\alpha)\max\left\{c_{2,*}^{q_1-2}\|\Delta u\|_2^{q_1}, c_{2,*}^{q_2-2}\|\Delta u\|_2^{q_2}\right\}\\ &\leq \alpha\max\left\{c_{1,*}^{q_1}\|\nabla u\|_2^{q_1}, c_{1,*}^{q_2}\|\nabla u\|_2^{q_2}\right\}\times\|\nabla u\|_2^2\\ &\quad + (1-\alpha)\max\left\{c_{2,*}^{q_1}\|\Delta u\|_2^{q_1}, c_{2,*}^{q_2}\|\Delta u\|_2^{q_2}\right\}\times\|\Delta u\|_2^2\\ &\leq \alpha\max\left\{c_{1,*}^{q_1}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_1-2}{2}}, c_{1,*}^{q_2}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_2-2}{2}}\right\}\times\|\nabla u\|_2^2\\ &\quad + (1-\alpha)\max\left\{c_{2,*}^{q_1}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_1-2}{2}}, c_{2,*}^{q_2}\left(\frac{2q_1}{q_1-2}E(0)\right)^{\frac{q_2-2}{2}}\right\}\\ &\quad \times\|\Delta u\|_2^2\end{aligned}. \tag{36}$$

Then, we have the following equation:

$$\int_\Omega |u|^{q(x)}dx \leq \theta_1\|\nabla u\|^2+\theta_2\|\Delta u\|^2 \text{ for all } t\in[0,T_*]. \tag{37}$$

Since $\theta_1+\theta_2<1$, then, we obtain the following equation:

$$\int_\Omega |u|^{q(x)}dx < \|\nabla u\|^2+\|\Delta u\|^2 \text{ for all } t\in[0,T_*]. \tag{38}$$

This implies that:

$$I(t) > 0, \text{ for all } t \in [0, T_*]. \tag{39}$$

By repeating the aforementioned process, we can extend T_* to T. □

3. Local Existence

This section is dedicated to establishing the local existence of problem (1). We will employ the Faedo–Galerkin method approximation.

Theorem 7. *Suppose that* $p, q \in C(\overline{\Omega})$ *and satisfies Equation* (6)*. Then, for any* $(u_0, u_1) \in H^2(\Omega) \cap H^4(\Omega) \times L^2(\Omega)$*, problem* (1) *has a unique weak local solution:*

$$\begin{aligned} &u \in L^\infty(0, T), H_0^2(\Omega), \\ &u_t \in L^\infty(0, T), H_0^2(\Omega) \cap L^{m(\cdot)}(\Omega \times (0, T)). \end{aligned} \tag{40}$$

Proof. Let $\{v_l\}_{l=1}^\infty$ be a basis of $H_0^2(\Omega)$ that forms a complete orthonormal system in $L^2(\Omega)$. Denote $V_k = span\{v_1, v_2, \ldots, v_k\}$ as the subspace generated by the first k vectors from the basis $\{v_l\}_{l=1}^\infty$. Due to normalization, we have $\|v_l\| = 1$. For a given integer k, we consider the approximated solution:

$$u_k(t) = \sum_{l=1}^k u_{lk}(t) v_l, \tag{41}$$

where $u_k(t)$ is the solutions to the following Cauchy problem:

$$\begin{aligned} &\left(u_k''(t), v_l\right) - \left(\Delta u_k(t), v_l\right) - \left(\Delta u_k''(t), v_l\right) + \left(\Delta^2 u_k(t), v_l\right) - \left(\Delta u_k'(t), v_l\right) \\ &+ \left(\left|u_k'(t)\right|^{p(x)-2} u_k'(t), v_l\right) = \left(\left|u_k(t)\right|^{q(x)-2} u_k(t), v_l\right), \quad l = 1, 2, \ldots, k, \end{aligned} \tag{42}$$

$$u_k(0) = u_{0k} = \sum_{l=1}^k (u_k(0), v_l) v_l \longrightarrow u_0 \text{ in } H_0^2(\Omega) \cap H^4(\Omega), \tag{43}$$

$$u_k'(0) = u_{1k} = \sum_{l=1}^k \left(u_k'(0), v_l\right) v_l \longrightarrow u_1 \text{ in } H_0^2(\Omega). \tag{44}$$

It is worth noting that the systems (42)–(44) can be solved using Picard's iteration method for ordinary differential equations. As a result, a solution exists within the interval $[0, T_*)$ for some $T_* > 0$, and we can extend this solution to the whole interval $[0, T)$ for any given $T > 0$ by utilizing the a priori estimates provided below.

The first estimate: Multiplying Equation (42) by $u_{lk}'(t)$ and summing over l from 1 to k:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\left(\left\|u_k'\right\|^2 + \|\nabla u_k\|^2 + \left\|\Delta u_k'\right\|^2 + \|\Delta u_k\|^2\right) - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx\right) \\ &= -\int_\Omega \left|\nabla u_k'\right|^2 dx - \int_\Omega \left|u_k'\right|^{p(x)} dx. \end{aligned} \tag{45}$$

Then, we obtain the following equation:

$$E'(u_k(t)) = -\int_\Omega \left|\nabla u_k'\right|^2 dx - \int_\Omega \left|u_k'\right|^{p(x)} dx \le 0. \tag{46}$$

By integrating Equation (45) over the interval $(0, t)$, we derive the estimate the following equation:

$$\begin{aligned} &\frac{1}{2}\left\|u_k'\right\|^2 + \frac{1}{2}\|\nabla u_k\|^2 + \frac{1}{2}\left\|\Delta u_k'\right\|^2 + \frac{1}{2}\|\Delta u_k\|^2 \\ &- \int_0^t \int_\Omega \frac{1}{q(x)} |u_k|^{q(x)} dx \le E(0). \end{aligned} \tag{47}$$

Then, from Equation (38), the inequality (47) becomes:

$$\begin{aligned} &\frac{1}{2} \sup_{t \in (0,T)} \left\|u_k'\right\|^2 + \frac{q_1 - 2}{2q_1} \sup_{t \in (0,T)} \|\nabla u_k\|^2 + \frac{q_1 - 2}{2q_1} \sup_{t \in (0,T)} \|\Delta u_k\|^2 \\ &+ \frac{1}{2} \sup_{t \in (0,T)} \left\|\Delta u_k'\right\|^2 + \int_0^t \int_\Omega \left|u_t^k(x, s)\right|^{p(x)} dx ds \\ &\le E(0). \end{aligned} \tag{48}$$

From Equation (48), we conclude that:

$$\begin{cases} \{u_k\} \text{ is a bounded sequence in } L^\infty((0, T); H_0^2(\Omega)), \\ \{u_k'\} \text{ is a bounded sequence in } L^\infty((0, T); H_0^2(\Omega)) \cap L^{p(\cdot)}(\Omega \times (0, T)). \end{cases} \tag{49}$$

Since $\{u'_k\}$ is uniformly bounded in $L^{p(x)}(\Omega\times[0,T])$, then $\{|u'_k|^{p(x)-2}u'_k\}$ is bounded in $L^{\frac{p(x)}{p(x)-1}}(\Omega\times[0,T])$; hence, up to a subsequence, $|u'_k|^{p(x)-2}u'_k\rightharpoonup\Phi$ weakly in $L^{\frac{p(x)}{p(x)-1}}(\Omega\times[0,T])$. As in Messaoudi et al.'s [17] study, we have to show that $\Phi=|u'|^{p(x)-2}u'$.

Furthermore, from Lemma 3 and Equation (49), we obtain the following equation:

$$\{|u_k|^{q(x)-2}u_k\} \text{ is uniformly bounded in } L^\infty([0,T]),\ L^2(\Omega). \tag{50}$$

From Equations (49) and (50), we deduce the existence of a subsequence of u_k (still denoted by the same symbol) and a function u such that:

$$\begin{cases} u_k \longrightarrow u \text{ weakly star in } L^\infty([0,T];H_0^2(\Omega)),\\ u'_k \longrightarrow u' \text{ weakly star in } L^\infty([0,T]);H_0^2(\Omega) \text{ and weakly in } L^{p(\cdot)}(\Omega\times(0,T)),\\ |u_k|^{q(x)-2}u_k\rightharpoonup\psi \text{ weakly } L^\infty([0,T]),\ L^2(\Omega). \end{cases} \tag{51}$$

By the Aubin–Lions compactness Lemma [18], we conclude from Equation (51) that:

$$u_k \longrightarrow u \text{ strongly in } C([0,T];H_0^2(\Omega)), \tag{52}$$

which implies:

$$u_k \longrightarrow u \text{ everywhere in } \Omega\times[0,T]. \tag{53}$$

It follow from Equations (51) and (53) that:

$$|u_k|^{q(x)-2}u_k\rightharpoonup|u|^{q(x)-2}u \text{ weakly in } L^\infty([0,T]),\ L^2(\Omega). \tag{54}$$

The second estimate: Now, we would like to get more estimates. In doing so, differentiating Equation (42) with respect to t, we get the following equation:

$$\begin{aligned}&(u'''_k(t),v_l)-(\Delta u'_k(t),v_l)-(\Delta u'''_k(t),v_l)+(\Delta^2u'_k(t),v_l)-(\Delta u''_k(t),v_l)\\ &\quad+((p(x)-1)|u''_k(t)|^{p(x)-2}u''_k(t),v_l)\\ &=((q(x)-1)|u_k(t)|^{q(x)-2}u'_k(t),v_l),\ l=1,2,\dots,k.\end{aligned} \tag{55}$$

Next, multiplying Equation (55) by $u''_{lk}(t)$ and summing over l from 1 to k, we get the following equation:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k\|^2+\|\nabla u'_k\|^2+\|\nabla u''_k\|^2+\|\Delta u'_k\|^2\right)\\ &\quad+\int_\Omega(p(x)-1)|u'_k|^{p(x)-1}|u''_k|^2dx+\int_\Omega|\nabla u''_k|^2dx\\ &=-\int_\Omega(q(x)-1)|u_k|^{q(x)-2}u'_ku''_kdx.\end{aligned} \tag{56}$$

We have the following equation from Hölder's inequality:

$$\begin{aligned}&\left|\int_\Omega(q(x)-1)|u_k|^{q(x)-2}|u'_k||u''_k|dx\right|\\ &\le(q_2-1)\|u_k\|^{q(x)-2}_{2(q(x)-1)}\|u'_k\|_{2(q(x)-1)}\|u''_k\|_2.\end{aligned} \tag{57}$$

We have, then $u_k\in L^\infty([0,T],H_0^2(\Omega))$, then the following equation:

$$\int_\Omega|u_k|^{2q(x)-2}dx\le\int_\Omega|u_k|^{2q_1-2}dx+\int_\Omega|u_k|^{2q_2-2}dx<+\infty, \tag{58}$$

since, $2(q_1-1)\le2(q(x)-1)\le2(q_2-1)\le2\frac{n}{n-2}$. The inequality (57), becomes the equation as follows:

$$\left|\int_\Omega(q(x)-1)|u_k|^{q(x)-2}|u'_k||u''_k|dx\right|\le c_1\|u'_k\|_{2(q(x)-1)}\|u''_k\|_2. \tag{59}$$

We have the following equation from Young's inequality and Poincáre's inequality:

$$\left|\int_\Omega(q(x)-1)|u_k|^{q(x)-2}|u'_k||u''_k|dx\right|\le c_\delta\|\nabla u'_k\|^2+\delta\|u''_k\|^2. \tag{60}$$

Substituting Equation (60) into Equation (56) and integrating over $(0,t)$ for all $t\in[0,T]$, we obtain the following equation:

$$\int_{\Omega}\left(|u_k''|^2+|\nabla u_k'|^2+|\nabla u_k''|^2+|\Delta u_k'|^2+|\nabla u_k''|^2\right)dx$$
$$\leq\left(\|u_k''(0)\|^2+\|\nabla u_k'(0)\|^2+\|\nabla u_k''(0)\|^2+\|\Delta u_k'(0)\|^2\right)$$
$$+c_2\int_0^t\left(\|\nabla u_k'\|^2+\|u_k''\|^2\right)ds. \tag{61}$$

It follows from Equation (44) and the fact $\|\nabla u_k'(0)\|^2\leq c_3\|\Delta u_k'(0)\|^2$ that:

$$\|\nabla u_k'(0)\|^2+\|\Delta u_k'(0)\|^2\leq c_4, \tag{62}$$

where c_4 is a positive constant independent of k.

By multiplying both sides of Equation (42) by $u_{lk}''(t)$, summing over l from 1 to k and setting $t=0$, we obtain the following equation:

$$\|u_k''(0)\|^2-\left(\Delta u_k(0),u_k''(0)\right)-\left(\Delta u_k''(0),u_k''(0)\right)+\left(\Delta^2 u_k(0),u_k''(0)\right)-\left(\Delta u_k'(0),u_k''(0)\right)$$
$$+\left(|u_k'(0)|^{p(x)-2}u_k'(0),u_k''(0)\right)=\left(|u_k(0)|^{q(x)-2}u_k(0),u_k''(0)\right),\quad l=1,2,\ldots,k. \tag{63}$$

Utilizing Young's inequality along with Equations (43) and (44), we have:

$$\|u_k''(0)\|^2\leq c_5, \tag{64}$$

where c_5 is a positive constant independent of k.

By Equations (62) and (64), Equation (61) becomes:

$$\int_{\Omega}|u_k''|^2dx+\int_{\Omega}|\nabla u_k'|^2dx+\int_{\Omega}|\nabla u_k''|^2dx+\int_{\Omega}|\Delta u_k'|^2dx+\int_{\Omega}|\nabla u_k''|^2dx$$
$$\leq c_6+c_7\int_0^t\left(|u_k''|^2+|\nabla u_k'|^2+|\nabla u_k''|^2+|\Delta u_k'|^2+|\nabla u_k''|^2\right)ds. \tag{65}$$

We deduce from Equation (65) and Gronwall's lemma that:

$$\|u_k''\|^2+\|\nabla u_k'\|^2+\|\nabla u_k''\|^2+\|\Delta u_k'\|^2+\|\nabla u_k''\|^2\leq c_8, \tag{66}$$

for all $t\in[0,T]$, where c_8 is a positive constant independent of k.

We can infer from Equation (66) that:

$$\begin{cases}\{u_k'\}\text{ is uniformly bounded in } L^\infty([0,T],H_0^2(\Omega)),\\ \{u_k''\}\text{ is uniformly bounded in } L^\infty([0,T],H_0^1(\Omega)).\end{cases} \tag{67}$$

Similarly, we have the following equation:

$$\begin{cases}u_k'\text{ is uniformly bounded in } L^\infty([0,T],H_0^2(\Omega)),\\ u_k''\text{ is uniformly bounded in } L^\infty([0,T],H_0^1(\Omega)).\end{cases} \tag{68}$$

Setting up $k\longrightarrow\infty$ and passing to the limit in Equation (42), we obtain the following equation:

$$(u''(t),v_l)-(\Delta u(t),v_l)-(\Delta u''(t),v_l)+(\Delta^2 u(t),v_l)-(\Delta u'(t),v_l)$$
$$+\left(|u'(t)|^{p(x)-2}u'(t),v_l\right)=\left(|u(t)|^{q(x)-2}u(t),v_l\right),\quad l=1,2,\ldots,k. \tag{69}$$

Given that $\{v_l\}_{l=1}^{\infty}$ is a basis of $H_0^2(\Omega)$, we can deduce that u satisfies Equation (1). From Equation (51), Equation (68) and Lemma 3.1.7 in Zheng's [19] study with $B=H_0^2(\Omega)$ and $L^2(\Omega)$, respectively, we infer that:

$$\begin{cases} u_k' \text{ is uniformly bounded in } H_0^2(\Omega), \\ u_k'' \text{ is uniformly bounded in } H_0^1(\Omega). \end{cases} \tag{70}$$

We get from Equations (43), (44), and (70) that $u(0) = u_0$, $u(0) = u_1$.

Consequently, the proof of existence is now concluded.

Uniqueness of the solution: Now it remains to prove uniqueness. Let y and z be two solutions in the class described in the statement of this theorem, and $w = y - z$.

Then, w satisfies the following equation:

$$\begin{aligned} &w_{tt} - \Delta w - \Delta w_{tt} + \Delta^2 w - \Delta w_t + |y_t|^{p(x)-2} y_t - |z_t|^{p(x)-2} z_t \\ &= |y|^{q(x)-2} y - |z|^{q(x)-2} z, \end{aligned} \tag{71}$$

and

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x). \tag{72}$$

Multiplying Equation (71) by w_t, then integrating with respect to x, we get the following equation:

$$\begin{aligned} &\frac{1}{2}\int_\Omega (|w_t|^2 + |\nabla w|^2 + |\nabla w_t|^2 + |\Delta w|^2) dx \\ &+ \int_0^t \int_\Omega |\Delta w|^2 dx + \int_0^t \int_\Omega \left(|y_t|^{p(x)-2} y_t - |z_t|^{p(x)-2} z_t\right) w_t dx ds \\ &= \int_0^t \int_\Omega \left(|y|^{q(x)-2} y - |z|^{q(x)-2} z\right) w_t dx ds. \end{aligned} \tag{73}$$

By using the inequality:

$$\left(|a|^{p(x)-2} a - |b|^{p(x)-2} b\right)(a - b) \geq 0, \tag{74}$$

for all $a, b \in R$ and a.e. $x \in \Omega$.

This implies:

$$\begin{aligned} &\|w_t\|^2 + \|\nabla w\|^2 + \|\nabla w_t\|^2 + \|\Delta w\|^2 \\ &\leq C \int_0^t \int_\Omega \left(|y|^{q(x)-2} y - |z|^{q(x)-2} z\right) w_t dx ds. \end{aligned} \tag{75}$$

By repeating the estimate as in Messaoudi's [20] study, we arrive the following equation:

$$\begin{aligned} &\int_\Omega (|w_t|^2 + |\nabla w|^2 + |\nabla w_t|^2 + |\Delta w|^2) dx \\ &\leq C \int_0^t \int_\Omega (|w_t|^2 + |\nabla w|^2) dx ds. \end{aligned} \tag{76}$$

Then:

$$\begin{aligned} &\int_\Omega (|w_t|^2 + |\nabla w|^2 + |\nabla w_t|^2 + |\Delta w|^2) dx \\ &\leq C \int_0^t \int_\Omega (|w_t|^2 + |\Delta w|^2 + |\nabla w|^2 + |\Delta w|^2) dx ds. \end{aligned} \tag{77}$$

Gronwall's inequality yields the following equation:

$$\|w_t\|^2 + \|\nabla w\|^2 + \|\nabla w_t\|^2 + \|\Delta w\|^2 = 0. \tag{78}$$

Thus, $w = 0$. The shows the uniqueness. □

4. Blow-Up

In this section, we examine the blow-up of the solution to problem (1). To begin, we introduce the following [20].

Lemma 8. *If $q : \Omega \longrightarrow [1, \infty)$ is a measurable function and*

$$\begin{cases} 2 \leq q_1 \leq q(x) \leq q_2 < \infty \text{ for } n \leq 4, \\ 2 \leq q_1 \leq q(x) \leq q_2 < \dfrac{2n}{n-4} \text{ for } n > 4, \end{cases} \tag{79}$$

holds. Then, *we have the following inequalities:*

$$\rho_{q(x)}^{\frac{s}{q_1}}(u) \leq C\left(\|\Delta u\|^2 + \rho_{q(\cdot)}(u)\right). \tag{80}$$

Lemma 9. *Suppose the conditions of Lemma 8 hold and let u be the solution of Equation* (1). *Then:*

(i)

$$\|u\|_{q_1}^s \leq C\left(\|\Delta u\|^2 + \|u\|_{q_1}^{q_1}\right), \tag{81}$$

(ii)

$$\rho_{q(\cdot)}^{\frac{s}{q_1}}(u) \leq C\left(\mathscr{H}(t) + \|u_t\|^2 + \|\nabla u_t\|^2 + \rho_{q(\cdot)}(u)\right), \tag{82}$$

(iii)

$$\|u\|_{q_1}^s \leq C\left(\mathscr{H}(t) + \|u_t\|^2 + \|\nabla u_t\|^2 + \|u\|_{q_1}^{q_1}\right), \tag{83}$$

(iv)

$$\|u\|_{q_1}^{q_1} \leq C\rho_{q(\cdot)}(u), \tag{84}$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq q_1$. Where $\rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(\cdot)} dx$ and $C > 1$ a positive constant and $\mathscr{H}(t) = -E(t)$.

Then, functions $\mathscr{H}(t)$ and $E(t)$ will be defined later. Now, we state and prove our blow-up result.

Theorem 10. *Under the conditions of Lemma 9. Also, let initial energy satisfy $E(t) < 0$ and the exponents $p(\cdot)$ and $q(\cdot)$ satisfy the following equation:*

$$2 \leq p_1 \leq p(x) \leq p_2 \leq q_1 \leq q(x) \leq q_2 < 2\frac{(n-2)}{n-4}, \; n > 4. \tag{85}$$

Then, the solution of Equation (1) blows up in finite time T^*, in the following sense:

$$\Psi(t) \longrightarrow \infty \text{ as } t \longrightarrow T^* \leq \frac{1-\alpha}{\xi\alpha\Psi^{-\frac{\alpha}{1-\alpha}}(0)}, \tag{86}$$

here $\xi \in (0,1)$, $\Psi(t)$ and σ will given later in Equations (91) and (94), respectively.

Proof. When we multiply both sides by u_t and integrate over the domain Ω, the result is as follows:

$$\frac{d}{dt}\left[\frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\Delta u\|^2) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx\right] = -\int_\Omega |\nabla u_t|^2 dx - \int_\Omega |u_t|^{p(x)} dx, \tag{87}$$

$$E'(t) = -\int_\Omega |\nabla u_t|^2 dx - \int_\Omega |u_t|^{p(x)} dx, \tag{88}$$

where

$$E(t) = \frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\Delta u\|^2) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx. \tag{89}$$

By setting $\mathscr{H}(t) = -E(t)$, we establish that $E(t) < 0$. Referring to Equation (88), it follows that $\mathscr{H}(t) \geq \mathscr{H}(0) > 0$:

$$\begin{aligned} \mathscr{H}(t) &= -\frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\Delta u\|^2) + \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx. \\ &\leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx \\ &\leq \frac{1}{q_1}\rho_{q(\cdot)(u)}. \end{aligned} \tag{90}$$

We then define the following equation:

$$\Psi(t) = \mathscr{H}^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon}{2}\|\nabla u\|^2, \tag{91}$$

for small ε that will be selected later, and

$$0 < \alpha \leq \min\left\{\frac{q_1 - 2}{q_1}, \frac{q_1 - p_1}{q_1(p_1 - 1)}\right\}. \tag{92}$$

By deriving Equation (91) and applying Equation (1), we acquire the following equation:

$$\Psi'(t) = (1-\alpha)\mathscr{H}^{-\alpha}(t)\mathscr{H}'(t) + \varepsilon\int_\Omega u_t^2 dx + \varepsilon\int_\Omega u u_{tt} dx, \tag{93}$$

$$\begin{aligned} \Psi'(t) + \frac{d}{dt}\left(\varepsilon\int_\Omega \nabla u \nabla u_t dx\right) &= (1-\alpha)\mathscr{H}^{-\alpha}(t)\mathscr{H}'(t) \\ &+ \varepsilon\int_\Omega (u_t^2 - |\nabla u|^2 + |\nabla u_t|^2 - |\Delta u|^2) dx \\ &+ \varepsilon\int_\Omega |u|^{q(x)} dx - \varepsilon\int_\Omega u|u_t|^{p(x)-2}u_t dx. \end{aligned} \tag{94}$$

We subsequently utilize Young's inequality for all, for all $\delta>0$, $\frac{1}{s}+\frac{1}{t}=1$:

$$XY \leq \frac{\delta^s}{s} X^s + \frac{\delta^{-t}}{t} Y^t,\ X, Y \geq 0, \tag{95}$$

to estimate the last term in Equation (94) as follows:

$$\int_\Omega u|u_t|^{p(x)} dx \leq \int_\Omega \frac{1}{p(x)} \delta^{p(x)} |u|^{p(x)} dx + \int_\Omega \frac{p(x)-1}{p(x)} \delta^{\frac{p(x)}{p(x)-1}} |u_t|^{p(x)} u_t dx, \tag{96}$$

which yields, by substitution in Equation (94):

$$\begin{aligned}\Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \nabla u \nabla u_t dx\right) \geq{}& (1-\alpha)\mathscr{H}^{-\alpha}(t)\mathscr{H}'(t) \\ &+ \varepsilon \int_\Omega (u_t^2 - |\nabla u|^2 + |\nabla u_t|^2 - |\Delta u|^2) dx \\ &+ \varepsilon \int_\Omega |u|^{q(x)} dx - \int_\Omega \frac{1}{p(x)} \delta^{p(x)} |u|^{p(x)} dx \\ &- \int_\Omega \frac{p(x)-1}{p(x)} \delta^{\frac{p(x)}{p(x)-1}} |u_t|^{p(x)} u_t dx.\end{aligned} \tag{97}$$

Therefore, by taking δ so that $\delta^{-p(x)/p(x)-1} = k\mathscr{H}^{-\alpha}(t)$, for large k to be specified later, and substituting in Equation (97), we arrive at the following equation:

$$\begin{aligned}\Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \nabla u \nabla u_t dx\right) \geq{}& \left[(1-\alpha) - \varepsilon k \frac{p_2 - 1}{p_2}\right] \mathscr{H}^{-\alpha}(t)\mathscr{H}'(t) \\ &+ \varepsilon \int_\Omega (u_t^2 - |\nabla u|^2 - |\nabla u_t|^2 - |\Delta u|^2) dx \\ &+ \varepsilon \int_\Omega |u|^{q(x)-1} u dx - \frac{k^{1-p_1}}{p_1} \mathscr{H}^{p_2 - 1}(t) \int_\Omega |u|^{p(x)} dx.\end{aligned} \tag{98}$$

Adding and subtracting $\varepsilon q_1 H(t)$ from the right-hand side of Equation (98), we obtain the following equation:

$$\begin{aligned}\Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \nabla u \nabla u_t dx\right) \geq{}& \left[(1-\alpha) - \varepsilon k \frac{p_2 - 1}{p_2}\right] \mathscr{H}^{-\alpha}(t)\mathscr{H}'(t) \\ &+ \varepsilon \left(1 + \frac{q_1}{2}\right) \int_\Omega u_t^2 dx + \varepsilon \left(\frac{q_1}{2} - 1\right) \int_\Omega |\nabla u|^2 dx \\ &+ \varepsilon \left(\frac{q_1}{2} - 1\right) \int_\Omega |\nabla u_t|^2 dx + \varepsilon \left(\frac{q_1}{2} - 1\right) \int_\Omega |\Delta u|^2 dx \\ &+ \varepsilon q_1 \mathscr{H}(t) - \varepsilon \frac{k^{1-p_1}}{p_1} \mathscr{H}^{\alpha(p_2 - 1)}(t) \int_\Omega |u|^{p(x)} dx.\end{aligned} \tag{99}$$

By exploiting Equation (99) and the inequality Lemma 9, we obtain the following equation:

$$\begin{aligned}\mathscr{H}^{\alpha(p_2-1)}(t)\int_\Omega |u|^{p(x)}dx &\le \mathscr{H}^{\alpha(p_2-1)}(t)C\left(\varrho(u)^{\frac{p_1}{q_1}}+\varrho(u)^{\frac{p_2}{q_1}}\right)\\ &\le \left(\frac{1}{q_1}\right)^{\alpha(p_2-1)}C\left(\|u\|_{q_1}^{p_1+\alpha q_1(p_2-1)}+\|u\|_{q_1}^{p_2+\alpha q_1(p_2-1)}\right)\end{aligned}, \tag{100}$$

hence, Equation (100) yields the following equation:

$$\begin{aligned}\Psi'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega \nabla u\nabla u_t dx\right) \ge &\left[(1-\alpha)-\varepsilon k\frac{p_2-1}{p_2}\right]\mathscr{H}^{-\alpha}(t)\mathscr{H}'(t)\\ &+\varepsilon\left(1+\frac{q_1}{2}\right)\int_\Omega u_t^2dx+\varepsilon\left(\frac{q_1}{2}-1\right)\int_\Omega|\nabla u|^2dx\\ &+\varepsilon\left(\frac{q_1}{2}-1\right)\int_\Omega|\nabla u_t|^2dx+\varepsilon\left(\frac{q_1}{2}-1\right)\int_\Omega|\Delta u|^2dx\\ &+\varepsilon q_1H(t)-\varepsilon\frac{k^{1-p_1}}{p_1}\left(\frac{1}{q_1}\right)^{\alpha(p_2-1)}\\ &\times C\left(\|u\|_{q_1}^{p_1+\alpha q_1(p_2-1)}+\|u\|_{q_1}^{p_2+\alpha q_1(p_2-1)}\right).\end{aligned} \tag{101}$$

We then use Lemma 8 and Equation (92), for $s=p_1+\alpha q_1(p_2-1)\le q_1$ and $s=p_2+\alpha q_1(p_2-1)\le q_1$, to deduce from Equation (101):

$$\begin{aligned}\Psi'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega \nabla u\nabla u_t dx\right) \ge &\left[(1-\alpha)-\varepsilon k\frac{p_2-1}{p_2}\right]\mathscr{H}^{-\alpha}(t)\mathscr{H}'(t)\\ &+\varepsilon\left(1+\frac{q_1}{2}\right)\int_\Omega u_t^2dx+\varepsilon\left(\frac{q_1}{2}-1\right)\int_\Omega|\nabla u|^2dx\\ &+\varepsilon\left(\frac{q_1}{2}-1\right)\int_\Omega|\nabla u_t|^2dx+\varepsilon\left(\frac{q_1}{2}-1\right)\int_\Omega|\Delta u|^2dx\\ &+\varepsilon\left[q_1\mathscr{H}(t)-k^{1-p_1}C_1\left(\mathscr{H}(t)+\|u_t\|^2+\|\nabla u_t\|^2+\|u\|_{q_1}^{q_1}\right)\right],\end{aligned} \tag{102}$$

where $C_1=\frac{2C}{p_1}\left(\frac{1}{q_1}\right)^{\alpha(p_2-1)}$. By noting that:

$$\mathscr{H}(t)=\frac{1}{q_1}\|u_t\|_{q_1}^{q_1}-\frac{1}{2}\left(\|u_t\|^2+\|\nabla u\|^2+\|\nabla u_t\|^2+\|\Delta u\|^2\right), \tag{103}$$

and writing $q_1=(q_1+2)/2+(q_1-2)/2$ yields the following equation:

$$\begin{aligned}N(t)\ge &\left[(1-\alpha)-\varepsilon k\frac{p_2-1}{p_2}\right]\mathscr{H}^{-\alpha}(t)\mathscr{H}'(t)\\ &+\varepsilon\left(\left(\frac{6+q_1}{4}\right)-k^{1-p_1}C_1\right)\|u_t\|^2+\varepsilon\left(\frac{q_1-2}{4}\right)\|\nabla u\|^2\\ &+\varepsilon\left(\frac{q_1-2}{4}-k^{1-p_1}C_1\right)\|\nabla u_t\|^2+\varepsilon\left(\frac{q_1-2}{4}\right)\|\Delta u\|^2\\ &\varepsilon\left(\frac{q_1+2}{2}-k^{1-p_1}C_1\right)\mathscr{H}(t)+\varepsilon\left(\frac{q_1-2}{2q_1}-k^{1-p_1}C_1\right)\|u\|_{q_1}^{q_1},\end{aligned} \tag{104}$$

where

$$N(t) = \Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \nabla u \nabla u_t dx\right). \tag{105}$$

At this point, we choose k large enough so that the coefficients of $\mathscr{H}(t)$, $\|u_t\|^2$, $\|\nabla u_t\|^2$, and $\|u\|_{q_1}^{q_1}$ in Equation (104) are strictly positive; hence, we get the following equation:

$$\Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \nabla u \nabla u_t dx\right) \geq \left[(1-\alpha) - \varepsilon k \frac{p_2 - 1}{p_2}\right] \mathscr{H}^{-\alpha}(t)\mathscr{H}'(t) + \varepsilon \left[\begin{matrix} \mathscr{H}(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 \\ + \|\Delta u\|^2 + \|u\|_{q_1}^{q_1} \end{matrix}\right], \tag{106}$$

where $\gamma > 0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha) - \varepsilon k(p_2 - 1)/p_2 \geq 0$ and

$$\Psi(0) = \mathscr{H}^{1-\alpha}(0) + \varepsilon \int_{\Omega} u_0 u_1 dx + \frac{\varepsilon}{2}\|\nabla u_0\|^2 > 0. \tag{107}$$

Therefore, Equation (106) takes the following form:

$$\Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \nabla u \nabla u_t dx\right) \geq \varepsilon\gamma\left[\mathscr{H}(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\Delta u\|^2 + \|u\|_{q_1}^{q_1}\right]. \tag{108}$$

Consequently, we have the following equation:

$$\Psi(t) \geq \Psi(0) > 0, \text{ for all } t \geq 0. \tag{109}$$

Next, we would like to show the following equation:

$$\Psi'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \nabla u \nabla u_t dx\right) \geq \Gamma \Psi^{1/(1-\alpha)}(t), \text{ for all } t \geq 0, \tag{110}$$

where Γ is a positive constant depending on $\varepsilon\gamma$ and C (the constant of Equation (81)). Once Equation (110) is established, we obtain in a standard way the finite time blow-up of $\Psi(t)$, hence of u (see Batle et al. [21] for instance).

To prove Equation (110), we first estimate the following equation:

$$\left|\int_{\Omega} u u_t(x,t) dx\right| \leq \|u\|_2 + \|u_t\|_2 \leq C\left(\|u\|_{q_1} + \|u_t\|_2\right), \tag{111}$$

which implies:

$$\left|\int_{\Omega} u u_t(x,t) dx\right|^{1/(1-\alpha)} \leq C\|u\|_{q_1}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}. \tag{112}$$

Again Young's inequality gives the following equation:

$$\left|\int_{\Omega} u u_t(x,t) dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{q_1}^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right], \tag{113}$$

for $\frac{1}{\theta} + \frac{1}{\mu} = 1$. We take $\theta = \frac{2}{1-\alpha}$, to get $\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq q_1$ by Equation (92). Therefore, Equation (113) becomes:

$$\left|\int_{\Omega} u u_t(x,t) dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{q_1}^{s} + \|u_t\|_2^{2}\right], \tag{114}$$

where $s = \frac{2}{1-2\alpha} \leq q_1$. By using Equation (83), we obtain for all $t \geq 0$:

$$\left|\int_{\Omega} u u_t(x,t) dx\right|^{1/(1-\alpha)} \leq C\left[\mathscr{H}(t) + \|u_t\|^2 + \|\nabla u_t\|^2 + \|u\|_{q_1}^{q_1}\right]. \tag{115}$$

Finally, by noting the following equation:

$$\Psi^{1/(1-\alpha)}(t) \leq C\left[\mathscr{H}^{1/(1-\alpha)}(t) + \varepsilon \int_{\Omega} u u_t(x,t) dx\right]^{1/(1-\alpha)} \leq 2^{1/(1-\alpha)} C\left[\mathscr{H}^{1/(1-\alpha)}(t) + \left|\int_{\Omega} u u_t(x,t) dx\right|^{1/(1-\alpha)}\right], \tag{116}$$

and combining it with Equations (108) and (115), the inequality (110) is established. A simple integration of Equation (110) over $(0,t)$, then yields the following equation:

$$\Psi^{1/(1-\alpha)}(t) \geq \frac{1}{\Psi^{-\alpha/(1-\alpha)}(t) - \Gamma t\alpha/(1-\alpha)}. \tag{117}$$

Therefore, Equation (117) shows that $\Psi(t)$ blows up in finite time:

$$\Psi^{*} \geq \frac{1-\alpha}{\Gamma\alpha[\Psi(0)]^{\alpha/(1-\alpha)}}, \tag{118}$$

where Γ and α are positive constant with $\alpha < 1$ and Ψ is given by Equation (91). This completes the proof. □

Data Availability

There are no underlying data supporting the results of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. G. Petrovsky, "Über das cauchysche problem für systeme von partiellen differential-gleichungen," *Математический сборник*, vol. 44, pp. 815–870, 1937.
[2] I. G. Petrovsky, "Sur l'ánalyticité des solutions des systémes d'équations différentielles," *Matematiceskij sbornik*, vol. 47, pp. 3–70, 1939.
[3] J. Ferreira, N. Irkıl, E. Pişkin, C. Raposo, and M. Shahrouzi, "blow up of solutions for a Petrovsky type equation with logarithmic nonlinearity," *Bulletin of the Korean Mathematical Society*, vol. 59, no. 6, pp. 1495–1510, 2022.
[4] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.
[5] L. Diening, P. Hasto, P. Harjulehto, and M. M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Springer-Verlag, Berlin Heidelberg, 2017.
[6] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, Springer, 2000.
[7] A. Ouaoua and W. Boughamsa, "Well-posedness and stability result for a class of nonlinear fourth-order wave equation with variable-exponents," *International Journal of Nonlinear Analysis and Applications*, vol. 14, no. 1, pp. 1769–1785, 2023.
[8] S. Antontsev, J. Ferreira, and E. Pişkin, "Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities," *Electronic Journal of Differential Equations*, vol. 2021, no. 6, pp. 1–18, 2021.
[9] Z. Tebba, S. Boulaaras, H. Degaichia, A. Allahem, and A. Farouk, "Existence and blow-up of a new class of nonlinear damped wave equation," *Journal of Intelligent & Fuzzy Systems*, vol. 38, no. 3, pp. 2649–2660, 2020.
[10] Z. Zhang and Q. Ouyang, "Global existence, blow-up and optimal decay for a nonlinear viscoelastic equation with nonlinear damping and source term," *Discrete and Continuous Dynamical Systems-B*, vol. 28, no. 9, pp. 4735–4760, 2023.
[11] Z. Zhang, Z. Liu, Y. Deng, J. Huang, and C. Huang, "Long time behavior of solutions to the damped forced generalized Ostrovsky equation below the energy space," *Proceedings of the American Mathematical Society*, vol. 149, no. 4, pp. 1527–1542, 2021.
[12] Z.-Y. Zhang and J.-H. Huang, "On solvability of the dissipative Kirchhoff equation with nonlinear boundary damping," *Bulletin of the Korean Mathematical Society*, vol. 51, no. 1, pp. 189–206, 2014.
[13] Z.-Y. Zhang, Z.-H. Liu, and X.-Y. Gan, "Global existence and general decay for a nonlinear viscoelastic equation with nonlinear localized damping and velocity-dependent material density," *Applicable Analysis*, vol. 92, no. 10, pp. 2021–2048, 2013.
[14] Z.-Y. Zhang, Z.-H. Liu, X.-J. Miao, and Y.-Z. Chen, "Global existence and uniform stabilization of a generalized dissipative Klein–Gordon equation type with boundary damping," *Journal of Mathematical Physics*, vol. 52, Article ID 023502, 2011.
[15] E. Pişkin and B. Okutmuştur, *An Introduction to Sobolev Spaces*, Bentham Science, 2021.
[16] V. Komornik, *Exact Controllability and Stabilization the Multiplier Method*, Masson-JohnWiley, Paris, 1994.
[17] S. A. Messaoudi, A. A. Talahmeh, and J. H. Al-Smail, "Nonlinear damped wave equation: existence and blow-up," *Computers & Mathematics with Applications*, vol. 74, no. 12, pp. 3024–3041, 2017.
[18] J. L. Lions, *Quelques Mthodes de Resolution des Problmes aux Limites Nonlinaires*, Dunod, Paris, 1969.
[19] S. Zheng, *Nonlinear Evolution Equations*, Chapman and Hall/CRC, Florida, 2004.
[20] S. A. Messaoudi, "Global existence and nonexistence in a system of Petrovsky," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 296–308, 2002.
[21] J. Batle, C. H. R. Ooi, A. Farouk, M. S. Alkhambashi, and S. Abdalla, "Global versus local quantum correlations in the Grover search algorithm," *Quantum Information Processing*, vol. 15, no. 2, pp. 833–849, 2016.