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Research Article

On a Multistable Type of Free Boundary Problem with a Flux at the Boundary

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This paper studies the free boundary problem of a multistable equation with a Robin boundary condition, which may be used to describe the spreading of the invasive species with the solution representing the density of species and the free boundary representing the boundary of the spreading region. The Robin boundary condition $u_x(t,0)=\tau u(t,0)$ means that there is a flux of species at $x=0$. By studying the asymptotic properties of the bounded solution, we obtain the following two situations: (i) four types of survival states: the solution is either big spreading (the solution converges to a big stationary solution defined on the half-line) or small spreading (the solution converges to a small stationary solution defined on the half-line) or small equilibrium state (the survival interval $[0,h(t)]$ tends to a finite interval and the solution tends to a small compactly supported solution) or vanishing happens (the solution and the interval $[0,h(t)]$ shrinks to 0 as $t \longrightarrow T$ for $T<+\infty$); (ii) a trichotomous survival states of solutions: big spreading, big equilibrium state, and vanishing.

1. Introduction

Now, we study the problem having multistable nonlinearity

$$\begin{cases} u_t = u_{xx} + f(u), & x \in [0,h(t)], \quad t>0, \\ u(t,h(t))=0, & u_x(t,0)=\tau u(t,0), \quad t>0, \\ h'(t) = -u_x(t,x) - \delta, & t>0, \quad x=h(t), \\ u(0,x)=u_0(x), & x\in[0,h_0], \quad h_0 := h(0), \end{cases} \tag{1}$$

where $x=h(t)$ is a moving boundary, $\tau>0$ and $\delta>0$ are given constants, and the initial function u_0 belongs to $X(h_0)$, when $h_0>0$, where

$$X(h_0) := \left\{\phi \in C^2([0,h_0]) \colon \phi'(0)=\tau\phi(0), \phi(x)\ge 0\right\}. \tag{2}$$

Here, f is a multistable nonlinearity, $f:[0,+\infty) \longrightarrow \mathbb{R}$ is a C^1 function, and there are constants $\theta_0=0<\theta_1<\theta_2<\theta_3<\theta_4=1$, which satisfies

$$\begin{cases} f(\theta_i)=0 (i=0,1,2,3,4), \quad f(u)<0, \quad \theta\in(\theta_0,\theta_1)\cup(\theta_2,\theta_3), \\ f(u)>0, \quad \theta\in(\theta_1,\theta_2)\cup(\theta_3,\theta_4), \\ f'(\theta_i)<0 (i=0,2,4), \quad f'(\theta_j)>0 (j=1,3), \\ \int_{\theta_0}^{\theta_2} f(u)du>0, \quad \int_{\theta_2}^{\theta_4} f(u)du>0. \end{cases} \tag{3}$$

From this condition, one can regard $f(u)$ as a combination of two bistable functions. $C^2([0,h_0])$ is the space of functions with second derivative in $[0,h_0]$, and C^1 is the space of functions with first derivative.

There are some explanations for our problem. Usually, such problem used to describe the invading of new species or the development of chemical substances. Since the survival region of the species depends on time t, we use the moving interval $[0, h(t)]$ representing such region; the spreading speed of front $h(t)$ satisfies the classical Stefan condition, but there is a decay rate caused by the environment at the left boundary. We are primarily focused on the spreading of the solution for direction of right side, and there is a flux on the left boundary.

In the case $\delta = 0$ and $\tau = 0$, under monostable type of nonlinearity (cf. $f(u) = u(1-u)$) [1, 2] used such problem to explain the expanding of a species moving into a new place. Such species's density is given by $u(t, x)$, and the interval $[0, h(t)]$ is a region occupied by the species at time t. They proved that, when $h_0 \geq \pi/2$, only spreading happens for (u, h); when $h_0 < \pi/2$, there is a spreading/vanishing dichotomy conclusion: spreading (the solution $u \longrightarrow 1$ and $h(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$) or vanishing (the solution $u \longrightarrow 0$ and $h(t)$ tends to a positive number when $t \longrightarrow +\infty$). For bistable and combustion types of nonlinearity, [2, 3] studied the asymptotic behavior of solutions (i.e., the limits of the solutions when the time t goes to $+\infty$) for such a free boundary problem (also, the case $\tau = \delta = 0$). Moreover, [4] also obtained a dichotomy result with a advection term but under the condition $\delta = 0$ and $u_x(t, 0) = \tau_u(t, 0)$ replaced by $u(t, g(t)) = 0$. Besides, [5] studied the free boundary problems in high-dimensional space. For the situation $\tau = 0$, [6, 7] studied the convergence of solutions for a free problem and obtained a trichotomy result. When the nonlinearity is Fisher-KPP, [8] studied problem (1) when $\delta = 0$ and obtained a trichotomy result. For bistable nonlinearity, [2] obtained a trichotomy result for a free boundary problem when $\delta = \tau = 0$. When $f(u)$ is composed with a monostable and a bistable type of function, [9] studied the free boudary problem with Dirichlet boundary and obtained a richer phenomenon; [10] obtained another convergence conclusion. When is multistable, some papers studied the different travelling waves of reaction-diffusion equation, such as [11–15], and they considered propagating terrace.

Here, we will study the longtime limits of solutions of free boundary problem with $\tau > 0$ and $\delta > 0$. The constant $\delta > 0$ is the decay rate of the species at the spreading boundary. Such a condition also widely is used in protocell growth models (cf. [16, 17]). This boundary condition can also be deduced by reaction-diffusion equations (cf. [18]). Additionally, the boundary condition $u_x(t, 0) = \tau u(t, 0)$ in (1) means that there exists a flux of the species at the boundary $x = 0$. Under a multi-bistable nonlinearity, we consider the spreading of solution when the left boundary has a flux while the right boundary producing a decay rate of the solution (or, say the species). There are two critical points δ_0^1 and δ_0^2 (see details in Section 2) playing important roles in the long-time behavior of solutions; when $\delta \in (0, \delta_0^1)$, we have four types of diffusion: big spreading, small spreading, small equilibrium, and vanishing; when $\delta \in (\delta_0^1, \delta_0^2)$, we have big spreading, big equilibrium state, and vanishing.

The results in this paper are different from others; there are some different methods compared before ones, such as, we use a special stationary solution and an upper solution to prove that vanishing happens within a finite time. In the spreading situation, we first give the upper and lower bound of the limit of $u(t, x)$, then use a solution of another fixed boundary problem as a lower solution to prove that spreading happens. As for the sufficient conditions for the small and big spreading, we construct a moving lower solution.

In this paper, we use a multistable nonlinearity which often used in the study of travelling waves. Besides this, we add the influence of the boundary on the spreading of solution. Moreover, since the species is spreading on the right side, there may be an influx of species at the left boundary; so, we use the third boundary condition.

According to previous discussions [1, 2], we proved that equation (1) has a unique solution (u, h) defined on $[0, T_*)$, and $u(t, x) \in C^{(1+\gamma)/2, 1+\gamma}([0, T_*] \times [0, h(t)])$, $h(t) \in C^{1+\gamma/2}$ $([0, T_*])$ for $\gamma \in (0, 1)$. Additional references [19–27] further support this claim. Furthermore, if $h(T_*) > 0$, the solution can be extended to a larger interval $[0, T)$ with $T > T_*$. Additionally, using ([7], Lemma 2.8), we get that $h_\infty =: \lim_{t \longrightarrow T_*} h(t) \in (0, +\infty]$ exists.

We mainly consider the influences of the flux and the decay rate at the boundary on the asymptotic behavior of the solutions. We firstly obtain the following conclusions.

Theorem 1. *Assume $\tau > 0$ and $\delta \in (0, \delta_0^1)$. Then, the solution (u, h) of the problem (1) is either*

(i) Big spreading: $(0, h_\infty) = (0, +\infty)$, $T_* = +\infty$,

$u(t, \cdot) \longrightarrow V_\tau^2(\cdot)$ as $t \longrightarrow +\infty$ locally uniformly in $(0, +\infty)$ with V_τ^2 is the solution of

$$\begin{cases} v'' + f(v) = 0, & x > 0, \\ v'(0) = \tau v(0), & v(+\infty) = 1, \end{cases} \tag{4}$$

or

(ii) Small spreading: $(0, h_\infty) = (0, +\infty)$, $T_* = +\infty$,

$u(t, \cdot) \longrightarrow V_\tau^1(\cdot)$ as $t \longrightarrow +\infty$ locally uniformly in $(0, +\infty)$,

where V_τ^1 is the solution of (4) with 1 replaced by θ_2; or

(iii) Vanishing: $T_* < +\infty$, $h(t) \longrightarrow 0$, $\max_{0 \leq x \leq h(t)} u(t, x) \longrightarrow 0$, as $t \longrightarrow T_*$

or

(iv) In the small equilibrium state case: $h_\infty = l_1^*$ and

$$\lim_{t \longrightarrow +\infty} u(t, x) = v_1^*(x), x \in (0, h_\infty), \tag{5}$$

where $(l, v) = (l_1^*, v_1^*)$ is the smaller solution of

$$\begin{cases} v'' = f(v) = 0, & x \in (0, l), \\ v(l) = 0, & -v'(l) = \delta, \\ v'(0) = \tau v(0). \end{cases} \tag{6}$$

Theorem 2. *Assume $\tau > 0$ and $\delta \in (\delta_0^1, \delta_0^2)$, $u_0 = \rho\Phi$. Then,*

(i) *When $\rho > \rho^*$, big spreading happens*

(ii) *When $0 < \rho < \rho^*$, vanishing happens*

(iii) *When $\rho = \rho^*$, big equilibrium state happens: $h_\infty = l_2^*$ and*

$$\lim_{t\longrightarrow+\infty} u(t,x) = v_2^*(x), x \in (0, h_\infty), \tag{7}$$

where $(l, v) = (l_2^*, v_2^*)$ is the bigger solution of (6) (cf. Section 2).

Theorem 3. *Assume $\delta = \delta_0^1$; then, the solution is either vanishing or big spreading. When $\delta \geq \delta_0^2$, only vanishing happens for any solutions of (1).*

Remark 4. From the aspect of spreading for some species, $\delta \geq \delta_0^2$, that is, when the decay rate at the boundary is large, the environment at the boundary is so bad that the species cannot spread outside, and only vanishing happens.

The structure of this paper is organized as follows. In Section 2, we provide the stationary solutions of equation (1), in Section 3, we analyze the asymptotic behavior of solutions and present several sufficient conditions for spreading and vanishing in Section 4. In Section 5, we present the complete proof of our main theorems.

2. Stationary Solutions

This section focuses on the examination of stationary solutions for the given problem (1). Specifically, we consider

$$v'' + f(v) = 0, x > 0. \tag{8}$$

Let $p = v'$, then equation (8) is changed into

$$\frac{dp}{dv} = -\frac{f(v)}{p}. \tag{9}$$

According to the phase plane analysis (cf. [28]), the stationary solutions of (1) have the following cases (see Figure 1):

(i) Constant solutions: $\theta_0 = 0, \theta_1, \theta_2, \theta_3, \theta_4 = 1$

(ii) Increasing solutions defined on the half-line V_τ^i: $v(x) = V_\tau^i$ is the unique solution of (8) and satisfies

$$v'(0) = \tau v(0), v(+\infty) = \chi_i \text{ in } [0,+\infty), \tag{10}$$

where $\chi_i = \theta_2 (i = 1)$ or 1 $(i = 2)$. By the phase plane analysis, V_τ^i always exists for all $\tau > 0$ (cf. points B and D in Figure 1)

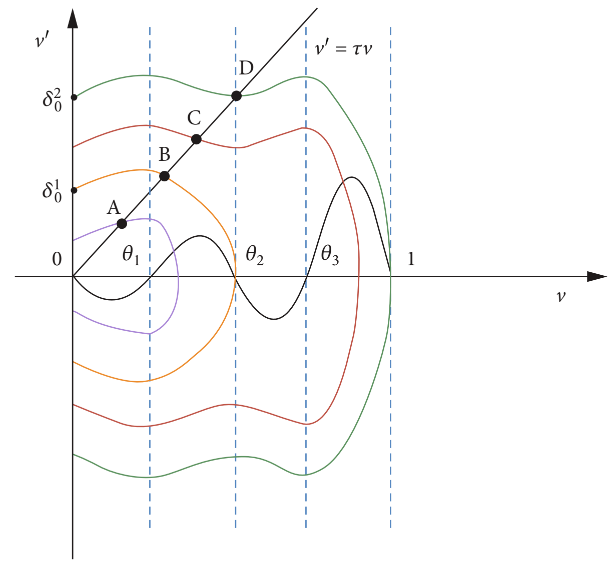

Figure 1: Points A and C are compactly supported solutions; points B and D are small and big increasing solutions defined on the half-line, respectively.

(iii) Decreasing solutions defined on the half-line U_0^i: $v(x) = U_0^i$ is the unique solution of (8) and satisfies

$$v(0) = 0, v(-\infty) = \chi_i \text{ in } [0,+\infty), \tag{11}$$

where $\chi_i = \theta_2 (i = 1)$ or 1 $(i = 2)$, denoted by

$$\delta_0^i := -\left(U_0^i\right)'(0). \tag{12}$$

(iv) Compactly supported solutions: on the phase plane, for any $\gamma \in (0, \delta_0^1) \cup (\delta_0^1, \delta_0^1)$, the problem

$$\begin{cases} v'' + f(v) = 0, & z \in (0, l), \\ v'(0) = \tau v(0), v(l) = 0, v'(l) = -\gamma, & v(z) \in (0, l), \end{cases} \tag{13}$$

has a unique solution (v_γ, l_γ). In addition, when $\gamma = \delta$, denoted its solution as (l_δ^*, v_δ^*)(cf. points A and C in Figure 1); when $\delta \in (0, \delta_0^1)$, denoted (l_δ^*, v_δ^*) as (l_1^*, v_1^*) (cf. point A in Figure 1); when $\delta \in (\delta_0^1, \delta_0^2)$, denoted (l_δ^*, v_δ^*) as (l_2^*, v_2^*) (cf. point C in Figure 1). Besides, for any $\gamma_1 < \gamma_2 \in (0, \delta_0^1) \cup (0, \delta_0^2)$, we have $l_{\gamma_1} < l_{\gamma_2}$.

(v) Compactly supported travelling wave $(v_c(x - ct), \ell_c)$: consider the problem

$$\begin{cases} v'' + cv' + f(v) = 0, & 0 < x < \ell, \\ v(0) = 0, v(\ell) = 0, & v'(\ell) = -\delta - c. \end{cases} \tag{14}$$

When $\beta \in (0, \delta_0^1) \cup (\delta_0^1, \delta_0^2)$, for any $c > 0$ smaller than the speed of the travelling wave of the equation $v'' + f(v) = 0$, then (14) has a unique solution, denoted by $(v_c(x - ct), \ell_c)$. Besides, (8) has other solutions, such as travelling wave and travelling semiwave solutions, groundstate solution which are not used in this paper.

3. Asymptotic Behavior of Solutions

Using the same method as in ([7], Lemma 2.5) with minor modifications, we have the following estimates.

Lemma 5. *Assume (3). Let (u, h) be a solution of (1) defined for $t \in [0, T)$, where $t \in [0, T)$. Then, there exists a positive constant M (depends on u_0 and h_0) such that $0 < u(t, x) \le M$, and $|u_x(t, x)| \le M$ for $t \in [0, T)$, and $x \in [0, h(t)]$.*

Also, there exists C depending on M but independent of T, such that $-\delta < h'(t) \le C$, for $t \in (0, T)$.

Proof. We first prove the bondedness of $u(t, x)$. By the property of f (i.e., (3)) and the comparison principle shows that $u(t, x) < \max\{\|u_0\|_{L^\infty}, 1\}$ for all $t \in [0, T)$ and $x \in [0, h(t)]$.

Hence, $u(t, x) \le M_1 := \max\{1, \|u_0\|_{L^\infty}\}$, for $t \in [0, T)$, and $x \in [0, h(t)]$.

We next consider the estimate of $h(t)$. Choose a large L satisfying

$$L := \max\left\{\frac{M_1 + \delta + \sqrt{(M_1 + \delta)^2 + N_1/2}, 2\|u_0\|_{C^1([0,h_0])}}{M_1}\right\}, \tag{15}$$

with $N_1 := \max_{0 \le u \le M_1} |f'(u)|$.

Construct

$$G(t, x) := M_1 L(h(t) - x)[2 - L(h(t) - x)], \tag{16}$$

for $0 < t < T$ and $x \in [h(t) - L^{-1}, h(t)]$. Then, $G(t, h(t)) = u(t, h(t)) = 0$ for $t \in (0, T)$.

The definitions of L and N_1 derive that

$$G_t - G_{xx} - f(G) \ge M_1\left[2L^2 - 4(M_1 + \delta)L - N_1\right] \ge 0, \tag{17}$$

for $0 < t < T$, $h(t) - L^{-1} < x < h(t)$.

Moreover,

$$G\left(t, h(t) - L^{-1}\right) = M_1 \ge u\left(t, h(t) - L^{-1}\right). \tag{18}$$

The classical comparison principle implies that

$$u(t, x) \le G(t, x), 0 < t < T, h(t) - L^{-1} < x < h(t). \tag{19}$$

Note that

$$h'(t) = -u_x(t, h(t)) - \delta \le -G_x(t, h(t)) - \delta \le 2MC_1 - \delta. \tag{20}$$

Denoted by $C := 2MC_1$ and $-u_x(t, h(t)) \le C$, according to the classical parabolic estimates, there exists a constant $M_2 > 0$ such that $|u_x(t, x)| < M_2$ when $0 < x < h(t)$ and define $M := \max\{M_1, M_2\}$; we get the conclusion. □

Lemma 6. *Let (u, h) be the solution of problem (1) for $t \in [0, T)$; if $h(t) \longrightarrow 0$ as $t \longrightarrow T$, then*

$$\lim_{t \longrightarrow T} \|u(t, \cdot)\|_{L^\infty(0,h(t))} = 0, \tag{21}$$

and $T \in (0, +\infty)$.

Proof. From $h(t) \longrightarrow 0$ and the estimates in Lemma 5, we can derive that, for any $\gamma < \delta_0^1$, there is a time T_γ such that, when $t > T_\gamma$, $u(t, x) \le v_\tau(x)$ for $x \in [0, h(t)]$.

By the property $v_\gamma \longrightarrow 0$ as $\gamma \longrightarrow 0$ (cf. Section 2), we have $u(t, x) \longrightarrow 0$ as $t \longrightarrow T$. And there is $T_1 < T$ such that $u(t, x) < 1$ for $x \in [0, h(t)]$.

Now, we show $T < +\infty$. Define

$$N := \max\left\{\delta + \sqrt{\delta^2 + 2F_0}, 4\varepsilon\right\}, \tag{22}$$

with $F_0 := \max_{0 \le u \le 1} |f'(u)|$. We choose a small $\varepsilon > 0$ such that

$$4N\varepsilon < \frac{\delta}{2}. \tag{23}$$

Also, there is $T_2 > T_1$ such that $u(t, x) < \varepsilon$ for $x \in [0, h(t)]$ and all $t \ge T_2$.

Construct a function

$$\bar{U}(t, x) := \varepsilon(h(t) - x)\left[2N + N^2(x - h(t))\right], \tag{24}$$

defined on

$$\Sigma := \left\{(t, x): \max\left\{0, h(t) - N^{-1}\right\} < x < h(t), t > T_2\right\}. \tag{25}$$

By calculation, $\bar{U}$ is an upper solution. From the Hopf lemma and the choice of ε, we have

$$-u_x(t, h(t)) < -\bar{U}(t, h(t)) = 4N\varepsilon < \frac{\delta}{2}. \tag{26}$$

Therefore,

$$h'(t) = -u_x(t, h(t)) - \delta < -\frac{\delta}{2}, \tag{27}$$

so $h(t) \longrightarrow 0$ as $t \longrightarrow T < 2(h_0/\delta)$. □

Lemma 7. *Assume $\delta \in (0, \delta_0^1) \cap (\delta_0^1, \delta_0^2)$, $0 < h_\infty < +\infty$ and (u, h) is the solution of (1). Then, $h_\infty = l_\delta^*$ and $\lim_{t \longrightarrow \infty} u(t, \cdot) = v_\delta^*$ in any subset of $(0, h_\infty)$, where (l_δ^*, v_δ^*) is the solution of (6).*

Remark 8. When $\delta = 0$ and $\tau = 0$, the convergence results in the above lemma never happens. Besides, if $f(u) = u(1 - u)$, as $\delta \longrightarrow 0$, we can show that $v_\delta^* \longrightarrow 0$ and $l_\delta^* \longrightarrow \pi$. This is the result in ([1], Lemma 3.1): vanishing happens when $h_\infty \le \pi$.

Proof. Since $0 < h_\infty < +\infty$, we deduce that, for any given $\nu > 0$, there is $t_\nu > 0$, such that $h(t) \in (h_\infty - \nu, h_\infty + \nu)$ for $t > t_\nu$. We now define an upper solution $V^+(t, x)$ which is the solution of

$$\begin{cases} V_t = V_{xx} + f(V), & t > t_\nu, \quad x \in (0, h_\infty + \nu), \\ V_x(t, 0) = \tau V(t, 0), & V(t, h_\infty + \nu) = 0, \quad t > t_\nu, \\ V(t_\nu, x) = V_0(x), & x \in (0, h_\infty + \nu), \end{cases} \tag{28}$$

where $V_0(x) \ge u(t_\nu, x)$ for $x \in (0, h_\infty)$. By the convergence result of the solution (cf. [2], Theorem 1.1), we have $V^+(t, x) \longrightarrow v_\nu^+(x)$ as $t \longrightarrow \infty$, where $v_\nu^+(x)$ is the solution of

$$\begin{cases} v'' + f(v) = 0, & 0 < x < h_\infty + \nu, \\ v'(0) = \tau v(0), & v(h_\infty + \nu) = 0. \end{cases} \tag{29}$$

Since V^+ is an upper solution, so $u(t, x) < V^+(x)$ for $x \in [0, h(t)]$ and $t > 0$. Therefore,

$$\lim_{t \longrightarrow +\infty} \sup u(t, \cdot) \le v_\nu^+(\cdot) \text{ in } [0, h_\infty]. \tag{30}$$

On the other hand, we can similarly prove

$$\lim_{t \longrightarrow +\infty} \inf u(t, \cdot) \ge v_\nu^-(\cdot) \text{ in } [0, h_\infty - \nu], \tag{31}$$

where v_ν^- is the solution of

$$\begin{cases} v'' + f(v) = 0, & 0 < x < h_\infty - \nu, \\ v(h_\infty - \nu) = 0, & v'(0) = \tau v(0). \end{cases} \tag{32}$$

We derive from the standard compactness and uniqueness argument that, as $\nu \longrightarrow 0$,

$$v_\nu^\pm(x) \longrightarrow w(x) \text{ in } C_{loc}^2([0, h_\infty]), \tag{33}$$

where w satisfies

$$\begin{cases} w'' + f(w) = 0, & 0 < x < h_\infty, \\ w(h_\infty) = 0, & w'(0) = \tau w(0). \end{cases} \tag{34}$$

Change $[0, h(t)]$ to $[0, h_0]$; furthermore utilizing the standard regularity theory of parabolic equation, we can obtain

$$\lim_{t \longrightarrow +\infty} \| u(t, \cdot) - w(\cdot) \|_{C^2([0, h(t)])} = 0. \tag{35}$$

This implies that, as $t \longrightarrow +\infty$,

$$u_x(t, h(t)) \longrightarrow w'(h_\infty). \tag{36}$$

From $0 < h_\infty < +\infty$, we must have $h'(t) \longrightarrow 0 (t \longrightarrow +\infty)$. This and the boundary condition $h'(t) = -u_x(t, h(t)) - \delta$ means that $-w'(h_\infty) = \delta$. From Section 2, the solution of problem (34) with $-w'(h_\infty) = \delta$ is nothing but $v_\delta^*(x)$, that is,

$$h_\infty = l_\delta^*, w(x) \equiv v_\delta^*(x) \text{ in } (0, h_\infty). \tag{37}$$

Moreover, when $\delta \in (0, \delta_0^1)$, $(l_\delta^*, v_\delta^*) = (l_1^*, v_1^*)$; when $\delta \in (\delta_0^1, \delta_0^2)$, $(l_\delta^*, v_\delta^*) = (l_2^*, v_2^*)$. □

Lemma 9. *Let (u, h) be a solution of problem (1); if $(0, h_\infty) = (0, +\infty)$, then*

$$V_\tau^1(x) \le \lim_{t \longrightarrow +\infty} \inf u(t, x) \le \lim_{t \longrightarrow +\infty} \sup u(t, x) \le V_\tau^2(x) \text{ in } (0, +\infty), \tag{38}$$

where V_τ^1 is the solution of (4) with 1 replaced by θ_2; V_τ^2 is the solution of (4).

Proof. Step 1. Firstly, choose $\gamma \in (\delta_0^1, \delta_0^2)$; let $v_\gamma(x)$ be the solution of (13) (cf. Section 2); we define an upper solution $u^+(t, x)$ which is the solution of

$$q_t - q_{xx} - f(q) = 0, x > 0, \quad t > 0, \tag{39}$$

with conditions

$$q(0, x) \ge \max \{u_0(x), v_\gamma(x)\}, x > 0; q_x(t, 0) = \tau q(t, 0), \quad t > 0. \tag{40}$$

Then, the comparison principle derives that $u(t, x) \le u^+(t, x)$, for $x \in [0, h(t)]$. By the convergence theorem of the solution, we have $u^+(t, x) \longrightarrow V_\tau^2(x)$ as $t \longrightarrow +\infty$ locally uniformly for $x \in (0, +\infty)$. Therefore,

$$\lim_{t \longrightarrow +\infty} \sup u(t, x) \le V_\tau^2(x), \text{for } x \in (0, +\infty). \tag{41}$$

Step 2. From $h_\infty = +\infty$, for any fixed large $X > h_0$, there is $T_X > 0$, such that $h(T_X) = X$ and $h(t) > X$ for $t > T_X$. We now define a lower solution $u_X^-(t,x)$ which is the solution of the problem

$$\begin{cases} v_t = v_{xx} + f(v), & 0 < x < X, \quad t > T_X, \\ v_x(t,0) = \tau v(t,0), & v(t,X) = 0, \quad t > T_X, \\ v(T_X, x) = u(T_X, x), & 0 < x < X. \end{cases} \tag{42}$$

Then, we deduce from the comparison principle that

$$u(t,x) \geq u_X^-(t,x) \text{ for } (t,x) \in (T_X, +\infty) \times [0, X]. \tag{43}$$

By the convergence result, $u_X^-(t,x) \longrightarrow u_X^*(x)$ as $t \longrightarrow +\infty$ locally uniformly in $[0,X]$, where u_X^* is the solution of

$$\begin{cases} \eta'' + f(\eta) = 0, & 0 < x < X, \quad t > 0, \\ \eta'(0) = \tau\eta(0), & \eta(X) = 0. \end{cases} \tag{44}$$

Therefore $\lim_{t\longrightarrow+\infty} \inf u(t,x) \geq u_X^*(x)$ locally uniformly in $\mathbb{R}$.

Step 3. As $X \longrightarrow \infty$, $u_X^*(x) \longrightarrow V_\tau^1(x)$ or V_τ^2 uniformly in any compact subset of $\mathbb{R}$.

Hence,

$$\lim_{t\longrightarrow+\infty} \inf u(t,x) \geq V_\tau^1(x), x \in [0, +\infty). \tag{45}$$

Therefore, (38) follows from (41) and (45). □

4. Sufficient Conditions

It is commonly known that the asymptotic behavior of solutions is affected by the initial data; here, we only give simple sufficient conditions for small (big) spreading, vanishing, and equilibrium state; these conditions will be utilized in the proof of the main theorems.

Lemma 10. *Let $u(t,x)$ be the solution of (1) with initial data $\tilde{u}_0 \in X(h_0)$.*

(i) Assume $\delta \in (0, \delta_0^1)$. If $\tilde{u}_0(x) > v_\delta^*(x)$ and $\tilde{u}_0(x) < V_\tau^1(x)$ for $x \in [0, h_0]$, then small spreading happens

(ii) Assume $\delta \in (0, \delta_0^2)$. Choose $\beta \in (\delta_0^1, \delta_0^2)$; if $\tilde{u}_0(x) > v_\beta(x)$ for $x \in [0, h_0]$, then big spreading happens

(iii) Assume $\delta \in (0, \delta_0^1) \cup (\delta_0^1, \delta_0^2)$. If $\tilde{u}_0(\cdot) \equiv v_\delta^*(\cdot)$ in $[0, h_0]$, then small (big) equilibrium state happens

(iv) For any $\delta > 0$. Choose $\gamma < \delta$; if $\tilde{u}_0(\cdot) < v_\gamma(x)$ for $x \in [0, h_0] \subset (0, l_\gamma)$, then vanishing happens

Proof.

(i) From the comparison principle and the definition of v_δ^* (which satisfies the boundary condition, cf. Section 2), we have $u(t,x) > v_\delta^*(x)$ for $x \in [0, h(t)] \supset [0, l_\delta^*]$. Moreover, let $(\ell_c, v_c(x - ct))$ be the compactly supported travelling wave in Section 2; we have, for some small c, $u_0(x) > v_c(x)$ for $x \in [0, h_0] \supset [0, \ell_c]$. Then, it follows from the comparison principle that

$$u(t,x) > v_c(x - ct), \text{for } x \in [ct, ct + \ell_c] \subset [0, h(t)], t > 0. \tag{46}$$

So, $h(t) > ct + \ell_c \longrightarrow +\infty$ as $t \longrightarrow \infty$.

Moreover, from $\tilde{u}_0(x) < V_\tau^1(x)$, we have $\lim\sup_{t\longrightarrow+\infty} u(t,x) \leq V_\tau^1(x)$ for $x \in [0, +\infty)$; combining this and Lemma 9, we get that small spreading happens.

(ii) By the comparison principle, we have

$$h(t) > l_\beta \text{ and } u(t,x) > v_\beta(x), \text{for all } t > 0, x \in [0, h(t)]. \tag{47}$$

So, $h_\infty > l_\beta > l_\delta^*$ (note that $\beta > \delta$). If $h_\infty < +\infty$, it then follows from Lemma 7 that the solution (h, u) converges to (l_δ^*, v_δ^*), but this is impossible since $h_\infty > l_\beta > l_\delta^*$. Therefore, $h_\infty = +\infty$. However, from Figure 1 in Section 2, we see that $\max_{x\in[0,l_\beta]} v_\beta(x) > \theta_2$; combining this, (47) and Lemma 9, we get that big spreading happens.

(iii) By the comparison principle, $h(t) \equiv l_\delta^*$ for all $t > 0$ and $u(t,x) \equiv v_\delta^*(x)$, for $x \in [0, l_\delta^*]$. This means that equilibrium happens

(iv) By the comparison principle,

$$\begin{aligned} h(t) < l_\tau, u(t,x) < v_\tau(x) \\ < v_\delta^*(x), \text{for all } t \in \mathbb{R}, x \in [0, h(t)]. \end{aligned} \tag{48}$$

By Section 2 (cf. Figure 1), $\max_{x\in[0,l_\tau]} v_\tau(x) < \theta_2$. So, small and big spreading cannot happen. But Lemma 7 implies that $h_\infty < +\infty$ is impossible since $h_\infty \leq l_\tau < l_\delta^*$ (notice that $\gamma < \delta$). So, $h(t)$ converges to 0 within a finite time. This means that vanishing happens. □

5. Proof of Main Theorems

Proof of Theorem 1. Theorem 1 follows from Lemmas 6, 7, and 9. □

Proof of Theorem 2. Vanishing, big equilibrium state, and big spreading follow from Lemmas 6, 7, and 9, respectively. To complete the proof of Theorem 1, we only need to prove the sharp result. Denote $u(t,x)$ by $u(t,x;\rho\phi)$ and $h(t)$ by $h(t;\rho\phi)$. Define

$$\rho^* = \sup\{\mu \ge 0 : (u(t,x;\rho\phi), h(t;\rho\phi))\text{vanishes}\}. \quad (49)$$

□

Proof. We divide the proof into three cases.

Case 1. $\rho < \rho^*$. We deduce from the definition of ρ^* and the classical comparison principle that vanishing happens for $u(t,x;\rho\phi)$.

Case 2. $\rho = \rho^*$. If vanishing happens, then there exists $T_1 > 0$ such that $u(T_1,x;\rho^*\phi) < \tilde{u}_0(x)$, where $\tilde{u}_0$ is given in Lemma 10 (iv). For any given sufficiently small $\varepsilon > 0$, we have

$$u(T_1,x;\rho_\varepsilon^+\phi) < \tilde{u}_0(x), 0 < x < h(T_1;\rho_\varepsilon^+\phi), \quad (50)$$

where $\rho_\varepsilon^+ := \rho^* + \varepsilon$. From Lemma 10, $u(t+T_1,x;\rho_\varepsilon^+\phi)$ vanishes. This also contradicts ρ^*. If big spreading happens when $\rho = \rho^*$.

This means $u(t,x;\rho^*\phi) \longrightarrow V_\tau^2(x)$ and $h(t;\rho^*\phi) \longrightarrow +\infty$ as $t \longrightarrow \infty$. So, there exists T_2 such that $h(T_2;\rho^*\phi) > l_2^*$ and $u(T_2,x;\rho^*\phi) > v_2^*(x)$ for $x \in [0,l_2^*]$. Also, there exists $\varepsilon > 0$ sufficiently small such that

$$\begin{aligned} u(T_2,x;(\rho^*-\varepsilon)\phi) > v_2^*(x), \text{and}\, h(T_2;\rho^*\phi) > l_2^*, x \in [0,l_2^*] \\ \subset [0,h(T_2;(\rho^*-\varepsilon)\phi)]. \end{aligned} \quad (51)$$

Lemma 9 implies big spreading for $u(t+T_2,x;(\rho^*-\varepsilon)\phi)$, a contradiction with ρ^*. Therefore, big spreading and vanishing cannot happen. Combine this and $\delta \in (\delta_0^1, \delta_0^2)$; we have big equilibrium state happens.

Case 3. $\rho > \rho^*$. The definition of ρ^* implies that vanishing cannot happen. We now prove that big equilibrium state is impossible. Otherwise, there is some $\rho_1 > \rho^*$ such that $u(t,x;\rho_1\phi) \longrightarrow v_2^*(x)$ as $t \longrightarrow +\infty$. By the comparison principle, for any $t > 0$,

$$h(t;\rho^*\phi) < h(t;\rho_1\phi) \text{ and } u(t,x;\rho^*\phi) < u(t,x;\rho_1\phi). \quad (52)$$

Since $u(t,x;\rho_1\phi)$ converges to v_2^*, for small $\varepsilon > 0$, there is large t_0 such that

$$\begin{aligned} h(t_0;\rho^*\phi) < l_{\delta-\varepsilon} < h(t_0;\rho_1\phi) \text{ and } u(t_0,x;\rho^*\phi) \\ < v_{\delta-\varepsilon}(x) < u(t_0,x;\rho_1\phi), \end{aligned} \quad (53)$$

where $(l_{\delta-\varepsilon}, v_{\delta-\varepsilon})$ is the compactly supported solution (cf. Section 2). By the comparison principle, for $t > 0$,

$$h(t+t_0;\rho^*\phi) < l_{\delta-\varepsilon} \text{ and } u(t+t_0,x;\rho^*\phi) < v_{\delta-\varepsilon}(x). \quad (54)$$

This contradicts case 2 that $u(t,x;\rho^*\phi)$ converges to the big equilibrium state. Therefore, combining this and Lemma 10, we have big spreading which happens when $\rho > \rho^*$. □

Proof of Theorem 3.

(i) We first prove that, when $\delta \ge \delta_0^2$, only vanishing happens. Actually, small and big equilibrium states are impossible when $\delta \ge \delta_0^2$, since there is no compactly supported solutions (cf. Section 2). We next show that big spreading is also impossible, and the proof for small spreading follows a similar approach. Suppose on the contrary that spreading happens, then

$$u(t,x) \longrightarrow V_\tau^2(x) \text{ as } t \longrightarrow +\infty. \quad (55)$$

Furthermore, the property $V_\tau^2(+\infty) = 1$ and ([29], Proposition A) imply that, for some $M_1 > 0$, $T_1 > 0$, and some $\nu \in (0, -f'(1))$, the following holds:

$$u(t,x) \le 1 + M_1 e^{-\nu t}, \text{for} x \in [0,h(t)], t \ge T_1. \quad (56)$$

For some $M' > M_1, \sigma > 0, X_0 > 2h(T_1)$, construct an upper solution:

$$\begin{aligned} \bar{h}(t) = X_0 + M'\sigma\left(1 - e^{-\nu t}\right), \bar{u}(t,x) \\ := \left(1 + M' e^{-\nu t}\right) U_0^2\left(x - \bar{h}(t)\right). \end{aligned} \quad (57)$$

Therefore, $h_\infty =: \lim_{t\longrightarrow+\infty} h(t) \le \lim_{t\longrightarrow+\infty} \bar{h}(t) < +\infty$. So, big spreading cannot happen. Suppose $h_\infty > 0$, it follows from Lemma 7 that equilibrium state happens, but this is impossible since there are no compactly supported solutions when $\delta > \delta_0^2$. Therefore, only vanishing happens.

(ii) When $\delta = \delta_0^1$, vanishing follows from Lemma 6, small spreading cannot happen using the same method as in the proof of (i). From Lemma 10 (ii), big spreading happens

□

6. Conclusions

This paper explores the free boundary problem with multistable nonlinearity and presents two cases for the solution's spreading. The first one is four types of diffusion: big spreading, small spreading, small equilibrium and vanishing; the second one is a trichotomy result: big spreading, big equilibrium state and vanishing. Additionally, it is a good idea to consider the free boundary problem of fractional differential equation (see also [30]). Also, we will consider the problem in time/space environment. Besides, we will find some interesting models from other background (cf. [31–33]).

Data Availability

All the data and formulas are in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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