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Research Article

Existence and Nonexistence of Traveling Wave Solutions for a Reaction–Diffusion Preys–Predator System with Switching Effect

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In this paper, we are concerned with traveling wave solutions for two preys–one predator system with switching effect. First, we discuss that there is no traveling wave solution for this system by using linearization method. Second, applying super-sub solution method we establish the existence of semitraveling wave solutions with the minimal speed explicitly defined. Moreover, using the method of Lyapunov function and LaSalle's invariance principle, under certain conditions, we obtain that the semitraveling wave solutions connect the only positive equilibrium point at infinity, are actually traveling wave solutions. Finally, the numerical experiments support the validity of our theoretical results.

1. Introduction

Saha and Samanta [1] considered the following two preys–one predator system with switching effect:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \alpha_1 x_1\left(1-\dfrac{x_1}{K_1}\right) - \dfrac{\beta_1 x_1^2 y}{1+x_1+x_2}, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \alpha_2 x_2\left(1-\dfrac{x_2}{K_2}\right) - \dfrac{\beta_2 x_2^2 y}{1+x_1+x_2}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\tau_1\beta_1 x_1^2 y + \tau_2\beta_2 x_2^2 y}{1+x_1+x_2} - by, \end{cases} \tag{1}$$

where $x_i(t)$ $(i=1,2)$ and $y(t)$ are the densities of two preys and one predator, respectively. $\alpha_1, \alpha_2, K_1, K_2, \beta_1, \beta_2, \tau_1, \tau_2$ and b are positive constants, b is natural mortality. For more specific background details on this system, we can take a look at [1].

Intrapopulation competition of the predator is a key factor in accurately predicting the population spread of the model. Moreover, due to the uneven distribution of preys and predators in different spaces, in the current paper, we study the following PDE:

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = d_1 \triangle x_1 + \alpha_1 x_1\left(1-\dfrac{x_1}{K_1}\right) - \dfrac{\beta_1 x_1^2 y}{1+x_1+x_2}, \\ \dfrac{\partial x_2}{\partial t} = d_2 \triangle x_2 + \alpha_2 x_2\left(1-\dfrac{x_2}{K_2}\right) - \dfrac{\beta_2 x_2^2 y}{1+x_1+x_2}, \\ \dfrac{\partial y}{\partial t} = d_3 \triangle y + \dfrac{\tau_1\beta_1 x_1^2 y + \tau_2\beta_2 x_2^2 y}{1+x_1+x_2} - by - my^2, \end{cases} \tag{2}$$

where m is reduction rate of intrapopulation competition, d_1, d_2, d_3 denote the diffusion coefficients, respectively, which are positive constants.

If $\alpha_2\beta_1K_1 = \alpha_1\beta_2K_2$, by direct calculation, the system (2) has the planar equilibrium point and interior equilibrium point $E_1(K_1, K_2, 0)$ and $E_2(\eta_1^*, \eta_2^*, \theta^*)$, and

$$\theta^* = \frac{\alpha_1}{\beta_1\eta_1^*}(1+\eta_1^*+\eta_2^*)\left(1-\frac{\eta_1^*}{K_1}\right), \ \eta_2^* = \frac{K_2}{K_1}\eta_1^*, \tag{3}$$

where η_1^* is a real and positive root of the equation $a_1x_1^3 + a_2x_1^2 + a_3x_1 + a_4 = 0$, with

$$\begin{aligned}
a_1 &= \tau_1\beta_1^2 + \tau_2\beta_1\beta_2\frac{K_2^2}{K_1^2} + \frac{m\alpha_1}{K_1}\left(1+\frac{K_2}{K_1}\right)^2,\\
a_2 &= -\beta_1 b\left(1+\frac{K_2}{K_1}\right) + \frac{2m\alpha_1}{K_1}\left(1+\frac{K_2}{K_1}\right) - m\alpha_1\left(1+\frac{K_2}{K_1}\right)^2,\\
a_3 &= -b\beta_1 + \frac{m\alpha_1}{K_1} - 2m\alpha_1\left(1+\frac{K_2}{K_1}\right),\\
a_4 &= -m\alpha_1.
\end{aligned} \tag{4}$$

In the past three decades, the existence and asymptotic behavior of solutions for some models had been studied by many scholars. Zhang and Ouyang [2] proved the existence of global weak solutions for a viscoelastic wave equation with memory term, nonlinear damping and source term by using the potential well method combined with Galerkin approximation procedure. Zhang and Miao [3], using Glerkin method and the multiplier technique, obtained the existence and asymptotic behavior of strong and weak solutions for nonlinear wave equation with nonlinear damped boundary conditions, respectively.

Population ecology has been well-developed as an important branch of biomathematics, in which the existence and nonexistence of traveling wave solutions of biological system, is one of the most in-depth researches by scholars, where the Lotka–Volterra model has attracted much attentions. Dunbar [4, 5] in the known papers proved the existence of traveling wave solutions to a special prey–predator model by applying Lyapunov function. He proposed a two-step method for the existence of traveling wave solutions of some specific systems for prey and predator interactions. The first step, applying shooting argument, he demonstrated the existence of semitraveling wave solutions. The second step, he proved the semitraveling wave solutions actually connect to the positive equilibrium point by using the Lyapunov functions method. Lin et al. [6] studied the one prey–two predators model, and proved existence of traveling wave front connecting the trivial equilibrium point and the positive equilibrium point with some certain conditions by using the cross iteration method. Due to the variety and inhomogeneity of ecosystems, the study of the general diffusive prey–predator model has more important significance. Wang and Fu [7], by establishing Lyapunov function, proved the existence of traveling waves solutions to the reaction–diffusion prey–predator models with kinds of functional responses, may be decided by the predator and prey populations at the same time. Hsu and Lin [8] considered general diffusive prey–predator models. First, using the method of counter evidence they proved that the general diffusive predator–prey models has no positive traveling wave solutions under specific conditions. Then, applying the method of super-sub solutions, they proved existence of semitraveling wave solutions. Final, establishing the strictly contracting rectangles they concluded existence of traveling wave solutions. Huang and Ruan [9] studied the existence of traveling wave solutions for a reaction–diffusion system. Ai et al. [10] by constructing Lyapunov function and using the squeeze method proved a similar general existence result. For more results, we can see [11–18] and the references therein.

A solution $(x_1(\mu,t), x_2(\mu,t), y(\mu,t))$ for system (2) is called a traveling wave solution when it has the special form

$$(x_1(\mu,t), x_2(\mu,t), y(\mu,t)) = (X_1(\xi), X_2(\xi), Y(\xi)), \xi = \mu + ct, \tag{5}$$

where the wave speed c is positive constant, and (X_1, X_2, Y) satisfies the following ODE system:

$$\begin{cases}
cX_1' = d_1X_1'' + \alpha_1X_1\left(1-\frac{X_1}{K_1}\right) - \frac{\beta_1X_1^2Y}{1+X_1+X_2},\\
cX_2' = d_2X_2'' + \alpha_2X_2\left(1-\frac{X_2}{K_2}\right) - \frac{\beta_2X_2^2Y}{1+X_1+X_2},\\
cY' = d_3Y'' + Y\frac{\tau_1\beta_1X_1^2 + \tau_2\beta_2X_2^2}{1+X_1+X_2} - bY - mY^2,
\end{cases} \tag{6}$$

and the boundary conditions as follows:

$$0 < X_i(\xi) \le K_i,\ (i=1,2), 0 < Y(\xi) \le Y_0, \forall\xi\in\mathbb{R}, \tag{7}$$

$$(X_1, X_1', X_2, X_2', Y, Y')(-\infty) = (K_1, 0, K_2, 0, 0, 0), \tag{8}$$

$$(X_1, X_1', X_2, X_2', Y, Y')(\infty) = (\eta_1^*, 0, \eta_2^*, 0, \theta^*, 0), \tag{9}$$

where Y_0 is a positive constant.

In this paper, based on the idea from Ai et al. [10], we consider traveling wave solutions for two preys–one predator systems (2) with switching effect. We prove that the nonexistence and existence of traveling wave solutions of system (2), namely, we show the nonexistence and existence of positive solutions of system (6) satisfying (7), (8) and (9). Let us point out that although this idea has been used by the others, our application is new. Our problem is more difficult to solve, and we need more precise calculations.

The structure of the paper is organized as follows. Section 2 is devoted to the proof of nonexistence of semitraveling wave solutions for the system (2) by using linearization method. Section 3 is concerned with existence of semitraveling wave solutions by method of the super-sub solution and Schauder fixed point theorem. Such semitraveling wave solutions connect the planar equilibrium point $E_1(K_1, K_2, 0)$ at $\xi\to-\infty$. In Section 4, utilizing the Lyapunov function techniques, we show, with the aid of LaSalle's invariance principle, that semitraveling wave solutions of system (2) are traveling wave solutions. These traveling wave solutions connect the only positive equilibrium point $E_2(\eta_1^*, \eta_2^*, \theta^*)$ at $\xi\to\infty$ under the additional conditions. In Section 5, the numerical experiments support the validity of our theoretical results.

Hereafter, for convenience, we shall apply i to represent the number 1, 2.

2. Nonexistence of Semitraveling Wave Solutions

We pay attention to the nonexistence of semitraveling solutions for the system (2) in the section.

Let

$$s = \frac{\tau_1\beta_1 K_1^2 + \tau_2\beta_2 K_2^2}{1 + K_1 + K_2} - b,\ c^* = 2\sqrt{d_3 s}. \tag{10}$$

Our main result is as following.

Theorem 1. *Suppose $s>0$ holds. For $c<c^*$, the system (6) does not have positive solutions satisfying (8).*

Proof. Linearizing the last equation of system (6) around $(K_1, K_2, 0)$, we get

$$cY' = d_3 Y'' + Y\left(\frac{\tau_1\beta_1 K_1^2 + \tau_2\beta_2 K_2^2}{1 + K_1 + K_2} - b\right). \tag{11}$$

Thus the characteristic equation of (11) is as follows:

$$d_3\lambda^2 - c\lambda + s = 0. \tag{12}$$

Suppose λ_1 and λ_2 are two eigenvalues of (12), namely

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d_3 s}}{2d_3}, \lambda_2 = \frac{c + \sqrt{c^2 - 4d_3 s}}{2d_3}. \tag{13}$$

For contradiction, we suppose (X_1, X_2, Y) is a positive solution of system (6) with $c<c^* = 2\sqrt{d_3 s}$ satisfying (8). If $c \leq -2\sqrt{d_3 s}$, so that $\lambda_i<0$, then positive solution of (11) is unbounded as $\xi \longrightarrow -\infty$. Suppose $|c|<2\sqrt{d_3 s}$, then λ_1 and λ_2 form a complex conjugate pair: $p \pm qi$, where $p = c/(2d_3)$, $q = \sqrt{4d_3 s - c^2}/(2d_3)$. So the positive solutions of (11) are $e^{p\xi}\cos q\xi$ and $e^{p\xi}\sin q\xi$, and they cannot be of the same sign as ξ near negative infinity. Since both eigenvalues have nonzero real parts, the stability of the original equation at equilibrium $(K_1, K_2, 0)$ is the same as that of the linearized equation at equilibrium $(K_1, K_2, 0)$, yielding a contradiction. This proves Theorem 1. □

3. Existence of Semitraveling Wave Solutions

In order to prove the existence of semitraveling wave solutions for system (2), we first give the definition super-sub solutions, then we construct a pair of super-sub solutions of system (2), and finally we prove the existence of semitraveling wave solutions for system (2) by applying method of super-sub solution and Schauder fixed point theorem.

The definition of super-sub solutions of (6) as following.

Definition 1. The functions $(\overline{X}_1, \overline{X}_2, \overline{Y})$ and $(\underline{X}_1, \underline{X}_2, \underline{Y})$ on $\mathbb{R}$ are called a pair of super-sub solutions of (6) if the following

(i)

$$0 \leq \underline{X}_i \leq \overline{X}_i \leq U_{i0}, \qquad 0 \leq \underline{Y} \leq \overline{Y} \leq Y_0\ , \tag{14}$$

hold, where U_{i0} are positive constants, $(\overline{X}_1, \overline{X}_2, \overline{Y})$, $(\underline{X}_1, \underline{X}_2, \underline{Y})$ on $\mathbb{R}$ are continuous functions.

(ii) There is a finite set $\mathbb{B}$ so that:

(a) $\overline{X}_i, \underline{X}_i, \overline{Y}, \underline{Y} \in C^2(\mathbb{R}/\mathbb{B})$.

(b) The limits to $\underline{X}_i', \overline{X}_i', \underline{Y}', \overline{Y}'$, $\forall \xi \in \mathbb{B}$ satisfy:

$$\begin{aligned}&\underline{X}_i'(\xi-) \leq \underline{X}_i'(\xi+),\ \overline{X}_i'(\xi-) \geq \overline{X}_i'(\xi+),\\ &\underline{Y}'(\xi-) \leq \underline{Y}'(\xi+),\ \overline{Y}'(\xi-) \geq \overline{Y}'(\xi+).\end{aligned} \tag{15}$$

(iii) For continuous functions (X_1, X_2, Y) with $\underline{X}_i \leq X_i \leq \overline{X}_i$, $\underline{Y} \leq Y \leq \overline{Y}$, $\forall \xi \in \mathbb{R}/\mathbb{B}$ satisfy:

$$\begin{aligned}
&d_1\overline{X}_1''(\xi) - c\overline{X}_1'(\xi) + \alpha_1\overline{X}_1(\xi)\left(1 - \frac{\overline{X}_1(\xi)}{K_1}\right) - \frac{\beta_1\overline{X}_1^2(\xi)Y(\xi)}{1 + \overline{X}_1(\xi) + X_2(\xi)} \leq 0,\\
&d_2\overline{X}_2''(\xi) - c\overline{X}_2'(\xi) + \alpha_2\overline{X}_2(\xi)\left(1 - \frac{\overline{X}_2(\xi)}{K_2}\right) - \frac{\beta_2\overline{X}_2^2(\xi)Y(\xi)}{1 + X_1(\xi) + \overline{X}_2(\xi)} \leq 0,\\
&d_3\overline{Y}''(\xi) - c\overline{Y}'(\xi) + \overline{Y}(\xi)\frac{\tau_1\beta_1 X_1^2(\xi) + \tau_2\beta_2 X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} - b\overline{Y}(\xi) - m\overline{Y}^2(\xi) \leq 0,\\
&d_1\underline{X}_1''(\xi) - c\underline{X}_1'(\xi) + \alpha_1\underline{X}_1(\xi)\left(1 - \frac{\underline{X}_1(\xi)}{K_1}\right) - \frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1 + \underline{X}_1(\xi) + X_2(\xi)} \geq 0,\\
&d_2\underline{X}_2''(\xi) - c\underline{X}_2'(\xi) + \alpha_2\underline{X}_2(\xi)\left(1 - \frac{\underline{X}_2(\xi)}{K_2}\right) - \frac{\beta_2\underline{X}_2^2(\xi)Y(\xi)}{1 + X_1(\xi) + \underline{X}_2(\xi)} \geq 0,\\
&d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + \underline{Y}(\xi)\frac{\tau_1\beta_1 X_1^2(\xi) + \tau_2\beta_2 X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} - b\underline{Y}(\xi) - m\underline{Y}^2(\xi) \geq 0.
\end{aligned} \tag{16}$$

The following will provide the super-sub solutions required to show the existence of semitraveling wave solutions of the system (6) on $c>c^*$ and $c=c^*$, respectively.

Assume

$$\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2\le me^{\lambda x},\forall x<\frac{\ln Y_0}{\lambda}, \tag{17}$$

where $Y_0=(\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2-b)/m$, and

$$eY_0(1+K_1+K_2)\ge K_1+K_2, \tag{18}$$

hold.

Lemma 1. *Assume that $c>c^*$, $s>0$, and (17) is satisfied. $\lambda=(c-\sqrt{c^2-4d_3s})/(2d_3)$. Constants ω,κ,ζ,B one by one in the following order such that the inequalities*

$$\begin{aligned}
&0<\omega<\min\{\lambda,\omega_1^+,\omega_2^+\},\\
&\kappa>\max\left\{\left(\frac{QK_1^{\lambda/\omega-1}}{\kappa_1}\right)^{\omega/\lambda},\left(\frac{QK_2^{\lambda/\omega-1}}{\kappa_2}\right)^{\omega/\lambda},K_1\left(\frac{1}{Y_0}\right)^{\omega/\lambda},K_2\left(\frac{1}{Y_0}\right)^{\omega/\lambda},K_1,K_2\right\},\\
&0<\zeta<\omega,\ -d_3(\lambda+\zeta)^2+c(\lambda+\zeta)-s>0,\\
&B>\max\left\{\left(\frac{\kappa}{K_1}\right)^{\zeta/\omega},\left(\frac{\kappa}{K_2}\right)^{\zeta/\omega},\frac{S(1+2\kappa)}{-d_3(\lambda+\zeta)^2+c(\lambda+\zeta)-s}\right\},
\end{aligned} \tag{19}$$

hold, where

$$\omega_i^+=\frac{c}{d_i},\ \kappa_i=c\omega-d_i\omega^2,\ Q=\max\{\beta_1K_1^2,\beta_2K_2^2\}, \tag{20}$$

$$S=\max\left\{\frac{2\tau_1\beta_1K_1}{1+K_1+K_2},\frac{2\tau_2\beta_2K_2}{1+K_1+K_2},m\right\}. \tag{21}$$

We define $\overline{X}_i(\xi),\underline{X}_i(\xi),\overline{Y}(\xi),\underline{Y}(\xi)$ on $\mathbb{R}$ as follows:

$$\begin{aligned}
&\overline{X}_i(\xi)\equiv K_i,\ \underline{X}_i(\xi)=\begin{cases}K_i-\kappa e^{\omega\xi}, & \forall\ \xi\le a_i,\\ 0, & \forall\ \xi>a_i,\end{cases}\\
&\overline{Y}(\xi)=\begin{cases}e^{\lambda\xi}, & \forall\ \xi\le a_3,\\ Y_0, & \forall\ \xi>a_3,\end{cases}\quad \underline{Y}(\xi)=\begin{cases}e^{\lambda\xi}(1-Be^{\zeta\xi}), & \forall\ \xi\le a_0,\\ 0, & \forall\ \xi>a_0,\end{cases}
\end{aligned} \tag{22}$$

where

$$a_0=-\frac{1}{\zeta}\ln B,\quad a_i=-\frac{1}{\omega}\ln\frac{\kappa}{K_i},\quad a_3=\frac{1}{\lambda}\ln Y_0. \tag{23}$$

Then the system (6) has a pair of super-sub solutions $(\overline{X}_1,\overline{X}_2,\overline{Y})$ and $(\underline{X}_1,\underline{X}_2,\underline{Y})$.

Proof. Now we prove the above constants are well defined. First, we have

$$0<\frac{c-\sqrt{c^2-4d_3s}}{2d_3}=\lambda, \tag{24}$$

so that ω is well defined. Since choice of ω yields that $c\omega-d_i\omega^2>0$, the κ_i is well defined, so κ is well defined.

Due to the assumptions of ω,κ,ζ,B, we have $a_0<\max\{a_1,a_2\}<\min\{0,a_3\}$. According to the definitions of $\underline{X}_i,\overline{X}_i,\underline{Y},\overline{Y}$, it is clear that $\underline{X}_i(\xi)<\overline{X}_i(\xi)$ and $\underline{Y}(\xi)<\overline{Y}(\xi)$, $\forall\xi\in\mathbb{R}$, and

$$\begin{aligned}
&\underline{X}_i'(a_i-)=-\omega K_i<0=\underline{X}_i'(a_i+),\\
&\underline{Y}'(a_0-)=-\zeta e^{\lambda a_0}<0=\underline{Y}'(a_0+),\\
&\overline{Y}'(a_3-)=\lambda Y_0>0=\overline{Y}'(a_3+).
\end{aligned} \tag{25}$$

Let X_1,X_2,Y are continuous functions satisfying $\underline{X}_i\le X_i\le\overline{X}_i$ and $\underline{Y}\le Y\le\overline{Y}$.

Due to $\overline{X}_1\equiv K_1$, $\forall\xi\in\mathbb{R}$, so we obtain

$$\begin{aligned}
&d_1\overline{X}_1''(\xi)-c\overline{X}_1'(\xi)+\alpha_1\overline{X}_1(\xi)\left(1-\frac{\overline{X}_1(\xi)}{K_1}\right)-\frac{\beta_1\overline{X}_1^2(\xi)Y(\xi)}{1+\overline{X}_1(\xi)+X_2(\xi)}\\
&=-\frac{\beta_1K_1^2Y(\xi)}{1+K_1+K_2}\\
&<0.
\end{aligned} \tag{26}$$

If $\xi<a_1$, then $\underline{X}_1(\xi)=K_1-\kappa e^{\omega\xi}$, $d_1\underline{X}_1''(\xi)-c\underline{X}_1'(\xi)=\kappa\omega(c-d_1\omega)e^{\omega\xi}$. Combining with $\alpha_1\underline{X}_1(\xi)(1-\underline{X}_1(\xi)/K_1)\ge 0$, we have

$$\begin{aligned}
&\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\
&\ge-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\
&\ge-\beta_1K_1^2Y(\xi)\\
&\ge-Q\overline{Y}(\xi).
\end{aligned} \tag{27}$$

For $\xi<a_1$, since $a_1<a_3$, we have $\overline{Y}(\xi)=e^{\lambda\xi}$ and the inequality

$$\begin{aligned}
&d_1\underline{X}_1''(\xi)-c\underline{X}_1'(\xi)+\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\
&\geq \kappa\omega(c-d_1\omega)e^{\omega\xi}-Qe^{\lambda\xi}\\
&=\kappa e^{\omega\xi}\left[\omega(c-d_1\omega)-\frac{Qe^{(\lambda-\omega)\xi}}{\kappa}\right]\\
&\geq \kappa e^{\omega\xi}\left[\omega(c-d_1\omega)-\frac{Qe^{(\lambda-\omega)a_1}}{\kappa}\right]\\
&=\kappa e^{\omega\xi}\left[\omega(c-d_1\omega)-QK_1^{\lambda/\omega-1}\kappa^{-\lambda/\omega}\right].
\end{aligned} \tag{28}$$

Due to definition of κ, so we have

$$\begin{aligned}
&d_1\underline{X}_1''(\xi)-c\underline{X}_1'(\xi)+\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)\\
&-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\geq 0.
\end{aligned} \tag{29}$$

For $\xi>a_1$, we obtain $\underline{X}_1(\xi)=0$, and then

$$\begin{aligned}
&d_1\underline{X}_1''(\xi)-c\underline{X}_1'(\xi)+\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)\\
&-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}=0,
\end{aligned} \tag{30}$$

holds.

Similarly, we have

$$\begin{aligned}
&d_2\overline{X}_2''(\xi)-c\overline{X}_2'(\xi)+\alpha_2\overline{X}_2(\xi)\left(1-\frac{\overline{X}_2(\xi)}{K_2}\right)-\frac{\beta_2\overline{X}_2^2(\xi)Y(\xi)}{1+X_1(\xi)+\overline{X}_2(\xi)}\leq 0, \forall\xi\in\mathbb{R},\\
&d_2\underline{X}_2''(\xi)-c\underline{X}_2'(\xi)+\alpha_2\underline{X}_2(\xi)\left(1-\frac{\underline{X}_2(\xi)}{K_2}\right)-\frac{\beta_2\underline{X}_2^2(\xi)Y(\xi)}{1+X_1(\xi)+\underline{X}_2(\xi)}\geq 0, \forall\xi\in\mathbb{R}.
\end{aligned} \tag{31}$$

For $\xi<a_3$, we have $\overline{Y}(\xi)=e^{\lambda\xi}$, and by assumption (17), we infer

$$\begin{aligned}
&d_3\overline{Y}''(\xi)-c\overline{Y}'(\xi)+\overline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b\overline{Y}(\xi)-m\overline{Y}^2(\xi)\\
&=d_3\lambda^2e^{\lambda\xi}-c\lambda e^{\lambda\xi}+\left(\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b-me^{\lambda\xi}\right)e^{\lambda\xi}\\
&=e^{\lambda\xi}\left(d_3\lambda^2-c\lambda+\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b-me^{\lambda\xi}\right)\\
&=e^{\lambda\xi}\left(-\frac{\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2}{1+K_1+K_2}+\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-me^{\lambda\xi}\right)\\
&\leq e^{\lambda\xi}\left[\left(-\frac{1}{1+K_1+K_2}+1\right)(\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2)-me^{\lambda\xi}\right]\\
&\leq e^{\lambda\xi}[(\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2)-me^{\lambda\xi}]\\
&\leq 0.
\end{aligned} \tag{32}$$

For $\xi>a_3$, since $\overline{Y}(\xi)=Y_0$, so we have

$$\begin{aligned}
&\overline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b\overline{Y}(\xi)-m\overline{Y}^2(\xi)\\
&=Y_0\left(\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b-mY_0\right)\\
&\leq Y_0(\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2-b-mY_0)\\
&=0.
\end{aligned} \tag{33}$$

And hence

$$\begin{aligned}
&d_3\overline{Y}''(\xi)-c\overline{Y}'(\xi)+\overline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}\\
&-b\overline{Y}(\xi)-m\overline{Y}^2(\xi)\leq 0,
\end{aligned} \tag{34}$$

holds.

For $\xi < a_0$, we have

$$\underline{Y}(\xi) = e^{\lambda\xi}(1 - Be^{\zeta\xi}), \overline{Y}(\xi) = e^{\lambda\xi}. \tag{35}$$

By the definition of B, we obtain $\underline{Y}(\xi) \le \overline{Y}(\xi) \le e^{\lambda a_0}$. Thus, for $\xi < a_0$, it holds that

$$\begin{aligned}
&\underline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1^2(\xi)+X_2^2(\xi)} - b\underline{Y}(\xi) - m\underline{Y}^2(\xi)\\
&= s\underline{Y}(\xi) - \underline{Y}(\xi)\left(\frac{\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2}{1+K_1+K_2} - b - \frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}\right) - b\underline{Y}(\xi) - m\underline{Y}^2(\xi)\\
&= s\underline{Y}(\xi) - \left(\frac{\tau_1\beta_1K_1^2+\tau_2\beta_2K_2^2}{1+K_1+K_2} - \frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)} + m\underline{Y}(\xi)\right)\underline{Y}(\xi)\\
&\ge s\underline{Y}(\xi) - \left[\left(\frac{\tau_1\beta_1K_1^2}{1+K_1+K_2} - \frac{\tau_1\beta_1X_1^2(\xi)}{1+K_1+K_2}\right) + \left(\frac{\tau_2\beta_2K_2^2}{1+K_1+K_2} - \frac{\tau_2\beta_2X_2^2(\xi)}{1+K_1+K_2}\right) + m\underline{Y}(\xi)\right]\underline{Y}(\xi)\\
&\ge s\underline{Y}(\xi) - \left[\frac{\tau_1\beta_1}{1+K_1+K_2}2K_1(K_1 - X_1(\xi)) + \frac{\tau_2\beta_2}{1+K_1+K_2}2K_2(K_2 - X_2(\xi)) + m\underline{Y}(\xi)\right]\underline{Y}(\xi)\\
&\ge s\underline{Y}(\xi) - S[(K_1 - X_1(\xi)) + (K_2 - X_2(\xi)) + \underline{Y}(\xi)]\underline{Y}(\xi)\\
&\ge s\underline{Y}(\xi) - S[(K_1 - X_1(\xi)) + (K_2 - X_2(\xi)) + \overline{Y}(\xi)]\overline{Y}(\xi)\\
&\ge s\underline{Y}(\xi) - S(\kappa e^{\omega\xi} + \kappa e^{\omega\xi} + e^{\lambda\xi})\overline{Y}(\xi)\\
&= s\underline{Y}(\xi) - S\left(\kappa + \kappa + e^{(\lambda-\omega)\xi}\right)e^{\omega\xi}e^{\lambda\xi}\\
&\ge s\underline{Y}(\xi) - S(2\kappa+1)e^{\omega\xi}e^{\lambda\xi}.
\end{aligned} \tag{36}$$

Combining with the form of $\underline{Y}(\xi)$, for $a_0 < 0$ and $0 < \zeta < \omega$, we conclude

$$\begin{aligned}
&d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + \underline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1^2(\xi)+X_2^2(\xi)} - b\underline{Y}(\xi) - m\underline{Y}^2(\xi)\\
&\ge d_3\underline{Y}''(\xi) - cY'(\xi) + s\underline{Y}(\xi) - S(2\kappa+1)e^{\omega\xi}e^{\lambda\xi}\\
&= e^{(\lambda+\zeta)\xi}\left\{B[-d_3(\lambda+\zeta)^2 + c(\lambda+\zeta) - s] - S(2\kappa+1)e^{(\omega-\zeta)\xi}\right\}\\
&\ge 0.
\end{aligned} \tag{37}$$

For $\xi > a_0$, due to $\underline{Y}(\xi) = 0$, so we obtain

$$\begin{aligned}
&d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + \underline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1^2(\xi)+X_2^2(\xi)}\\
&-b\underline{Y}(\xi) - m\underline{Y}^2(\xi) = 0.
\end{aligned} \tag{38}$$

The proof is completed. □

Lemma 2. *Assume that $c = c^*$, $s > 0$, $\lambda = c/(2d_3)$, $M_1 = \lambda e Y_0$, and (17) and (18) are satisfied. Constants ω, κ one by one in the following order such that the inequalities*

$$\begin{aligned}
&0 < \omega < \min\{\lambda, \omega_1^+, \omega_2^+\},\\
&\kappa > \max\left\{e^{\frac{\omega}{\lambda-\omega}}, \frac{QM_1}{\kappa_1(\lambda-\omega)e}, \frac{QM_1}{\kappa_2(\lambda-\omega)e}\right\},
\end{aligned} \tag{39}$$

hold. There is a sufficiently large $N_0 > 0$ such that for $\forall N \ge N_0$, we define

$$a_i = \frac{1}{\omega}\ln\frac{K_i}{\kappa}, \quad a_3 = -\frac{1}{\lambda}, \quad a_0 = -\frac{N^2}{M_1^2}, \tag{40}$$

and

$$\overline{X}_i(\xi)\equiv K_i,\ \underline{X}_i(\xi)=\begin{cases}K_i-\kappa e^{\omega\xi}, & \forall\ \xi\le a_i,\\ 0, & \forall\ \xi>a_i,\end{cases}$$
$$\overline{Y}(\xi)=\begin{cases}M_1|\xi|e^{\lambda\xi}, & \forall\ \xi\le a_3,\\ Y_0, & \forall\ \xi>a_3,\end{cases}\ \underline{Y}(\xi)=\begin{cases}e^{\lambda\xi}\left(M_1|\xi|-N\sqrt{|\xi|}\right), & \forall\ \xi\le a_0,\\ 0, & \forall\ \xi>a_0.\end{cases}\tag{41}$$

Then the system (6) has a pair of super-sub solutions $(\overline{X}_1,\overline{X}_2,\overline{Y})$ *and* $(\underline{X}_1,\underline{X}_2,\underline{Y})$.

Proof. Similar Lemma 1, we can conclude that ω and κ are well defined. By the assumption of M_1,ω,κ and the a_i $(i=0,1,2,3)$, we have $a_0<a_i<a_3<0$ $(i=1,2)$, and N is sufficiently large.

Let X_1,X_2,Y are continuous functions on $\mathbb{R}$ satisfying $\underline{X}_i\le X_i\le\overline{X}_i$ and $\underline{Y}\le Y\le\overline{Y}$.

For $\xi<a_1$, combining with $\alpha_1\underline{X}_1(\xi)(1-\underline{X}_1(\xi)/K_1)\ge 0$, we infer that

$$\begin{aligned}&\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\&\ge-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\&\ge-\beta_1K_1^2Y(\xi)\\&\ge-Q\overline{Y}(\xi).\end{aligned}\tag{42}$$

Due to $\underline{X}_1=K_i-\kappa e^{\omega\xi}$ and $a_1<a_3$, so $\overline{Y}(\xi)=M_1|\xi|e^{\lambda\xi}$, one has

$$\begin{aligned}&d_1\underline{X}_1''(\xi)-c\underline{X}_1'(\xi)+\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\&\ge\kappa\omega(c-d_1\omega)e^{\omega\xi}-QM_1|\xi|e^{\lambda\xi}\\&\ge\kappa e^{\omega\xi}\left[\omega(c-d_1\omega)-\frac{QM_1|\xi|e^{(\lambda-\omega)\xi}}{\kappa}\right].\end{aligned}\tag{43}$$

Since derivative of $(|\xi|e^{(\lambda-\omega)\xi})'>0$ for $\xi\in(-\infty,-1/(\lambda-\omega))$ and $a_1<-1/(\lambda-\omega)$, so that $|\xi|e^{(\lambda-\omega)\xi}\le 1/[(\lambda-\omega)e]$ for $\xi<a_1$. In combination with the constraint on κ, we get

$$\begin{aligned}&d_1\underline{X}_1''(\xi)-c\underline{X}_1'(\xi)+\alpha_1\underline{X}_1(\xi)\left(1-\frac{\underline{X}_1(\xi)}{K_1}\right)-\frac{\beta_1\underline{X}_1^2(\xi)Y(\xi)}{1+\underline{X}_1(\xi)+X_2(\xi)}\\&\ge\kappa e^{\omega\xi}\left[\omega(c-d_1\omega)-\frac{QM_1}{\kappa(\lambda-\omega)e}\right]\\&\ge 0.\end{aligned}\tag{44}$$

In addition,

$$\underline{X}_1'(a_1-)=-\kappa\omega e^{\omega a_1}\le 0=\underline{X}_1'(a_1+).\tag{45}$$

Similarly, we obtain

$$\begin{aligned}&d_2\underline{X}_2''(\xi)-c\underline{X}_2'(\xi)+\alpha_2\underline{X}_2(\xi)\left(1-\frac{\underline{X}_2(\xi)}{K_2}\right)\\&-\frac{\beta_2\underline{X}_2^2(\xi)Y(\xi)}{1+X_1(\xi)+\underline{X}_2(\xi)}\ge 0,\end{aligned}\tag{46}$$

and

$$\underline{X}_2'(a_2-)=-\kappa\omega e^{\omega a_2}\le 0=\underline{X}_2'(a_2+).\tag{47}$$

For $\xi<a_0$, we have

$$\underline{Y}(\xi)=\left(M_1|\xi|-N\sqrt{|\xi|}\right)e^{\lambda\xi},\overline{Y}(\xi)=M_1|\xi|e^{\lambda\xi}.\tag{48}$$

For all large N, it readily follows that

$$\begin{aligned}&\underline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b\underline{Y}(\xi)-m\underline{Y}^2(\xi)\\&\ge s\underline{Y}(\xi)-S[(K_1-X_1(\xi))+(K_2-X_2(\xi)+\underline{Y}(\xi))]\underline{Y}(\xi)\\&\ge s\underline{Y}(\xi)-S\left[(K_1-\underline{X}_1(\xi))+\left(K_2-\underline{X}_2(\xi)+\overline{Y}(\xi)\right)\right]\overline{Y}(\xi)\\&=s\underline{Y}(\xi)-S(\kappa e^{\omega\xi}+\kappa e^{\omega\xi}+M_1|\xi|e^{\lambda\xi})\overline{Y}(\xi)\\&=s\underline{Y}(\xi)-M_1S\left(2\kappa+M_1|\xi|e^{(\lambda-\omega)\xi}\right)e^{\omega\xi}|\xi|e^{\lambda\xi}\\&\ge s\underline{Y}(\xi)-M_1S(2\kappa+M_1|\xi|)|\xi|e^{\omega\xi}e^{\lambda\xi}.\end{aligned}\tag{49}$$

Since

$$d_3(M_1\xi e^{\lambda\xi})'' - c(M_1\xi e^{\lambda\xi})' + sM_1\xi e^{\lambda\xi} = 0, \tag{50}$$

so

$$\begin{aligned} &d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + s\underline{Y}(\xi) = d_3(\underline{Y}(\xi) + M_1\xi e^{\lambda\xi})'' \\ &-c(\underline{Y}(\xi) + M_1\xi e^{\lambda\xi})' + s(\underline{Y}(\xi) + M_1\xi e^{\lambda\xi}), \end{aligned} \tag{51}$$

thus we have

$$\begin{aligned} (\underline{Y}(\xi) + M_1\xi e^{\lambda\xi})' &= \left(\frac{N}{2\sqrt{-\xi}} - \lambda N\sqrt{-\xi}\right)e^{\lambda\xi} = N\left(\frac{1}{2\sqrt{-\xi}} - \sqrt{-\xi}\lambda\right)e^{\lambda\xi}, \\ (\underline{Y}(\xi) + M_1\xi e^{\lambda\xi})'' &= N\left(-\frac{1}{4\xi\sqrt{-\xi}} + \frac{1}{\sqrt{-\xi}}\lambda - \sqrt{-\xi}\lambda^2\right)e^{\lambda\xi}, \end{aligned} \tag{52}$$

and

$$\begin{aligned} &d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + s\underline{Y}(\xi) \\ &= N\left(-\frac{d_3}{4\xi\sqrt{-\xi}} + \lambda\frac{d_3}{\sqrt{-\xi}} - d_3\lambda^2\sqrt{-\xi} - \frac{c}{2\sqrt{-\xi}} + c\lambda\sqrt{-\xi}\right)e^{\lambda\xi} - sN\sqrt{-\xi}e^{\lambda\xi} \\ &= \frac{-d_3N}{4\xi\sqrt{-\xi}}e^{\lambda\xi}. \end{aligned} \tag{53}$$

Thus we obtain

$$\begin{aligned} &d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + \underline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi) + \tau_2\beta_2X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} - b\underline{Y}(\xi) - m\underline{Y}^2(\xi) \\ &\geq \left[-\frac{d_3N}{4\xi\sqrt{-\xi}} - M_1S(2\kappa + M_1|\xi|)|\xi|e^{\omega\xi}\right]e^{\lambda\xi}. \end{aligned} \tag{54}$$

Since N is large enough, so it holds that

$$\begin{aligned} &d_3\underline{Y}''(\xi) - c\underline{Y}'(\xi) + \underline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi) + \tau_2\beta_2X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} \\ &-b\underline{Y}(\xi) - m\underline{Y}^2(\xi) \geq 0. \end{aligned} \tag{55}$$

Furthermore, we have

$$\underline{Y}'(a_0-) = \left[-M_1 + \frac{N}{2\sqrt{|a_0|}}\right]e^{\lambda a_0} = -\frac{M_1}{2}e^{\lambda a^0} < 0 = \underline{Y}'(a_0+). \tag{56}$$

For $\xi < a_3$, now we check the $\overline{Y}(\xi)$. Since

$$\overline{Y}(\xi) = -M_1\xi e^{\lambda\xi},\ \overline{Y}'(\xi) = -M_1(1 + \lambda\xi)e^{\lambda\xi}. \tag{57}$$

So that $\overline{Y}'(a_3-) = 0 = \overline{Y}'(a_3+)$. we can apply (17) and (18) to get

$$\begin{aligned} &\overline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi) + \tau_2\beta_2X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} - b\overline{Y}(\xi) - m\overline{Y}^2(\xi) - s\overline{Y}(\xi) \\ &= -M_1\xi e^{\lambda\xi}\left(\frac{\tau_1\beta_1X_1^2(\xi) + \tau_2\beta_2X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} - b + mM_1\xi e^{\lambda\xi}\right) + M_1\xi e^{\lambda\xi}\left(\frac{\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2}{1 + K_1 + K_2} - b\right) \\ &= -M_1\xi e^{\lambda\xi}\left(\frac{\tau_1\beta_1X_1^2(\xi) + \tau_2\beta_2X_2^2(\xi)}{1 + X_1(\xi) + X_2(\xi)} + mM_1\xi e^{\lambda\xi} - \frac{\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2}{1 + K_1 + K_2}\right) \\ &\leq -M_1\xi e^{\lambda\xi}\left[\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2 + M_1\xi(\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2) - \frac{\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2}{1 + K_1 + K_2}\right] \\ &\leq -M_1\xi e^{\lambda\xi}\left[\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2 - \frac{M_1}{\lambda}(\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2) - \frac{\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2}{1 + K_1 + K_2}\right] \\ &= -M_1\xi e^{\lambda\xi}(\tau_1\beta_1K_1^2 + \tau_2\beta_2K_2^2)\left(1 - eY_0 - \frac{1}{1 + K_1 + K_2}\right) \\ &\leq 0. \end{aligned} \tag{58}$$

Thus

$$\overline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-b\overline{Y}(\xi)-m\overline{Y}^2(\xi)\le s\overline{Y}(\xi), \tag{59}$$

combining with

$$d_3\overline{Y}''(\xi)-c\overline{Y}'(\xi)+s\overline{Y}(\xi)=0, \tag{60}$$

it follows that

$$\begin{aligned}&d_3\overline{Y}''(\xi)-c\overline{Y}'(\xi)+\overline{Y}(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}\\&-b\overline{Y}(\xi)-m\overline{Y}^2(\xi)\le 0.\end{aligned} \tag{61}$$

The proof is completed. □

For convenience, let

$$\begin{aligned}F_i(X_1(\xi),X_2(\xi),Y(\xi))&=\alpha_iX_i(\xi)\left(1-\frac{X_i(\xi)}{K_i}\right)-\frac{\beta_iX_i^2(\xi)Y(\xi)}{1+X_1(\xi)+X_2(\xi)}\quad(i=1,2),\\F_3(X_1(\xi),X_2(\xi),Y(\xi))&=Y(\xi)\frac{\tau_1\beta_1X_1^2(\xi)+\tau_2\beta_2X_2^2(\xi)}{1+X_1(\xi)+X_2(\xi)}-bY(\xi)-mY^2(\xi).\end{aligned} \tag{62}$$

Then the system (6) can be written as follows:

$$\begin{cases}cX_1'(\xi)=d_1X_1''(\xi)+F_1(X_1(\xi),X_2(\xi),Y(\xi)),\\cX_2'(\xi)=d_2X_2''(\xi)+F_2(X_1(\xi),X_2(\xi),Y(\xi)),\\cY'(\xi)=d_3Y''(\xi)+F_3(X_1(\xi),X_2(\xi),Y(\xi)).\end{cases} \tag{63}$$

We can easy verify that the $F_1(X_1,X_2,Y)$, $F_2(X_1,X_2,Y)$, $G(X_1,X_2,Y)$ satisfy Lipschitz condition on $[0,K_1]\times[0,K_2]\times[0,Y_0]$, namely

$$\begin{aligned}&|F_1(X_{11},X_{21},Y_1)-F_1(X_{12},X_{22},Y_2)|+|F_2(X_{11},X_{21},Y_1)-F_2(X_{12},X_{22},Y_2)|\\&+|F_3(X_{11},X_{21},Y_1)-F_3(X_{12},X_{22},Y_2)|\le\Omega(|X_{11}-X_{12}|+|X_{21}-X_{22}|+|Y_1-Y_2|),\end{aligned} \tag{64}$$

where Ω is a positive constant.

We give the following existence result of system (63) on semitraveling wave solutions.

Theorem 2. *If (17) and (18) hold. Then the system (63) has a positive solution (X_1,X_2,Y) for every $c\ge c^*$, and satisfying*

$$\underline{X}_i(\xi)\le X_i(\xi)\le\overline{X}_i(\xi),\ \underline{Y}(\xi)\le Y(\xi)\le\overline{Y}(\xi),\ \forall\,\xi\in\mathbb{R}, \tag{65}$$

and X_i', X_i'', Y', Y'' are bounded on $\mathbb{R}$. Moreover, the solution (X_1,X_2,Y) satisfying (8).

Proof. Define the functions $\widehat{F}_i=F_i(X_1,X_2,Y)+\Omega X_i$ and $\widehat{F}_3=F_3(X_1,X_2,Y)+\Omega Y$, where Ω is the constant in (64). We can easily check that $\widehat{F}_1(X_1,X_2,Y)$ is nondecreasing in $X_1\in[0,K_1]$ for every fixed $(X_2,Y)\in[0,K_2]\times[0,Y_0]$, $\widehat{F}_2(X_1,X_2,Y)$ is nondecreasing in $X_2\in[0,K_2]$ for every fixed $(X_1,Y)\in[0,K_1]\times[0,Y_0]$, $\widehat{F}_3(X_1,X_2,Y)$ is nondecreasing in $Y\in[0,Y_0]$ for every fixed $(X_1,X_2)\in[0,K_1]\times[0,K_2]$, then (63) can be rewritten as follows:

$$\begin{cases}d_1X_1''-cX_1'-\Omega X_1+\widehat{F}_1(X_1,X_2,Y)=0,\ \xi\in\mathbb{R},\\d_2X_2''-cX_2'-\Omega X_2+\widehat{F}_2(X_1,X_2,Y)=0,\ \xi\in\mathbb{R},\\d_3Y''-cY'-\Omega Y+\widehat{F}_3(X_1,X_2,Y)=0,\ \xi\in\mathbb{R}.\end{cases} \tag{66}$$

Let

$$\begin{aligned}W=\{(X_1,X_2,Y)\in[C(\mathbb{R})]^3|\,&\underline{X}_i(\xi)\le X_i(\xi)\\&\le\overline{X}_i(\xi),\underline{Y}(\xi)\le Y(\xi)\le\overline{Y}(\xi),\forall\xi\in\mathbb{R}\},\end{aligned} \tag{67}$$

and

$$\lambda_i^{\pm}=\frac{1}{2d_i}\left(c\pm\sqrt{c^2+4d_i\Omega}\right),i=1,2,3. \tag{68}$$

We define the map $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3): W \to C[\mathbb{R}]^3$ by

$$\Gamma_i(X_1, X_2, Y)(\xi) = \frac{1}{\sqrt{c^2 + 4d_i\Omega}} \left(\int_{-\infty}^{\xi} e^{\lambda_i^-(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_i^+(\xi-s)} \right) \widehat{F}_i(X_1, X_2, Y)(s)ds. \tag{69}$$

By variation-of-parameters formula we obtain that $(\Phi_1, \Phi_2, \Psi) = \Gamma(X_1, X_2, Y)$ for each $(X_1, X_2, Y) \in W$ is a bounded solution to the following system

$$\begin{cases} d_1\Phi_1'' - c\Phi_1' - \Omega\Phi_1 + \widehat{F}_1(\Phi_1, \Phi_2, \Psi) = 0, \\ d_2\Phi_2'' - c\Phi_2' - \Omega\Phi_2 + \widehat{F}_2(\Phi_1, \Phi_2, \Psi) = 0, \\ d_3\Psi'' - c\Psi' - \Omega\Psi + \widehat{F}_3(\Phi_1, \Phi_2, \Psi) = 0. \end{cases} \tag{70}$$

Apparently, the fixed point of Γ in W is a solution of system (63). So, we are going to prove that Γ in W has a fixed point. Inspired by [10], we define the Banach space

$$X_\rho = \left\{ (X_1, X_2, Y) \in [C(\mathbb{R})]^3 : \|(X_1, X_2, Y)\|_\rho < \infty \right\}, \tag{71}$$

with the exponentially weighted norm

$$\begin{aligned} \|(X_1, X_2, Y)\|_\rho &= \sup_{\xi \in \mathbb{R}} |(X_1(\xi), X_2(\xi), Y(\xi))| e^{-\rho|\xi|} \\ &= \sup_{\xi \in \mathbb{R}} [|X_1(\xi)| + |X_2(\xi)| + |Y(\xi)|] e^{-\rho|\xi|}, \end{aligned} \tag{72}$$

here $0 < \rho < \min\{|\lambda_i^-|\}$ $(i = 1, 2, 3)$, and we can easily know this subset W is closed, bounded and convex in X_ρ.

Obviously, $\Gamma : W \to W$ is Lipschitz continuous, and compact on W. From the Schauder fixed point theorem, it follows that Γ has a fixed point (X_1, X_2, Y) in W. Next, we prove that the X_i', X_i'', Y' and Y'' are bounded.

Note that for $\xi \in \mathbb{R}$

$$X_i'(\xi) = \frac{1}{\sqrt{c^2 + 4d_i\Omega}} \left(\lambda_i^- \int_{-\infty}^{\xi} e^{\lambda_i^-(\xi-s)} + \lambda_i^+ \int_{\xi}^{\infty} e^{\lambda_i^+(\xi-s)} \right) \widehat{F}_i(X_1, X_2, Y)(s)ds, \tag{73}$$

$$Y'(\xi) = \frac{1}{\sqrt{c^2 + 4d_3\Omega}} \left(\lambda_3^- \int_{-\infty}^{\xi} e^{\lambda_3^-(\xi-s)} + \lambda_3^+ \int_{\xi}^{\infty} e^{\lambda_3^+(\xi-s)} \right) \widehat{F}_3(X_1, X_2, Y)(s)ds. \tag{74}$$

It follows that $|X_i'(\xi)| \le M_0/(\sqrt{c^2 + 4d_i\Omega})$, and $|Y'(\xi)| \le M_0/(\sqrt{c^2 + 4d_3\Omega})$ for $\xi \in \mathbb{R}$, where

$$M_0 = \max\left\{ \left| \widehat{F}_i(X_1, X_2, Y) \right| \ (i = 1, 2, 3) : 0 \le X_i \le K_i, 0 \le Y \le Y_0 \right\}. \tag{75}$$

This shows that X_i' and Y' are bounded on $\mathbb{R}$, and using the system (63), the boundedness of X_i'' and Y'' are obtained as well. Finally, we show that the solution satisfying (8). Summarizing the above results, we obtain a solution (X_1, X_2, Y) for (63) satisfying $\underline{X}_i(\xi) \le X_i(\xi) \le \overline{X}_i(\xi)$ and $\underline{Y}(\xi) \le Y(\xi) \le \overline{Y}(\xi)$. Then by the definitions of $\underline{X}_i(\xi), \overline{X}_i(\xi), \underline{Y}(\xi)$ and $\overline{Y}(\xi)$, we have $(X_1, X_2, Y)(\xi) \to (K_1, K_2, 0)$ as $\xi \to -\infty$. Using the expressions

$$X_i'(\xi) = e^{\frac{c\xi}{d_i}} X_i'(0) + \int_{\xi}^{0} e^{\frac{c(\xi-s)}{d_i}} F_i(X_1(s), X_2(s), Y(s))ds, \tag{76}$$

$$Y'(\xi) = e^{\frac{c\xi}{d_3}} Y'(0) + \frac{1}{d_3} \int_{\xi}^{0} e^{\frac{c(\xi-s)}{d_3}} F_3(X_1(s), X_2(s), Y(s))ds, \tag{77}$$

we know $(X_1'(\xi), X_2'(\xi), Y'(\xi)) \to (0, 0, 0)$ as $\xi \to -\infty$. Therefore, (X_1, X_2, Y) is a positive solution satisfying (8).

The proof of Theorem 2 is given. □

4. Existence of Traveling Wave Solutions

Summarizing the above results, Theorem 2 established that the system (63) has a positive solution $(X_1.X_2, Y)$, and $(X_1, X_1', X_2, X_2', Y, Y')(-\infty) = (K_1, 0, K_2, 0, 0, 0)$. In what follows, we aim to verify the solution $(X_1.X_2, Y)$ satisfying $(X_1, X_1', X_2, X_2', Y, Y')(\infty) = (\eta_1^*, 0, \eta_2^*, 0, \theta^*, 0)$ by applying method of Lyapunov function.

Theorem 3. *Suppose that all conditions in Theorem 2 are met. Furthermore, assume that $\alpha_2\beta_1K_1 = \alpha_1\beta_2K_2$, $\tau_1\beta_1\eta_1^* + \tau_2\beta_2\eta_2^* < b$, and $Y_0 > \theta^*$ hold. Then the system (63) has a positive solution (X_1, X_2, Y) satisfying (8) and (9) for every $c \ge c^*$.*

Proof. We konw that system (63) admits a positive solution (X_1, X_2, Y) satisfying (8) by Theorem 2. We shall show that the $(X_1, X_1', X_2, X_2', Y, Y')(\xi) \to (\eta_1^*, 0, \eta_2^*, 0, \theta^*, 0)$ as $\xi \to \infty$. We construct a Lyapunov function V as follows:

$$\begin{aligned} V(X_1, X_1', X_2, X_2', Y, Y') = {} & cH(X_1, X_2, Y) - d_1 \frac{\partial H}{\partial X_1} X_1' \\ & - d_2 \frac{\partial H}{\partial X_2} X_2' - d_3 \frac{\partial H}{\partial Y} Y', \end{aligned} \tag{78}$$

in the following region

$$M=\left\{(X_1,X_2,Y)\in\mathbb{R}^3: X_i>\frac{2\alpha_i K_i(1+K_1+K_2)}{\alpha_i(1+K_1+K_2)+K_i\beta_i\theta^*}, Y\geq\theta^*\right\}, \tag{79}$$

where

$$\begin{aligned} H(X_1,X_2,Y)= {} & \tau_1\left[X_1-\eta_1^*-\eta_1^*\ln\left(\frac{X_1}{\eta_1^*}\right)\right] \\ & +\tau_2\left[X_2-\eta_2^*-\eta_2^*\ln\left(\frac{X_2}{\eta_2^*}\right)\right] \\ & +\left[Y-\theta^*-\theta^*\ln\left(\frac{Y}{\theta^*}\right)\right]. \end{aligned} \tag{80}$$

We have

$$\begin{aligned} \frac{dV}{d\xi}= {} & \left(\frac{\partial H}{\partial X_1}F_1(X_1,X_2,Y)+\frac{\partial H}{\partial X_2}F_2(X_1,X_2,Y)+\frac{\partial H}{\partial Y}F_3(X_1,X_2,Y)\right) \\ & -d_1\frac{\partial^2 H}{\partial X_1^2}(X_1')^2-d_2\frac{\partial^2 H}{\partial X_2^2}(X_2')^2-d_3\frac{\partial^2 H}{\partial Y^2}(Y')^2, \end{aligned} \tag{81}$$

where

$$\begin{aligned} & \frac{\partial H}{\partial X_1}F_1(X_1,X_2,Y)+\frac{\partial H}{\partial X_2}F_2(X_1,X_2,Y)+\frac{\partial H}{\partial Y}F_3(X_1,X_2,Y) \\ & =\tau_1\left(1-\frac{\eta_1^*}{X_1}\right)\left[\alpha_1X_1\left(1-\frac{X_1}{K_1}\right)-\frac{\beta_1X_1^2Y}{1+X_1+X_2}\right]+\tau_2\left(1-\frac{\eta_2^*}{X_2}\right)\left[\alpha_2X_2\left(1-\frac{X_2}{K_2}\right)-\frac{\beta_2X_2^2Y}{1+X_1+X_2}\right] \\ & \quad+\left(1-\frac{\theta^*}{Y}\right)\left[\frac{\tau_1\beta_1X_1^2Y}{1+X_1+X_2}+\frac{\tau_2\beta_2X_2^2Y}{1+X_1+X_2}-bY-mY^2\right] \\ & =\tau_1\alpha_1X_1\left(1-\frac{X_1}{K_1}\right)+\tau_2\alpha_2X_2\left(1-\frac{X_2}{K_2}\right)-\tau_1\alpha_1\eta_1^*\left(1-\frac{X_1}{K_1}\right)-\tau_2\alpha_2\eta_2^*\left(1-\frac{X_2}{K_2}\right) \\ & \quad+\frac{\tau_1\beta_1\eta_1^*X_1Y}{1+X_1+X_2}+\frac{\tau_2\beta_2\eta_2^*X_2Y}{1+X_1+X_2}-\frac{\tau_1\beta_1X_1^2\theta^*}{1+X_1+X_2}-\frac{\tau_2\beta_2X_2^2\theta^*}{1+X_1+X_2}-bY+b\theta^*-mY^2\left(1-\frac{\theta^*}{Y}\right) \\ & \leq\tau_1\alpha_1X_1-\frac{\tau_1\alpha_1}{K_1}X_1^2+\tau_2\alpha_2X_2-\frac{\tau_2\alpha_2}{K_2}X_2^2-\tau_1\alpha_1\eta_1^*+\frac{\tau_1\alpha_1\eta_1^*}{K_1}X_1-\tau_2\alpha_2\eta_2^*+\frac{\tau_2\alpha_2\eta_2^*}{K_2}X_2 \\ & \quad+\tau_1\beta_1\eta_1^*Y+\tau_2\beta_2\eta_2^*Y-\frac{\tau_1\beta_1X_1^2\theta^*}{1+K_1+K_2}-\frac{\tau_2\beta_2X_2^2\theta^*}{1+K_1+K_2}-b(Y-\theta^*) \\ & \leq 2\tau_1\alpha_1X_1+2\tau_2\alpha_2X_2-\frac{\tau_1\alpha_1}{K_1}X_1^2-\frac{\tau_2\alpha_2}{K_2}X_2^2-\tau_1\alpha_1\eta_1^*-\tau_2\alpha_2\eta_2^*+Y(\tau_1\beta_1\eta_1^*+\tau_2\beta_2\eta_2^*-b) \\ & \quad-\frac{\tau_1\beta_1X_1^2\theta^*}{1+K_1+K_2}-\frac{\tau_2\beta_2X_2^2\theta^*}{1+K_1+K_2} \\ & =\tau_1X_1\left[2\alpha_1-X_1\left(\frac{\alpha_1}{K_1}+\frac{\beta_1\theta^*}{1+K_1+K_2}\right)\right]+\tau_2X_2\left[2\alpha_2-X_2\left(\frac{\alpha_2}{K_2}+\frac{\beta_2\theta^*}{1+K_1+K_2}\right)\right] \\ & \quad+Y(\tau_1\beta_1\eta_1^*+\tau_2\beta_2\eta_2^*-b)-(\tau_1\alpha_1\eta_1^*+\tau_2\alpha_2\eta_2^*) \\ & <0. \end{aligned} \tag{82}$$

Table 1: Parametric values.

$d_1 = 1$	$d_2 = 1$	$d_3 = 1$	$\alpha_1 = 0.1$	$\alpha_2 = 0.2$	$K_1 = 25$
$K_2 = 100$	$\beta_1 = 0.4$	$\beta_2 = 0.2$	$\tau_1 = 0.1$	$\tau_2 = 0.06$	$m = 0.00001$

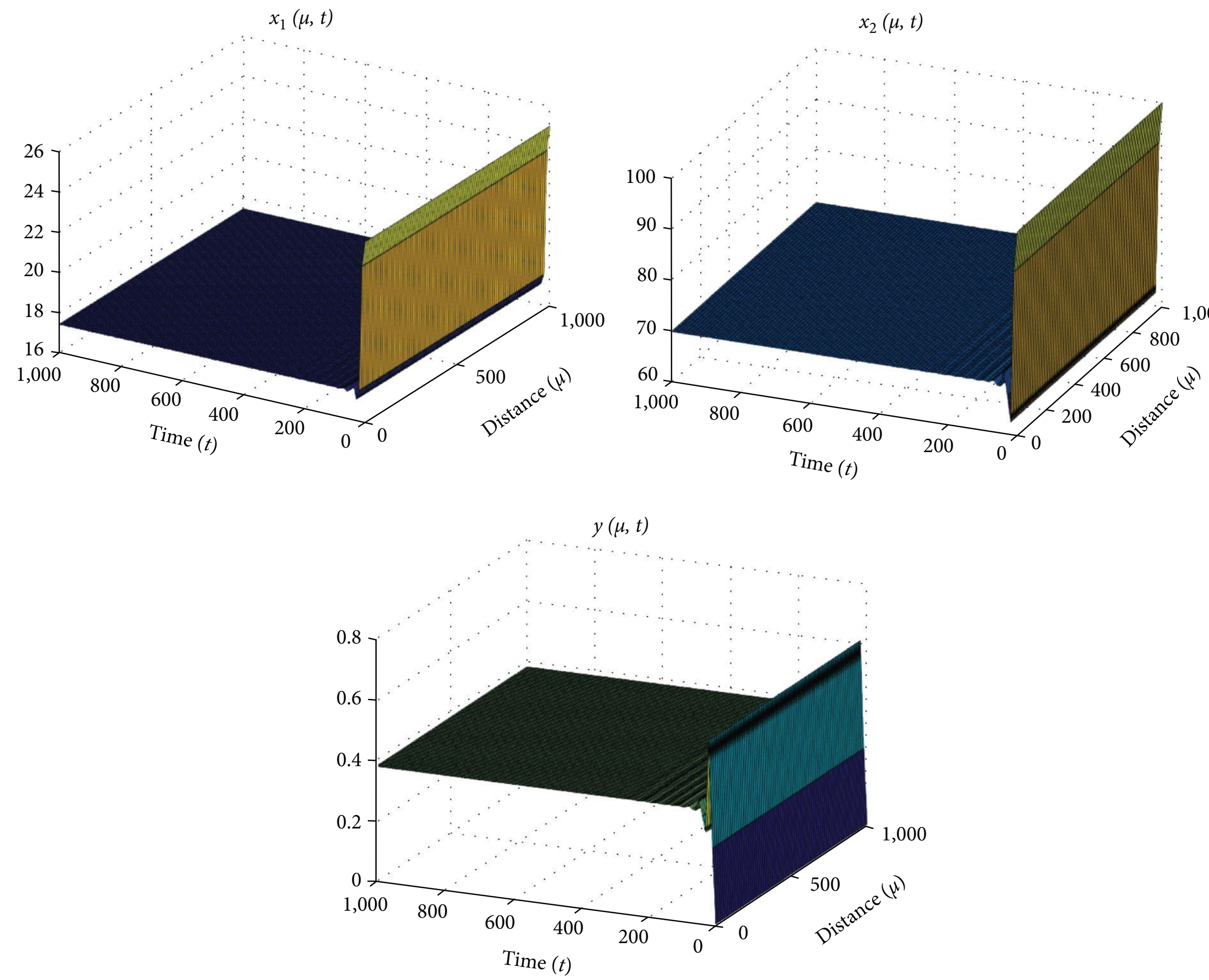

Figure 1: Profiles of the traveling wave solution to system (2) connecting E_2 for $b = 0.8$.

And we have

$$d_i \frac{\partial^2 H}{\partial X_i^2}(X_i')^2 = d_i\tau_i \frac{\eta_i^*}{X_i^2}(X_i')^2 = d_i\tau_i\eta_i^*\left(\frac{X_i'}{X_i}\right)^2, \quad (83)$$

$$d_3 \frac{\partial^2 H}{\partial Y^2}(Y')^2 = d_3 \frac{\theta^*}{Y^2}(Y')^2 = d_3\theta^*\left(\frac{Y'}{Y}\right)^2. \quad (84)$$

So

$$\frac{dV}{d\xi} < -d_1\tau_1\eta_1^*\left(\frac{X_1'}{X_1}\right)^2 - d_2\tau_2\eta_2^*\left(\frac{X_2'}{X_2}\right)^2 - d_3\theta^*\left(\frac{Y'}{Y}\right)^2 \le 0. \quad (85)$$

Let $\rho_i = \frac{X_i'}{X_i}$, $\rho_3 = \frac{Y'}{Y}$, ρ_i^+ be a positive constant solution of the following equation

$$\rho_i' = -\rho_i^2 + \frac{c}{d_i}\rho_i + \frac{\widehat{M}}{d_i}. \quad (86)$$

Then we have

$$\begin{aligned}\rho_i' &= \frac{X_i''X_i - (X_i')^2}{X_i^2} = \frac{X_i''}{X_i} - \rho_i^2 \\ &= \frac{c}{d_i}\rho_i - \frac{F_i}{d_iX_i} - \rho_i^2 \\ &\le -\rho_i^2 + \frac{c}{d_i}\rho_i + \frac{\widehat{M}}{d_i},\end{aligned} \quad (87)$$

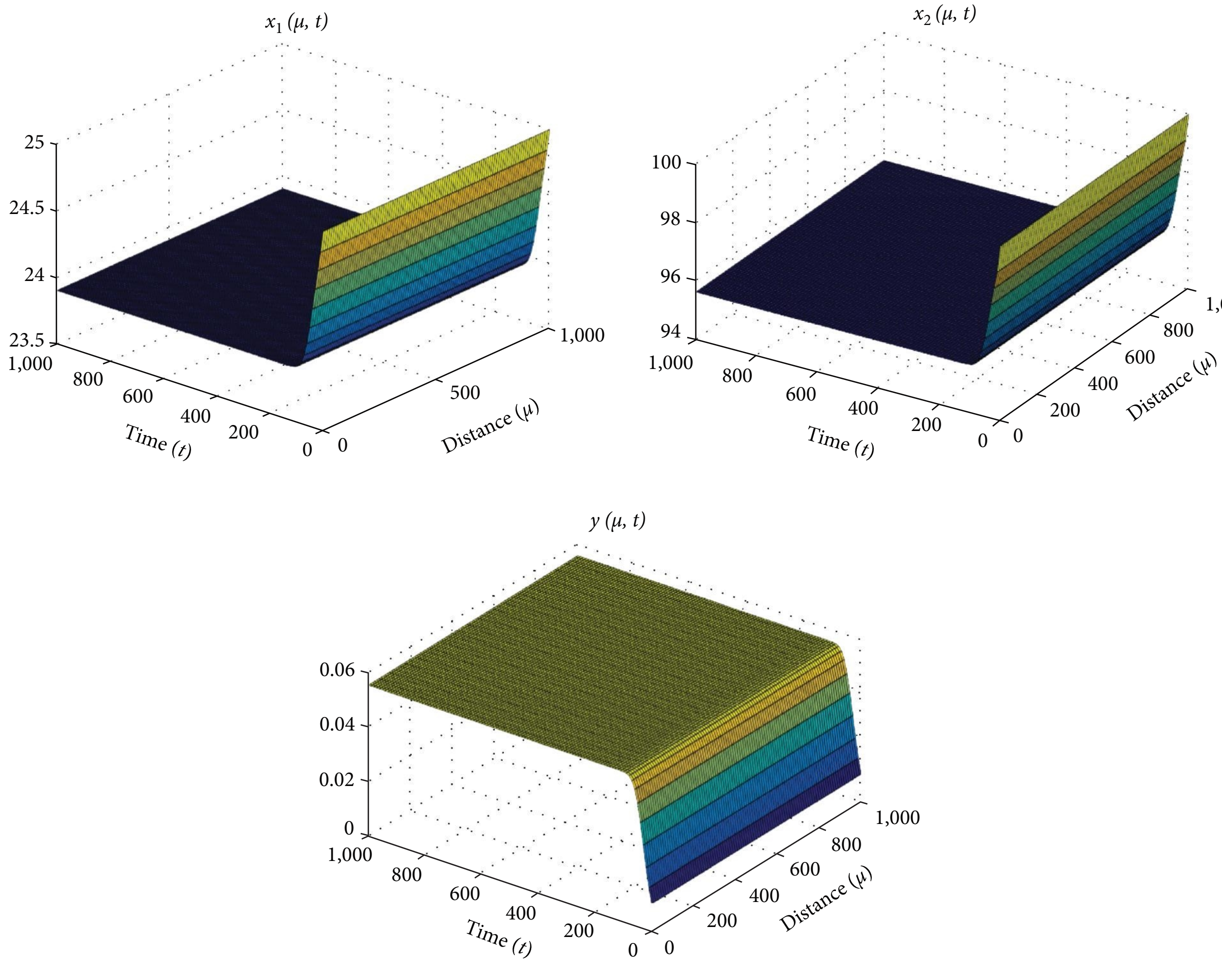

FIGURE 2: Profiles of the traveling wave solution to (2) connecting E_2 for $b = 1.1$.

and

$$\begin{aligned}\rho_3' &= \frac{Y''Y - (Y')^2}{Y^2} = \frac{Y''}{Y} - \rho_3^2 \\ &= \frac{c}{d_3}\rho_3 - \frac{F_3}{d_3 Y} - \rho_3^2 \\ &\leq -\rho_3^2 + \frac{c}{d_3}\rho_3 + \frac{\widehat{M}}{d_3},\end{aligned} \tag{88}$$

where $\widehat{M} = \max\{\frac{\alpha_i X_i}{K_i} + \frac{\beta_i X_i Y}{1+X_1+X_2}, b + mY\}$ $(i = 1, 2)$.

We can appeal to the comparison theorem to conclude $\rho_1(\xi) < \rho_1^+, \forall \xi \in \mathbb{R}$. If there exists ξ_0 such that $\rho_1(\xi_0) < -\rho_1^+$. We let $\rho(\xi)$ be the solution of $\rho'(\xi) = -\rho^2(\xi) + c/d_1\rho(\xi) + \widehat{M}/d_1$ with $\rho(\xi_0) = \rho_1(\xi_0)$ have a solution, then can apply the comparison theorem to derive that $\rho_1(\xi) \leq \rho(\xi), \forall \xi \geq \xi_0$. Notice

$$-\rho^2(\xi_0) + \frac{c}{d_1}\rho(\xi_0) - \frac{\widehat{M}}{d_1} < -(-\rho_1^+)^2 + \frac{c}{d_1}(-\rho_1^+) + \frac{\widehat{M}}{d_1} < 0. \tag{89}$$

This means that $\rho(\xi) \to -\infty$ as $\xi \to \xi_1$, where the ξ_1 is a finite number greater than ξ_0. It follows $\rho_1(\xi) \to -\infty$ as $\xi \to \xi_2$ for some $\xi_2 \in (\xi_0, \xi_1]$. Contradicting with the definition of ρ_1, we have $|\rho_1| \leq \rho_1^+$ for $\forall (X_1, X_2, Y) \in (0, K_1] \times (0, K_2] \times (0, Y_0]/\{(\eta_1^*, \eta_2^*, \theta^*)\}$.

Similarly, we can obtain constants $\rho_i^+ > 0$ such that $|\rho_i| \leq \rho_i^+$, $(i = 2, 3)$ for $\forall (X_1, X_2, Y) \in (0, K_1] \times (0, K_2] \times (0, Y_0]/\{(\eta_1^*, \eta_2^*, \theta^*)\}$. It shows that $dV/d\xi \leq 0$ and the equality hold only at $(\eta_1^*, 0, \eta_2^*, 0, \theta^*, 0)$.

Applying LaSalle's invariance principle, $(X_1, X_1', X_2, X_2', Y, Y')(\xi) \to (\eta_1^*, 0, \eta_2^*, 0, \theta^*, 0)$ as $\xi \to \infty$. Theorem 3 is proved. □

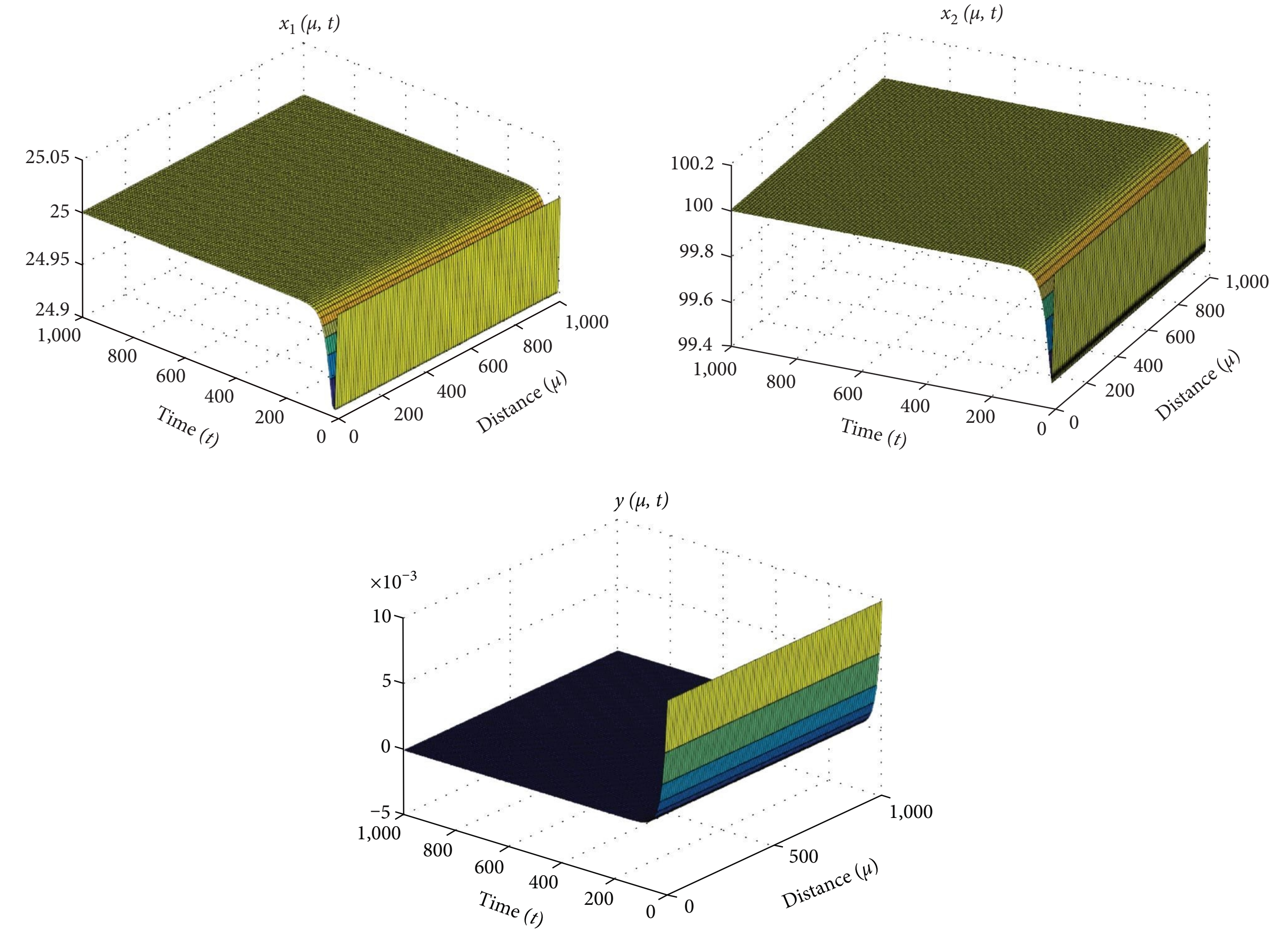

Figure 3: Profiles of the traveling wave solution to (2) connecting E_1 for $b = 1.2$.

5. Numerical Simulation

Numerical simulations are vital parts for the system (2) as they support the above theorem results. In current section, we provide the numerical simulations with the arithmetic software MATLAB.

We show that the values of the parameters in Table 1 are used to draw Figure 1 (for $b = 0.8$), Figure 2 (for $b = 1.1$), and Figure 3 (for $b = 1.2$). We assign the values as initial conditions $x_1(\mu, 0) = 25, x_2(\mu, 0) = 100, y(\mu, 0) = 0.01,$ and choose a small disturbance of steady-state $E_1(25, 100, 0)$ with only two preys to simulate a predator population invading a new resource habitat. Direct computations show that $E_2 = (17.3939, 69.5756, 0.3847)$ for $b = 0.8$, so we obtain that, from Figure 1, the system (2) admits a traveling wave solution, and the trajectory approaches connecting E_2. The Figures 2 and 3 show that if the predator mortality rate b decreases, and b is less than a certain value b^* $(1.1 \leq b^* < 1.2)$, then the three species approach toward coexistence.

6. Discussion

In this paper, we investigate the existence and nonexistence of traveling wave solutions for two preys–one predator system with switching effect. In order to be more practical and accurately predict the key factors of population dispersal, the spatial diffusive behavior of population and the internal competition of predator are considered. First, we use linearization method to discuss the nonexistence of semitraveling wave solutions with the wave speed $c < c^*$, and c^* is selected as the critical value (Theorem 1). Second, we apply super-sub solution method to obtain existence of semitraveling wave solutions, only connecting the planar equilibrium point $E_1(K_1, K_2, 0)$ with $c \geq c^*$ (Theorem 2). Moreover, utilizing method of Lyapunov function, we obtain traveling wave solutions to system (2), namely, the semitraveling wave solutions from Theorem 2 connect the only positive equilibrium point $E_2(\eta_1^*, \eta_2^*, \theta^*)$ at infinity (Theorem 3). Finally, we provide numerical experiments to demonstrate existence results of the traveling wave solutions to system (2), and we show that the three species approach toward coexistence when the predator mortality rate is less than a certain value.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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